IMGS-261 Solution Set #7

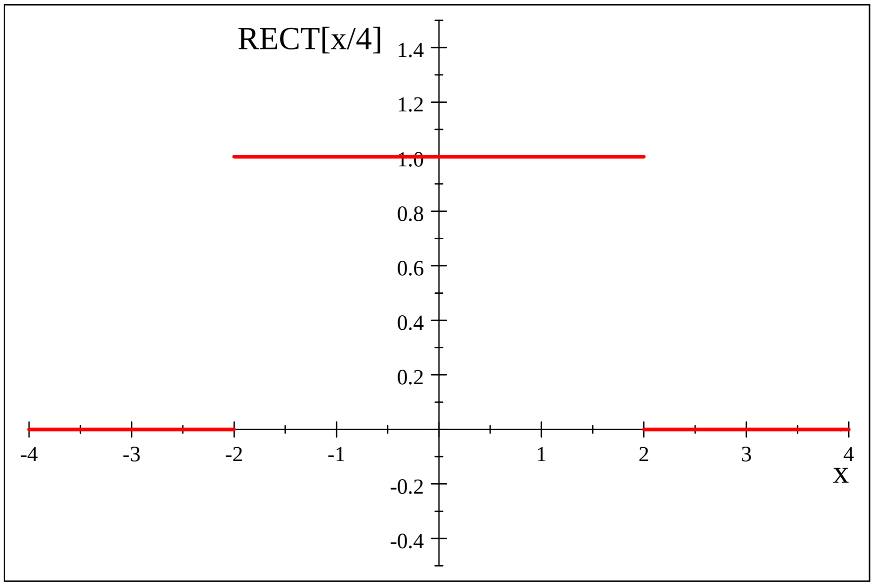
1. Sketch the following functions and evaluate their continuous Fourier transforms

   (a) $f_1[x] \equiv \begin{cases} 0 & \text{if} \quad |x| > 2 \\ \frac{1}{2} & \text{if} \quad |x| = 2 \\ 1 & \text{if} \quad |x| < 2 \end{cases}$

   Solution: *Another way to write this is in the form*

$$f_1[x] \equiv \begin{cases} 0 & \text{if} & -x > 2 > x < -2 \\ \frac{1}{2} & \text{if} & |x| = 2 \implies x = -2 \\ 1 & \text{if} & |x| < 2 \implies -2 < x < +2 \\ \frac{1}{2} & \text{if} & |x| = 2 \implies x = +2 \\ 0 & \text{if} & +x > 2 \end{cases}$$

So this is really $f_1[x] = RECT\left[\frac{x}{4}\right]$

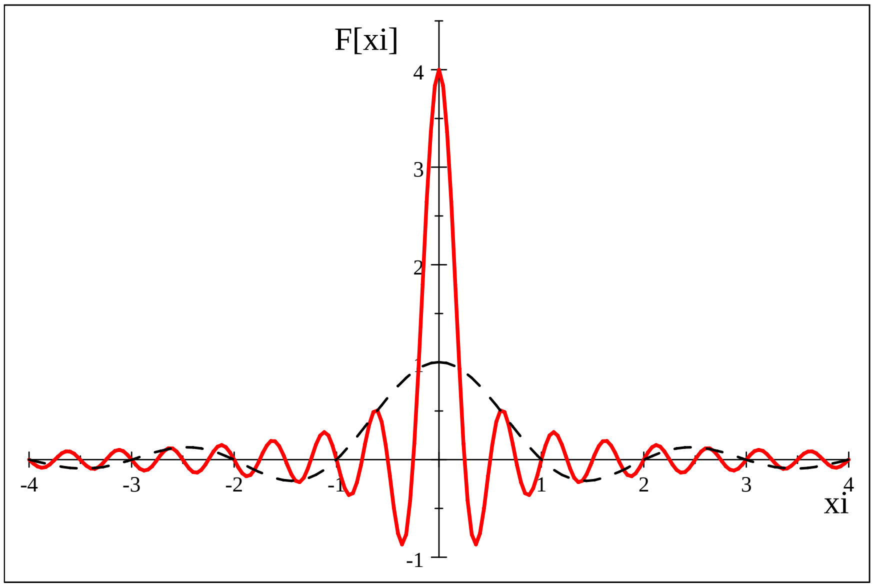


The Fourier transform is easy to derive by direct integration:

$$\mathcal{F}\left\{RECT\left[\frac{x}{4}\right]\right\} = \int_{x=-\infty}^{x=+\infty} RECT\left[\frac{x}{4}\right] \cdot \exp\left[-i \cdot 2\pi \cdot \xi x\right]\; dx$$

The function evaluates to zero outside the window ±2 *and to unity inside it:*

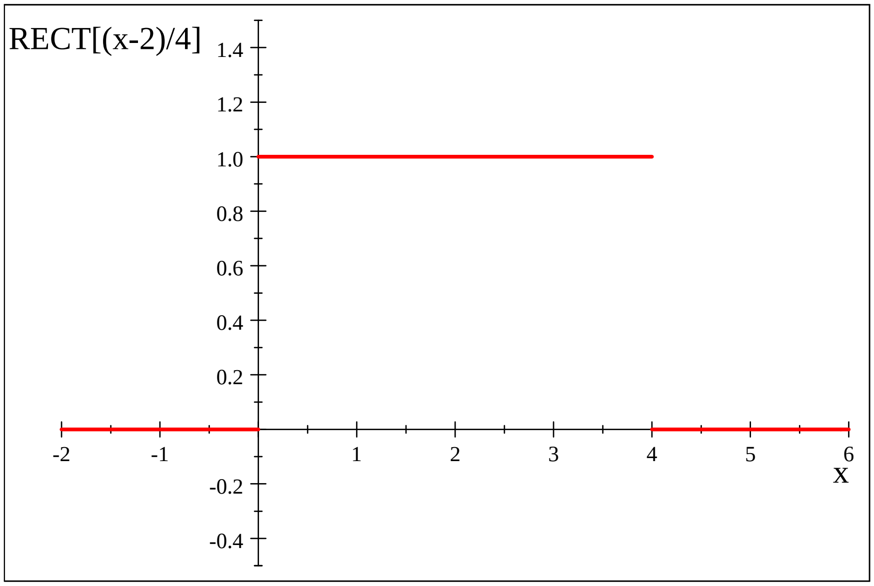
$$\begin{aligned}
\mathcal{F}\left\{RECT\left[\frac{x}{4}\right]\right\} &= \int_{x=-2}^{x=+2} 1\cdot\exp\left[-i\cdot 2\pi\cdot\xi x\right]\ dx \\
&= \left.\frac{\exp\left[-i\cdot 2\pi\cdot\xi x\right]}{-i\cdot 2\pi\cdot\xi}\right|_{x=-2}^{x=+2} \\
&= \frac{\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]}{-i\cdot 2\pi\cdot\xi} - \frac{\exp\left[-i\cdot 2\pi\cdot\xi\cdot(-2)\right]}{-i\cdot 2\pi\cdot\xi} \\
&= -\frac{\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]-\exp\left[-i\cdot 2\pi\cdot\xi\cdot(-2)\right]}{2i\cdot\pi\xi} \\
&= \frac{\exp\left[+i\cdot 2\pi\cdot\xi\cdot 2\right]-\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]}{2i}\cdot\frac{1}{\pi\xi} \\
&= \sin\left[4\pi\xi\right]\cdot\frac{1}{\pi\xi} \\
&= 4\cdot\frac{\sin\left[4\pi\xi\right]}{4\pi\xi} = 4\cdot\frac{\sin\left[\pi\cdot 4\xi\right]}{\pi\cdot 4\xi} \\
&= 4\cdot SINC\left[4\xi\right] \\
\mathcal{F}\left\{f_1\left[x\right]\right\} &= F_1\left[\xi\right] = 4\cdot SINC\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right]
\end{aligned}$$



