IMGS-261 Solution for HW#4

1. (a) (2')

$$z_1 = 4\exp\left(-i\frac{5\pi}{6}\right) = 4\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$$

$$= 4\left(-\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right) = -2\sqrt{3} - 2i \approx -3.4641 - 2i$$

(c) (1')

$$(z_1)^2 = 16\exp\left(-i\frac{5\pi}{3}\right) = 16\exp\left(i\frac{\pi}{3}\right)$$

$$= 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 16\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 8 + 8\sqrt{3}i \approx 8 + 13.8564i$$

(d) (1')

$$(z_1)^{\frac{1}{2}}_1 = 2\exp\left(-i\frac{5\pi}{12}\right) = 2\left[\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right]$$

$$= 2\cos\left(\frac{5\pi}{12}\right) - i2\sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{2} - \frac{\sqrt{6}+\sqrt{2}}{2}i$$

$$\approx 0.5176 - 1.9319i$$

$$(z_1)^{\frac{1}{2}}_2 = 2\exp\left(i\frac{7\pi}{12}\right) = 2\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$$

$$= \frac{-\sqrt{6}+\sqrt{2}}{2} + i\frac{\sqrt{6}+\sqrt{2}}{2} \approx -0.5176 + 1.9319i$$

1 b) 2′ c)1′ d) 1′

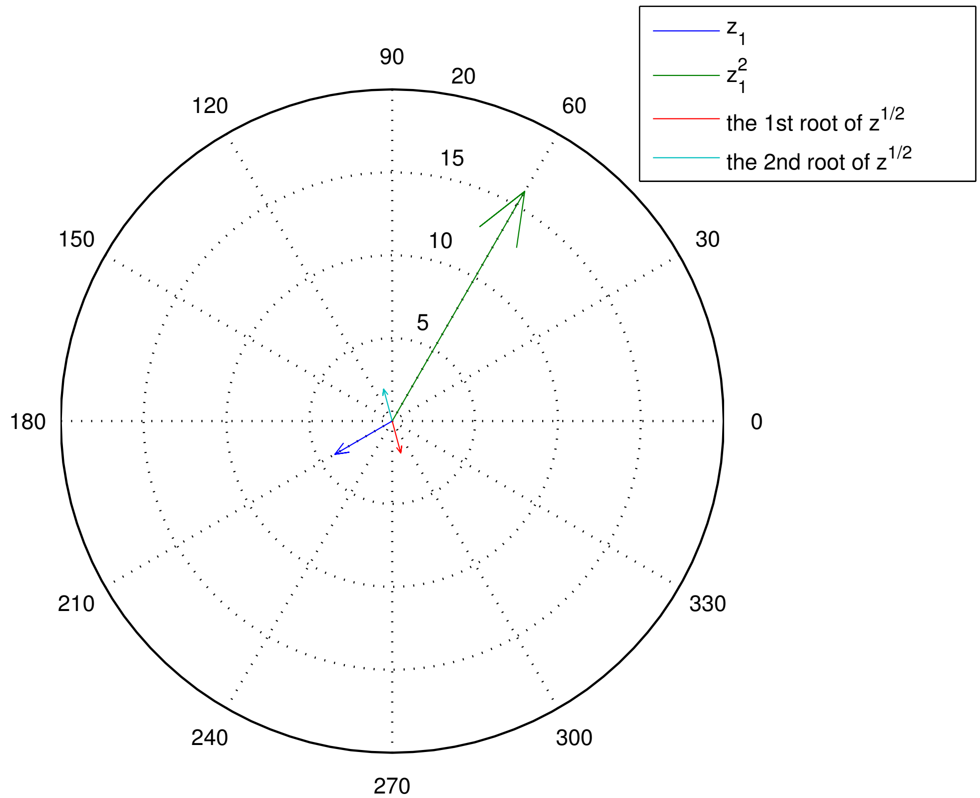

Figure 1: z_1, z_1^2 and two roots of $sqrtz_1$ on the Argand diagram

2. We could find a vector $\underline{x}$ that is orthogonal with each of following vectors and then normalized $\underline{x}$ to get $\hat{\underline{x}}$ with unit length.