$\mathcal{F}\left\{RECT\left[\frac{x}{4}\right]\right\} = 4\cdot SINC\left[4\xi\right]$ *(red solid) compared to* $\mathcal{F}\left\{RECT\left[x\right]\right\} = SINC\left[\xi\right]$ *(black dashed line), showing that the spectrum of the "wider" rectangle is "taller" and "skinnier."*

(b) $$f_2[x] \equiv \begin{cases} 0 & \text{if} \quad x > 4 \\ \frac{1}{2} & \text{if} \quad x = 4 \\ 1 & \text{if} \quad 0 < x < 4 \\ \frac{1}{2} & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Solution: *Again, draw the function first:*



Now evaluate the Fourier transform:

$$\begin{aligned}
\mathcal{F}\{f_2[x]\} &= \int_{x=0}^{x=4} 1[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x] \; dx \\
&= \left. \frac{\exp[-i \cdot 2\pi \cdot \xi x]}{-i \cdot 2\pi \cdot \xi} \right|_{x=0}^{x=4} \\
&= \frac{\exp[-i \cdot 2\pi \cdot \xi \cdot 4]}{-i \cdot 2\pi \cdot \xi} - \frac{\exp[-i \cdot 2\pi \cdot \xi \cdot 0]}{-i \cdot 2\pi \cdot \xi} \\
&= \frac{\exp[-i \cdot 2\pi \cdot \xi \cdot 4] - 1}{-i \cdot 2\pi \cdot \xi}
\end{aligned}$$

Note that there is a common factor here:

$$\begin{aligned}
&\frac{\exp\left[-i\cdot 2\pi\cdot\xi\cdot 4\right]-1}{-i\cdot 2\pi\cdot\xi} \\
= \;& \frac{-\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]\cdot\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]+\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]\cdot\exp\left[+i\cdot 2\pi\cdot\xi\cdot 2\right]}{+i\cdot 2\pi\cdot\xi} \\
= \;& \exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]\cdot\frac{\exp\left[+i\cdot 2\pi\cdot\xi\cdot 2\right]-\exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]}{2i\cdot\pi\xi} \\
= \;& \exp\left[-i\cdot 2\pi\cdot\xi\cdot 2\right]\cdot\frac{\sin\left[4\pi\xi\right]}{\pi\xi} \\
= \;& \exp\left[-i\cdot 4\pi\cdot\xi\right]\cdot 4\cdot\frac{\sin\left[\pi\cdot 4\xi\right]}{\pi\cdot 4\xi} \\
= \;& \exp\left[-i\cdot 4\pi\cdot\xi\right]\cdot 4\cdot SINC\left[4\xi\right] \\
= \;& \exp\left[-i\cdot 4\pi\cdot\xi\right]\cdot 4\cdot SINC\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right]
\end{aligned}$$

Note that the exponential may be recast by applying the Euler relation:

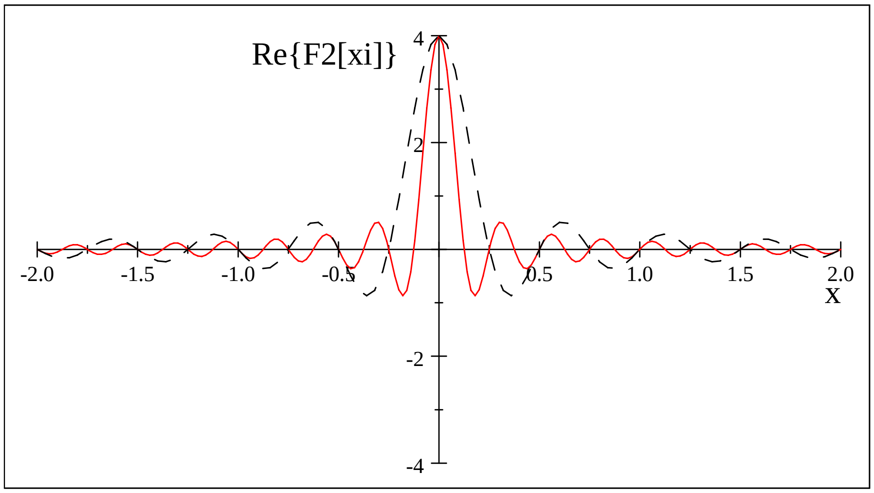
$$\begin{aligned}
\exp\left[-i\theta\right] &= \cos\left[\theta\right]-i\cdot\sin\left[\theta\right] \\
&\Longrightarrow \exp\left[-i\cdot 4\pi\cdot\xi\right]=\cos\left[4\pi\xi\right]-i\cdot\sin\left[4\pi\xi\right]
\end{aligned}$$

so the spectrum is:

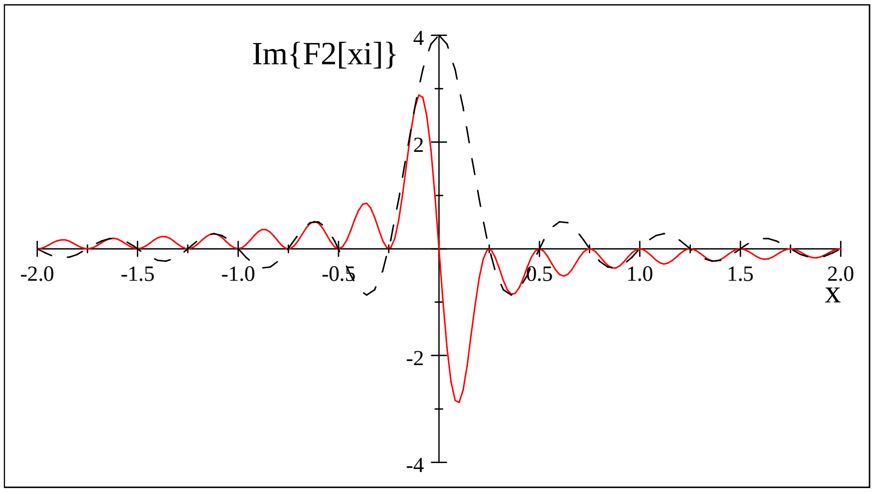
$$F_2\left[\xi\right]=\left(4\cdot SINC\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right]\cdot\cos\left[4\pi\xi\right]\right)-i\cdot\left(4\cdot SINC\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right]\cdot\sin\left[4\pi\xi\right]\right)$$

So we can graph the real part and imaginary part separately:

$$\operatorname{Re}\left\{F_2\left[\xi\right]\right\}=4\cdot\frac{\sin\left[\pi\frac{\xi}{\left(\frac{1}{4}\right)}\right]}{\pi\frac{\xi}{\left(\frac{1}{4}\right)}}\cdot\cos\left[4\pi\xi\right]$$



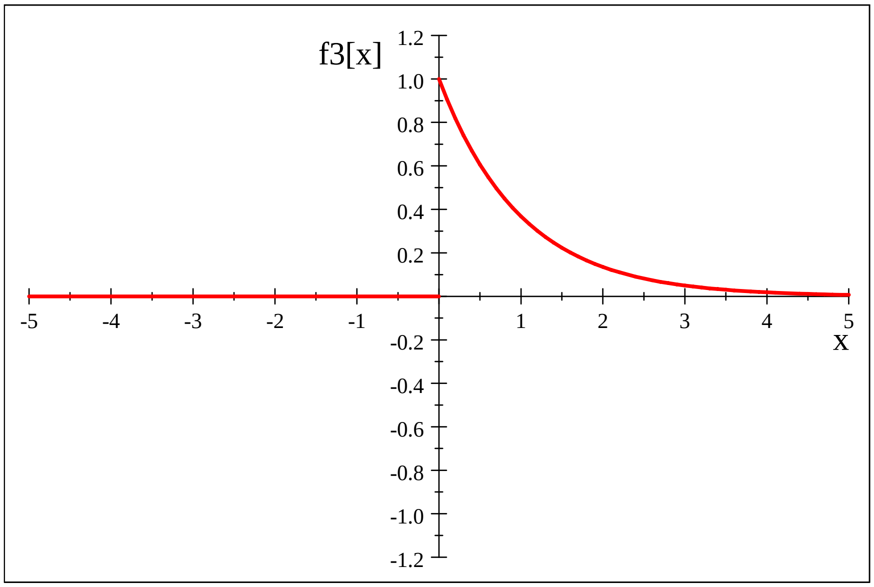
$\operatorname{Re}\left\{F_2\left[\xi\right]\right\}$ *in solid red and* $4\cdot SINC\left[4\xi\right]$ *in dashed black*



$\mathrm{Im}\left\{F_2\left[\xi\right]\right\}$ *in solid red and* $4 \cdot SINC\left[4\xi\right]$ *in dashed black*

(c) $f_3[x] = \exp[-x] \cdot STEP[x]$

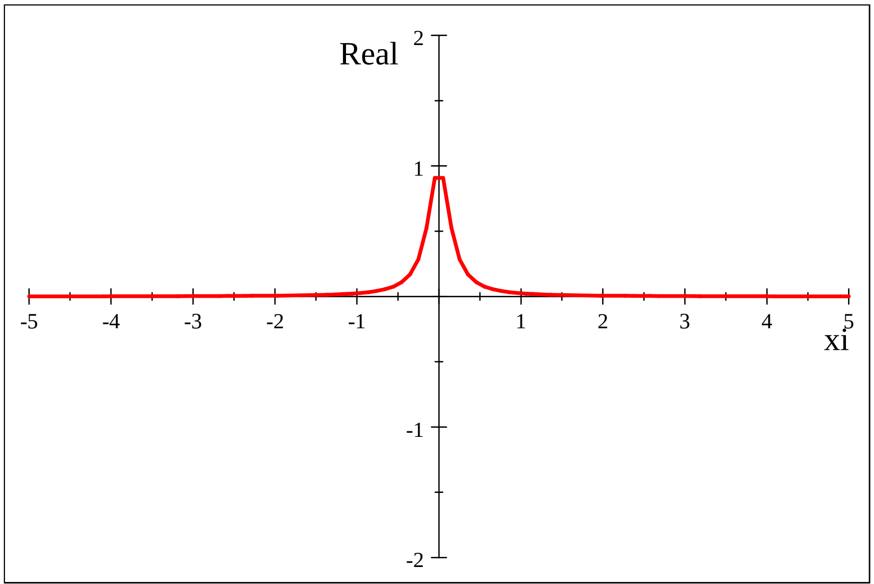
Solution: *We did this one in class:*



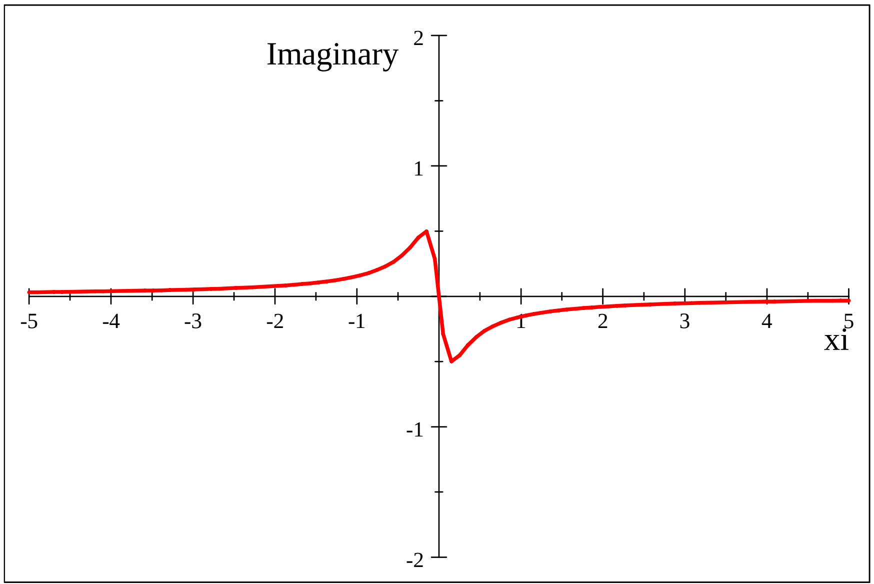
The spectrum of this function is:

$$\begin{aligned}
F_3[\xi] &= \mathcal{F}\{\exp[-x] \cdot STEP[x]\} \\
&= \int_{-\infty}^{+\infty} \exp[-x] \cdot STEP[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\; dx \\
&= \int_{0}^{+\infty} \exp[-x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\; dx \\
&= \int_{0}^{+\infty} \exp[-(1 + i \cdot 2\pi\xi) \cdot x]\; dx \\
&= \left.\frac{\exp[-(1 + i \cdot 2\pi\xi) \cdot x]}{-(1 + i \cdot 2\pi\xi)}\right|_{x=0}^{x=+\infty} \\
&= \frac{\exp[-(1 + i \cdot 2\pi\xi) \cdot \infty]}{-(1 + i \cdot 2\pi\xi)} - \frac{\exp[-(1 + i \cdot 2\pi\xi) \cdot 0]}{-(1 + i \cdot 2\pi\xi)} \\
&= \frac{1}{1 + i \cdot 2\pi\xi} \\
&= \frac{1}{1 + i \cdot 2\pi\xi} \cdot \left(\frac{1 - i \cdot 2\pi\xi}{1 - i \cdot 2\pi\xi}\right) \\
&= \frac{1 - i \cdot 2\pi\xi}{1 + (2\pi\xi)^2}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Re}\{F_3[\xi]\} &= \frac{1}{1 + (2\pi\xi)^2} \\
\operatorname{Im}\{F_3[\xi]\} &= \frac{-2\pi\xi}{1 + (2\pi\xi)^2}
\end{aligned}$$



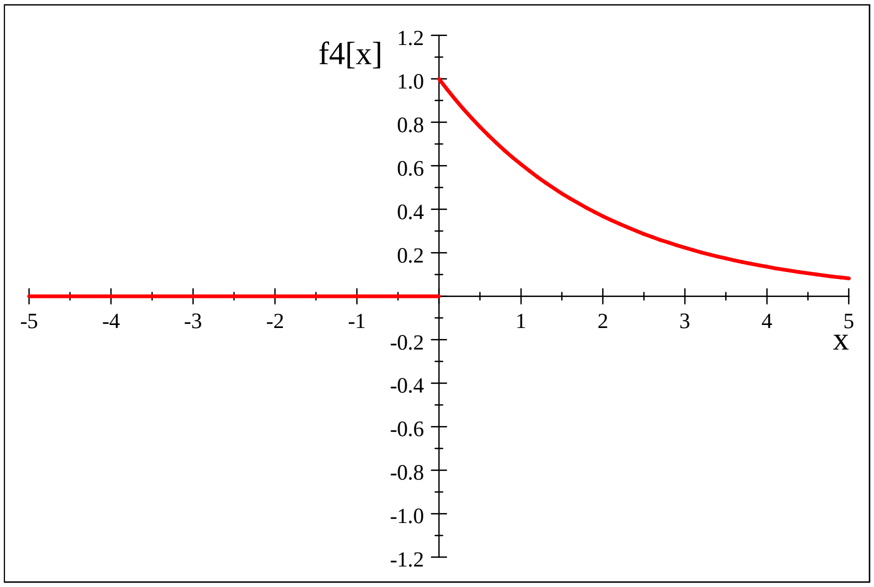
$$\mathrm{Re}\left\{F_3\left[\xi\right]\right\} = \frac{1}{1+\left(2\pi\xi\right)^2}$$



$$\mathrm{Im}\left\{F_3\left[\xi\right]\right\} = \frac{-2\pi\xi}{1+\left(2\pi\xi\right)^2}$$

(d) $f_4[x] = \exp\left[-\frac{x}{2}\right] \cdot STEP\left[\frac{x}{2}\right]$

Solution: *This is scaled version of (c); it is twice as wide.*



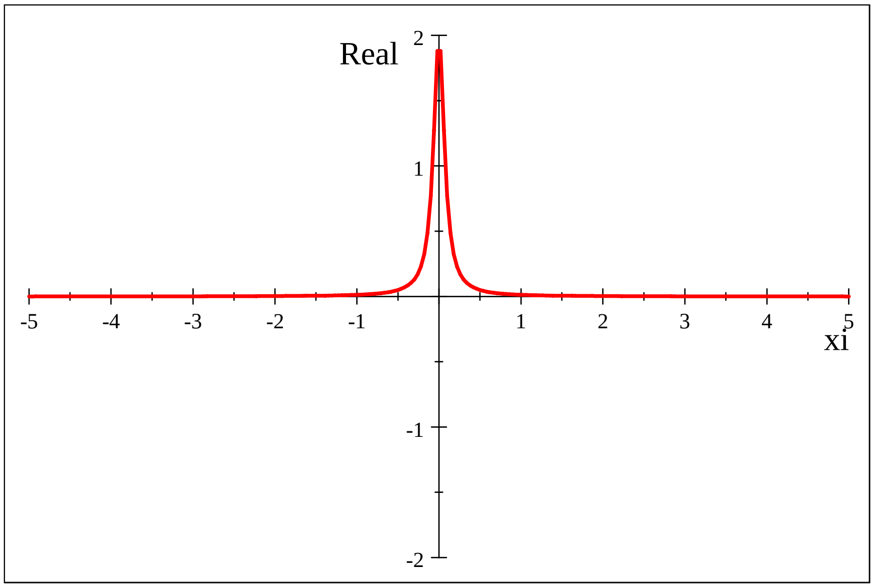
The spectrum of this function is easy to evaluate in the same way. First, note that

$$STEP\left[\frac{x}{2}\right] = STEP[x]$$

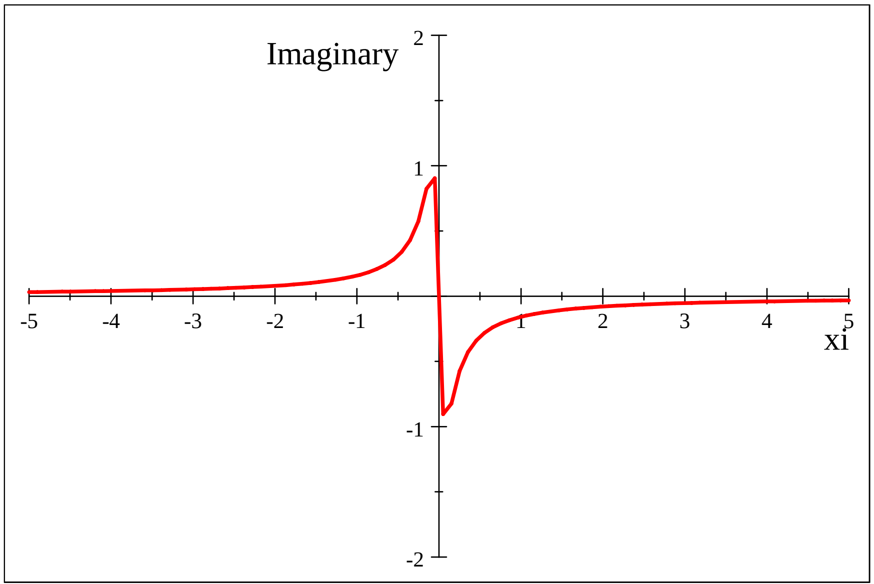
because the transition point does not "move"

$$\begin{aligned}
F_3[\xi] &= \mathcal{F}\left\{\exp\left[-\frac{x}{2}\right] \cdot STEP\left[\frac{x}{2}\right]\right\} = \mathcal{F}\left\{\exp\left[-\frac{x}{2}\right] \cdot STEP[x]\right\} \\
&= \int_{-\infty}^{+\infty} \exp\left[-\frac{x}{2}\right] \cdot STEP[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= \int_{0}^{+\infty} \exp\left[-\frac{x}{2}\right] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= \int_{0}^{+\infty} \exp\left[-\left(\frac{1}{2} + i \cdot 2\pi\xi\right) \cdot x\right]\ dx \\
&= \int_{0}^{+\infty} \exp\left[-\left(\frac{1 + i \cdot 4\pi\xi}{2}\right) \cdot x\right]\ dx \\
&= \left.\frac{\exp\left[-\left(\frac{1+i\cdot 4\pi\xi}{2}\right) \cdot x\right]}{-\left(\frac{1+i\cdot 4\pi\xi}{2}\right)}\right|_{x=0}^{x=+\infty} \\
&= \frac{\exp\left[-\left(\frac{1+i\cdot 4\pi\xi}{2}\right) \cdot \infty\right]}{-\left(\frac{1+i\cdot 4\pi\xi}{2}\right)} - \frac{\exp\left[-\left(\frac{1+i\cdot 4\pi\xi}{2}\right) \cdot 0\right]}{-\left(\frac{1+i\cdot 4\pi\xi}{2}\right)} \\
&= \frac{1}{\left(\frac{1+i\cdot 4\pi\xi}{2}\right)} = \frac{2}{1 + i \cdot 4\pi\xi} \\
&= \frac{2}{1 + i \cdot 4\pi\xi} \cdot \left(\frac{1 - i \cdot 4\pi\xi}{1 - i \cdot 4\pi\xi}\right) \\
&= \frac{2 - i \cdot 8\pi\xi}{1 + (4\pi\xi)^2}
\end{aligned}$$

$$\begin{aligned}\operatorname{Re}\{F_3[\xi]\} &= \frac{2}{1+(4\pi\xi)^2}\\ \operatorname{Im}\{F_3[\xi]\} &= \frac{-8\pi\xi}{1+(4\pi\xi)^2}\end{aligned}$$



$$\operatorname{Re}\{F_3[\xi]\} = \frac{1}{1+(2\pi\xi)^2}$$



$$\operatorname{Im}\{F_3[\xi]\} = \frac{-2\pi\xi}{1+(2\pi\xi)^2}$$

2. The operation of "convolution" of two functions $f[x]$ and $h[x]$ is defined:

$$f[x] * h[x] = \int_{-\infty}^{+\infty} f[\alpha] \cdot h[x-\alpha] \; d\alpha$$

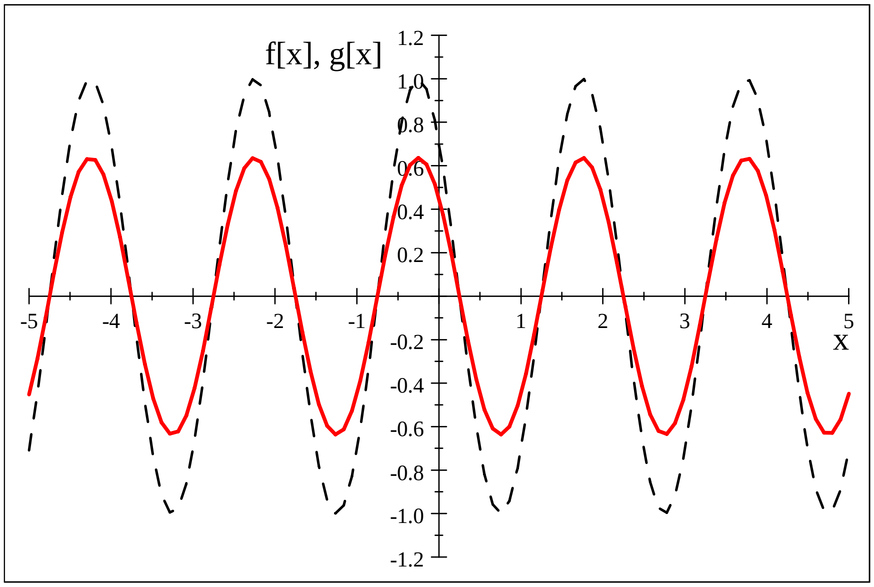
Evaluate the convolution of $f[x] = \cos\left[2\pi \frac{x}{2} + \frac{\pi}{4}\right]$ with the following functions for $h[x]$ and sketch (or plot) the results:

(a) $h_1[x] = RECT[x]$

$$\begin{aligned}
g_1[x] &= f[x] * h_1[x] \\
&= \cos\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right] * RECT[x] \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} \left(\cos\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]\right) \cdot RECT[x-\alpha] \; d\alpha \\
&= \int_{\alpha=x-\frac{1}{2}}^{\alpha=x+\frac{1}{2}} \left(\cos\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]\right) \cdot 1 \; d\alpha \\
&= \left. \frac{\sin\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]}{\pi} \right|_{\alpha=x-\frac{1}{2}}^{\alpha=x+\frac{1}{2}} \\
&= \frac{\sin\left[2\pi\frac{\left(x+\frac{1}{2}\right)}{2} + \frac{\pi}{4}\right]}{\pi} - \frac{\sin\left[2\pi\frac{\left(x-\frac{1}{2}\right)}{2} + \frac{\pi}{4}\right]}{\pi} \\
&= \frac{2}{\pi} \cdot \cos\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right]
\end{aligned}$$

so the input function is:

$$f[x] = \cos\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right]$$



$f[x]$ *as black dashed line and* $g[x]$ *has red solid line* *and the output function is:*

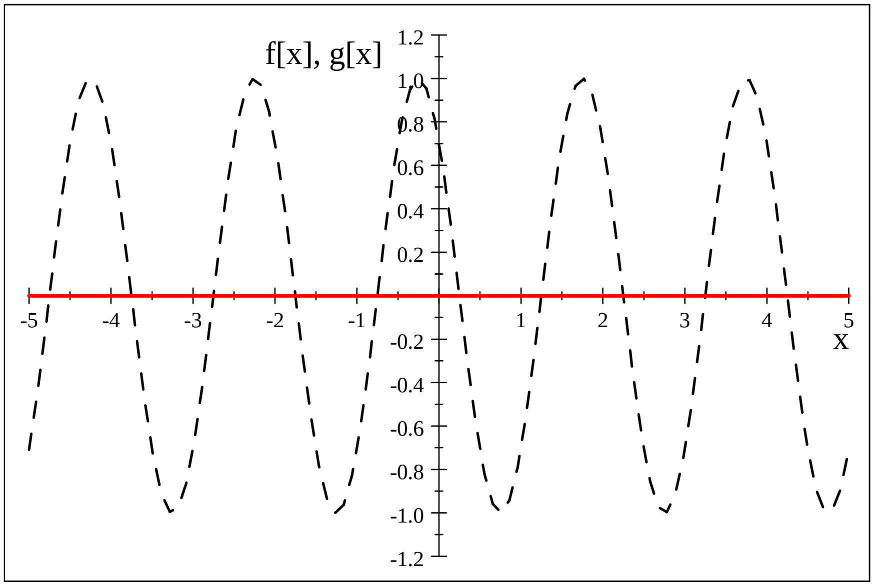
$$g\left[x\right] = \frac{2}{\pi} \cdot \cos\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right] = \frac{2}{\pi} \cdot f\left[x\right]$$

so the amplitude of the output function is reduced from unity to $\frac{2}{\pi} \approx 0.637$

(b) $h_2[x] = \frac{1}{2}RECT\left[\frac{x}{2}\right]$

$$\begin{aligned}
g_1[x] &= f[x] * h_1[x] \\
&= \cos\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right] * \frac{1}{2}RECT\left[\frac{x}{2}\right] \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} \left(\cos\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]\right) \cdot \frac{1}{2}RECT\left[\frac{x-\alpha}{2}\right]\, d\alpha \\
&= \int_{\alpha=x-1}^{\alpha=x+1} \left(\cos\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]\right) \cdot 1\, d\alpha \\
&= \left.\frac{\sin\left[2\pi\frac{\alpha}{2} + \frac{\pi}{4}\right]}{\pi}\right|_{\alpha=x-1}^{\alpha=x+1} \\
&= \frac{1}{\pi}\left(\sin\left[2\pi\frac{(x+1)}{2} + \frac{\pi}{4}\right] - \sin\left[2\pi\frac{(x-1)}{2} + \frac{\pi}{4}\right]\right) \\
&= \frac{1}{\pi}\left(\sin\left[2\pi\frac{x}{2} + \pi + \frac{\pi}{4}\right] - \sin\left[2\pi\frac{x}{2} - \pi + \frac{\pi}{4}\right]\right) \\
&= \frac{1}{\pi}\left(-\sin\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right] + \sin\left[2\pi\frac{x}{2} + \frac{\pi}{4}\right]\right) \\
&= 0[x]
\end{aligned}$$

So the local average of the sinusoid over the full cycle evaluates to 0:



$f[x]$ *as black dashed line and* $g[x]$ *has red solid line*

3. Evaluate the convolutions of the following functions by "direct integration" and sketch (or plot) the results:

   (a) $g_1[x] = RECT[x] * RECT[x]$

$$g_1[x] = \int_{-\infty}^{+\infty} RECT[\alpha] \cdot RECT[x-\alpha]\ d\alpha$$

Solution by direct integration:

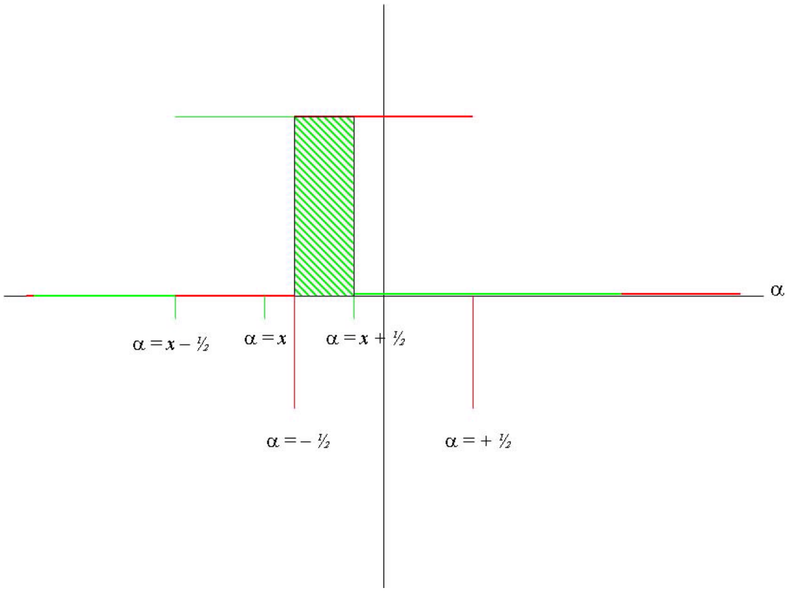
$$\begin{aligned}\int_{-\infty}^{+\infty} RECT[\alpha] \cdot RECT[x-\alpha]\ d\alpha &= \int_{\alpha=-\frac{1}{2}}^{\alpha=+\frac{1}{2}} 1 \cdot RECT[x-\alpha]\ d\alpha \\ &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} RECT[-(x-\alpha)]\ d\alpha \\ &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} RECT[\alpha - x]\ d\alpha \textit{ because } RECT[x] \textit{ is even}\end{aligned}$$

If the nonzero amplitude of the rectangle in the integral does not overlap the limits of the integral, then the area of the rectangle is zero:

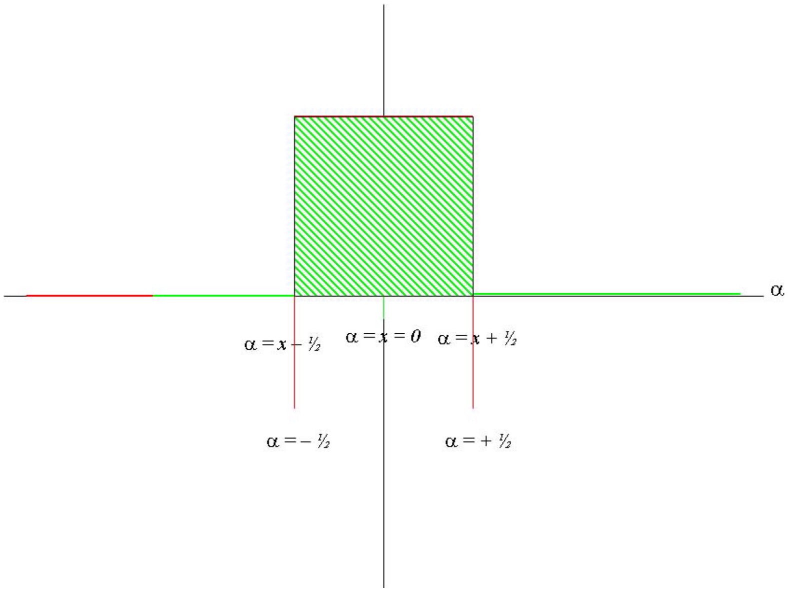
$$\begin{aligned}g[x<-1] &= \int_{-\frac{1}{2}}^{x+\frac{1}{2}} 0\ d\alpha = 0 \\ g[x>+1] &= \int_{-\frac{1}{2}}^{x+\frac{1}{2}} 0\ d\alpha = 0\end{aligned}$$

So we only need evaluate the ouput for two cases. In the first, the translation parameter of the rectangle is between -1 and 0, then only its upper edge lies within the limits of integration, and the upper limit of the integral becomes $x+\frac{1}{2}$:

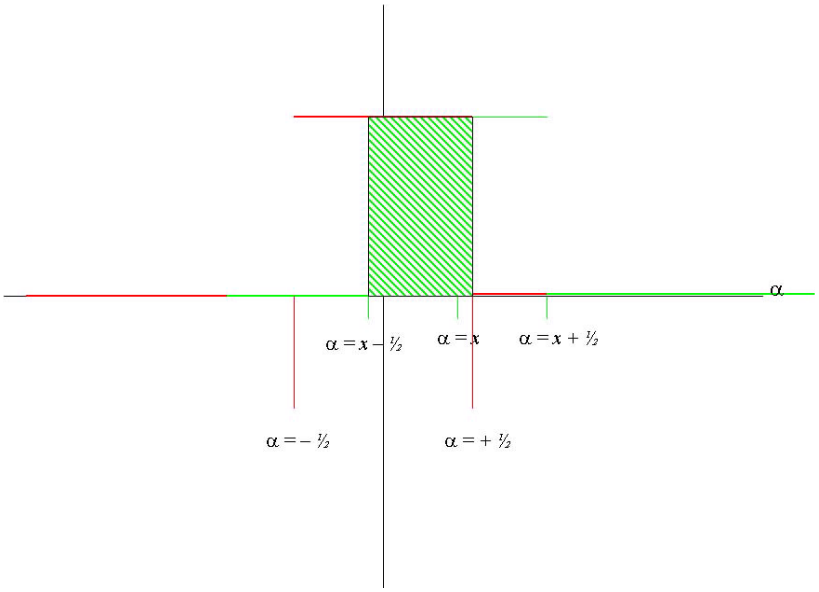
$$g[-1<x<0] = \int_{-\frac{1}{2}}^{x+\frac{1}{2}} 1\ d\alpha = \left(x+\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = x+1$$



The convolution of two rectangles for $-1 < x < 0$*: the red-line rectangle is* $RECT\,[\alpha]$ *and the green-line rectangle is* $RECT\,[x - \alpha]$ *where* $-1 < x < 0$*. The area of the product of the two rectangles is cross-hatched in green, which is the numerical value of the convolution for that* x



The convolution of two rectangles for $x = 0$*: the red-line rectangle is* $RECT\,[\alpha]$ *and the green-line rectangle is* $RECT\,[0 - \alpha]$ *. The unit area of the product of the two rectangles is cross-hatched in green.*



The convolution of two rectangles for $0 < x < +1$*: the red-line rectangle is* $RECT\,[\alpha]$ *and the green-line rectangle is* $RECT\,[x-\alpha]$*. The area of the product of the two rectangles is cross-hatched in green.*

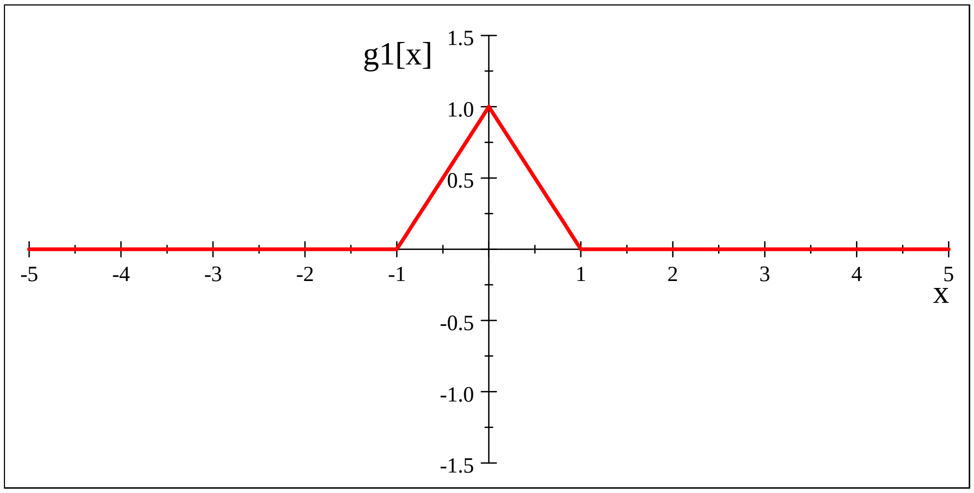
In the other case, the translation parameter of the rectangle is between 0 *and* $+1$*, then only its left edge lies within the limits of integration, and the lower limit of the integral becomes* $x - \frac{1}{2}$*:*

$$g\,[0 < x < 1] = \int_{x-\frac{1}{2}}^{+\frac{1}{2}} 1\; d\alpha = \left(\frac{1}{2}\right) - \left(x - \frac{1}{2}\right) = 1 - x$$

so the complete expression for the output is:

$$g\,[x] = \begin{cases} 0 & \text{if} \quad x > +1 \\ 1-x & \text{if} \quad 0 < x < +1 \\ 1+x & \text{if} \quad -1 < x < 0 \\ 0 & \text{if} \quad x < -1 \end{cases} \equiv TRI\,[x]$$

which we call the triangle function.



(b) $g_2[x] = RECT\left[\frac{x}{2}\right] * RECT\left[\frac{x}{2}\right]$

This is the same problem as (a) except that the rectangles are twice as wide:

$$g_1[x] = \int_{-\infty}^{+\infty} RECT\left[\frac{\alpha}{2}\right] \cdot RECT\left[\frac{x-\alpha}{2}\right]\, d\alpha$$

Solution by direct integration:

$$\begin{aligned}\int_{-\infty}^{+\infty} RECT\left[\frac{\alpha}{2}\right] \cdot RECT\left[\frac{x-\alpha}{2}\right]\, d\alpha &= \int_{\alpha=-1}^{\alpha=+1} 1 \cdot RECT\left[\frac{x-\alpha}{2}\right]\, d\alpha \\ &= \int_{-1}^{+1} RECT\left[-\left(\frac{x-\alpha}{2}\right)\right]\, d\alpha \\ &= \int_{-1}^{+1} RECT\left[\frac{\alpha-x}{2}\right]\, d\alpha\end{aligned}$$

(again because $RECT[x]$ is even). If the nonzero amplitude of the rectangle in the integral does not overlap the limits of the integral, then the area of the rectangle is zero:

$$\begin{aligned}g[x<-2] &= \int_{-1}^{x+1} 0\, d\alpha = 0 \\ g[x>+2] &= \int_{-1}^{x+1} 0\, d\alpha = 0\end{aligned}$$

So we only need evaluate the ouput for two cases. In the first, the translation parameter of the rectangle is between -1 and 0, then only its upper edge lies within the limits of integration, and the upper limit of the integral becomes $x+\frac{1}{2}$:

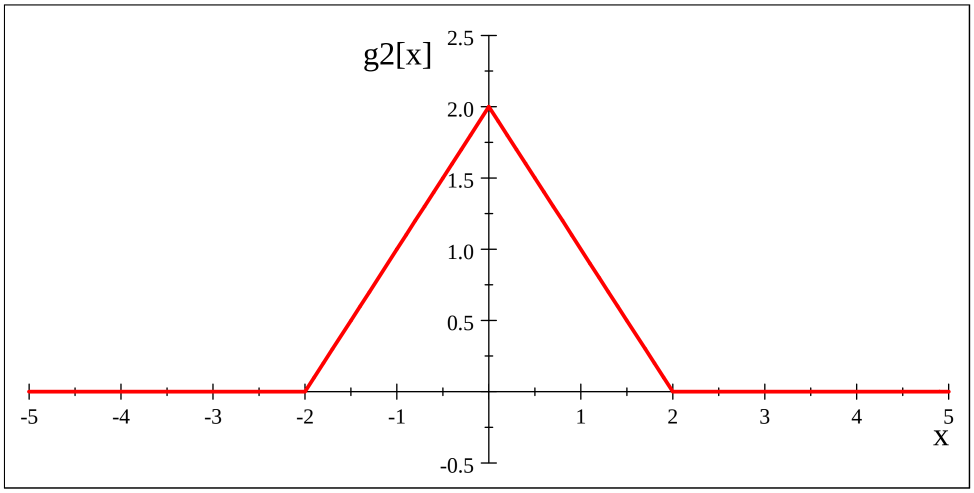
$$g[-2<x<0] = \int_{-1}^{x+1\frac{1}{2}} 1\, d\alpha = (x+1)-(-1) = x+2$$

In the other case, the translation parameter of the rectangle is between 0 and +1, then only its left edge lies within the limits of integration, and the lower limit of the integral becomes $x-\frac{1}{2}$:

$$g[0<x<+2] = \int_{x-1}^{+1} 1\, d\alpha = (1)-(x-1) = 2-x$$

so the complete expression for the output is:

$$g[x] = \begin{cases} 0 & \text{if} \quad x>+2 \\ 2-x & \text{if} \quad 0<x<+2 \\ 2+x & \text{if} \quad -2<x<0 \\ 0 & \text{if} \quad x<-2 \end{cases} \equiv 2\cdot TRI\left[\frac{x}{2}\right]$$



(c) $g_3[x] = RECT[x] * RECT[x] * RECT[x]$

From the facts that convolution is associative and commutative, we can see that

$$\begin{aligned} g_3[x] &= RECT[x] * RECT[x] * RECT[x] \\ &= TRI[x] * RECT[x] \\ &= RECT[x] * TRI[x] \end{aligned}$$

$$\begin{aligned} TRI[x] * RECT[x] &= \int_{-\infty}^{+\infty} TRI[\alpha] \cdot RECT[x-\alpha]\ d\alpha \\ &= \int_{-\infty}^{+\infty} \left\{ (1-|\alpha|) \cdot RECT\left[\frac{\alpha}{2}\right] \right\} \cdot RECT[x-\alpha]\ d\alpha \end{aligned}$$

because we can write the triangle as the product of $(1-|x|)$ *and the "window function"* $RECT\left[\frac{x}{2}\right]$. *SKETCHES WILL HELP YOU DETERMINE THE LIMITS IN THE FOLLOWING SECTIONS.*

The convolution evaluates to:

$$\begin{aligned} TRI[x] * RECT[x] &= \int_{-\infty}^{+\infty} (1-|\alpha|) \cdot RECT\left[\frac{\alpha}{2}\right] \cdot RECT[x-\alpha]\ d\alpha \\ &= \int_{-1}^{+1} (1-|\alpha|) \cdot RECT[x-\alpha]\ d\alpha \\ &= \int_{-1}^{0} (1+\alpha) \cdot RECT[x-\alpha]\ d\alpha + \int_{0}^{+1} (1-\alpha) \cdot RECT[x-\alpha]\ d\alpha \end{aligned}$$

Where the rectangle intercepts the "rising" part of the triangle, the integral eval-

uates to:

$$
\begin{aligned}
\textit{If } -\frac{3}{2} &< x < -\frac{1}{2} \Longrightarrow \int_{-1}^{x+\frac{1}{2}} (1+\alpha)\cdot 1 \; d\alpha = \left(\alpha + \frac{\alpha^2}{2}\right)\Bigg|_{\alpha=-1}^{\alpha=x+\frac{1}{2}} \\
&= \left(\left(x+\frac{1}{2}\right) + \frac{\left(x+\frac{1}{2}\right)^2}{2}\right) - \left(-1 + \frac{(-1)^2}{2}\right) \\
&= \left(\left(x+\frac{1}{2}\right) + \frac{1}{2}\cdot\left(x^2 + x + \frac{1}{4}\right)\right) + \left(1 - \frac{1}{2}\right) \\
&= \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8}
\end{aligned}
$$

which is a quadratic function of x, as you would expect since we are integrating a linear function of x.

The symmetric interval (where the rectangle intercepts the "falling" part of the triangle) yields a similar result:

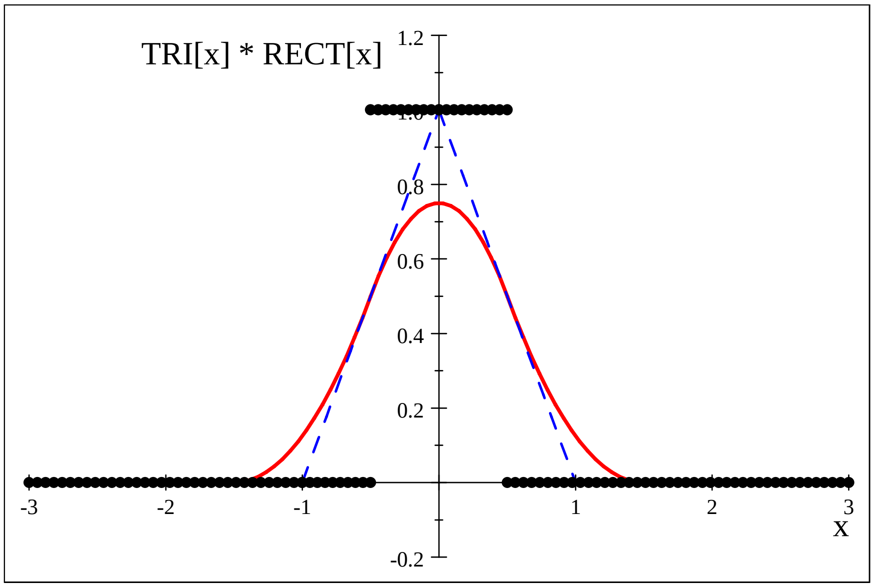
$$
\begin{aligned}
\textit{If } \frac{1}{2} &< x < \frac{3}{2} \Longrightarrow \int_{x-\frac{1}{2}}^{1} (1-\alpha)\cdot 1 \; d\alpha = \left(\alpha - \frac{\alpha^2}{2}\right)\Bigg|_{\alpha=x-\frac{1}{2}}^{\alpha=1} \\
&= \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8}
\end{aligned}
$$

In between these two, we have to include contributions from both sides of the triangle:

$$
\begin{aligned}
\textit{If } -\frac{1}{2} &< x < +\frac{1}{2} \Longrightarrow \int_{x-\frac{1}{2}}^{0} (1+\alpha) \; d\alpha + \int_{0}^{x+\frac{1}{2}} (1-\alpha) \; d\alpha \\
&= \left(-\frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{8}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{2}x + \frac{3}{8}\right) \\
&= \frac{3}{4} - x^2
\end{aligned}
$$

So the convolution is the sum of these three piecewise sections:

$$
TRI\,[x] * RECT\,[x] = \begin{cases} 0 & \text{if} \quad x > \frac{3}{2} \\ \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8} & \text{if} \quad -\frac{3}{2} < x < -\frac{1}{2} \\ \frac{3}{4} - x^2 & \text{if} \quad -\frac{1}{2} < x < +\frac{1}{2} \\ \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8} & \text{if} \quad \frac{1}{2} < x < \frac{3}{2} \\ 0 & \text{if} \quad x < -\frac{3}{2} \end{cases}
$$

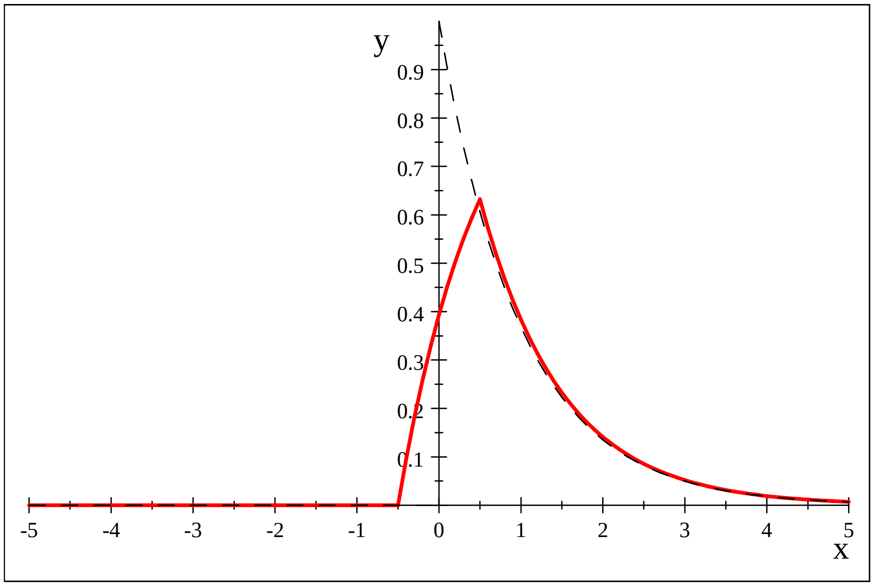


$TRI\,[x] * RECT\,[x]$ *in red solid,* $TRI\,[x]$ *in blue dashed, and* $RECT\,[x]$ *in black dot-dash.*

4. Evaluate and sketch (or plot) the following convolutions:

   (a) $(\exp[-x] \cdot STEP[x]) * RECT[x]$

$$
\begin{aligned}
& (\exp[-x] \cdot STEP[x]) * RECT[x] \\
= & \int_{\alpha=-\infty}^{\alpha=+\infty} (\exp[-\alpha] \cdot STEP[\alpha]) \cdot (RECT[x-\alpha])\ d\alpha \\
= & \int_{\alpha=0}^{\alpha=+\infty} \exp[-\alpha] \cdot (RECT[x-\alpha])\ d\alpha \\
= & \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ \int_{\alpha=0}^{\alpha=x+\frac{1}{2}} \exp[-\alpha]\ d\alpha & \text{if} \quad -\frac{1}{2} < x < +\frac{1}{2} \\ \int_{\alpha=x-\frac{1}{2}}^{\alpha=x+\frac{1}{2}} \exp[-\alpha]\ d\alpha & \text{if} \quad x > +\frac{1}{2} \end{cases} \\
= & \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ (-\exp[-\alpha])\big|_0^{x+\frac{1}{2}} = 1 - \exp\left[-\left(x+\frac{1}{2}\right)\right] & \text{if} \quad -\frac{1}{2} < x < +\frac{1}{2} \\ (-\exp[-\alpha])\big|_{x-\frac{1}{2}}^{x+\frac{1}{2}} = \exp\left[-\left(x-\frac{1}{2}\right)\right] - \exp\left[-\left(x+\frac{1}{2}\right)\right] & \text{if} \quad x > +\frac{1}{2} \end{cases} \\
= & \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ 1 - \exp[-x] \cdot \exp\left[-\frac{1}{2}\right] & \text{if} \quad -\frac{1}{2} < x < +\frac{1}{2} \\ \exp[-x] \cdot \left(-\exp\left[-\frac{1}{2}\right] + \exp\left[+\frac{1}{2}\right]\right) & \text{if} \quad x > +\frac{1}{2} \end{cases}
\end{aligned}
$$



(b) $(\exp[-x]\cdot STEP[x]) * (\exp[-x]\cdot STEP[x])$

We did this one in class, so you'd better get it right!

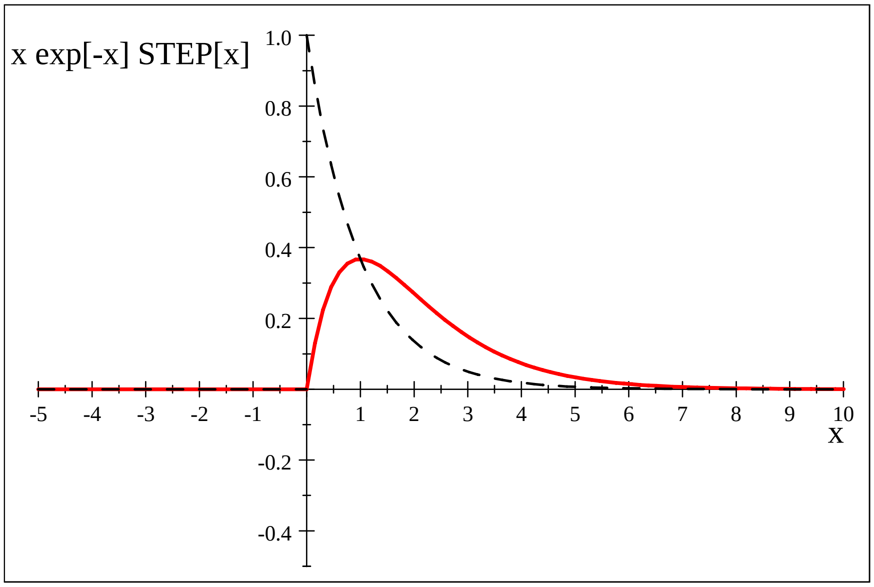
$$\begin{aligned}
&(\exp[-x]\cdot STEP[x]) * (\exp[-x]\cdot STEP[x])\\
= &\int_{\alpha=-\infty}^{\alpha=+\infty} (\exp[-\alpha]\cdot STEP[\alpha])\cdot(\exp[-(x-\alpha)]\cdot STEP[x-\alpha])\ d\alpha\\
= &\int_{\alpha=0}^{\alpha=+\infty} \exp[-\alpha]\cdot 1\cdot(\exp[-(x-\alpha)]\cdot STEP[x-\alpha])\ d\alpha\\
= &\int_{\alpha=0}^{\alpha=+\infty} \exp[-\alpha]\cdot 1\cdot \exp[-x]\cdot\exp[+\alpha]\cdot STEP[x-\alpha]\ d\alpha\\
= &\exp[-x]\cdot\int_{\alpha=0}^{\alpha=+\infty} (\exp[-\alpha]\cdot\exp[+\alpha])\cdot STEP[x-\alpha]\ d\alpha\\
= &\exp[-x]\cdot\int_{\alpha=0}^{\alpha=+\infty} 1\cdot STEP[x-\alpha]\ d\alpha
\end{aligned}$$

If $x\le 0$*, the intervals do not overlap and the area of the product is zero. If* $x>0$*, then the function evaluates to:*

$$\exp[-x]\cdot\int_{\alpha=0}^{\alpha=x} 1\ d\alpha = \exp[-x]\cdot(x-0) = x\cdot\exp[-x]\ \ \textit{if}\ \ x>0$$

So the full prescription is:

$$\begin{aligned}
(\exp[-x]\cdot STEP[x]) * (\exp[-x]\cdot STEP[x]) &= \begin{cases} x\cdot\exp[-x] & \text{if}\quad x>0\\ 0 & \text{if}\quad x\le 0\end{cases}\\
&= x\cdot\exp[-x]\cdot STEP[x]
\end{aligned}$$



$\exp[-x]\cdot STEP[x]$ *(black dashed line) and* $\exp[-x]\cdot STEP[x] * \exp[-x]\cdot STEP[x]$ *(red solid line)*