(a) (2') Assume $\underline{x} = \begin{bmatrix} a_1+b_1 i \\ a_2+b_2 i \end{bmatrix}$

$$\underline{x} \cdot \begin{bmatrix} +1+i \\ +1+i \end{bmatrix} = (a_1+b_1 i)^*(1+i) + (a_2+b_2 i)^*(1+i) = 0$$

$$\Rightarrow a_1+b_1+(a_1-b_1)i + a_2+b_2+(a_2-b_2)i = 0$$

$$\Rightarrow \begin{cases} a_1+b_1+a_2+b_2 = 0 \\ a_1-b_1+a_2-b_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1+a_2 = 0 \\ b_1+b_2 = 0 \end{cases}$$

choose a possible solution that $a_1 = -a_2 = 1$, $b_1 = b_2 = 0$

$$\therefore \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \hat{\underline{x}} = \frac{\underline{x}}{|\hat{\underline{x}}|} = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) (2') Assume $\underline{x} = \begin{bmatrix} a_1+b_1 i \\ a_2+b_2 i \end{bmatrix}$

$$\underline{x} \cdot \begin{bmatrix} -1+i \\ -2-i \end{bmatrix} = (a_1+b_1 i)^*(-1+i) + (a_2+b_2 i)^*(-2-i) = 0$$

$$\Rightarrow -a_1+b_1+(a_1+b_1)i \quad -2a_2-b_2+(-a_2+2b_2)i = 0$$

$$\Rightarrow \begin{cases} -a_1+b_1-2a_2-b_2 = 0 \\ a_1+b_1-a_2+2b_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = -2a_1-3b_2 \\ a_1 = -b_2-\frac{b_1}{3} \end{cases}$$

choose a possible solution that $b_1 = 0$, $b_2 = 1 \Rightarrow a_1 = -1, a_2 = -1$

$$\therefore \underline{x} = \begin{bmatrix} -1 \\ -1+i \end{bmatrix} \qquad \hat{\underline{x}} = \frac{\underline{x}}{|\underline{x}|}. \quad |\underline{x}| = \sqrt{(-1)^2+|-1+i|^2} = \sqrt{1+2} = \sqrt{3}$$

$$\therefore \hat{x} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1+i \end{bmatrix}$$

(c) (2') Let $\underline{X} = \begin{bmatrix} a_1+b_1 i \\ a_2+b_2 i \end{bmatrix}$, $\underline{X} \cdot \begin{bmatrix} -i \\ i \end{bmatrix} = (a_1+b_1 i)^*(-i) + (a_2+b_2 i)^*(-i) = 0$

$\Rightarrow -a_1 i - b_1 + a_2 i + b_2 = 0$

$\Rightarrow \begin{cases} a_2 - a_1 = 0 \\ b_2 - b_1 = 0 \end{cases}$ choose a possible solution $\begin{cases} a_1 = a_2 = 1 \\ b_1 = b_2 = 0 \end{cases}$

$\therefore \underline{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\hat{\underline{X}} = \frac{X}{|X|} = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) (2') Let $\underline{X} = \begin{bmatrix} r_1 \exp[i\theta_1] \\ r_2 \exp[i\theta_2] \end{bmatrix}$ $\therefore \underline{X} \cdot \begin{bmatrix} \exp[-i\frac{\pi}{3}] \\ \exp[+i\frac{\pi}{4}] \end{bmatrix} = r_1 \exp[-i\theta_1]\exp[-i\frac{\pi}{3}] + r_2 \exp[-i\theta_2]\exp[+i\frac{\pi}{4}] = 0$

$\Rightarrow r_1 \exp[-i(\theta_1 + \frac{\pi}{3})] = -r_2 \exp[i(\frac{\pi}{4} - \theta_2)] = r_2 \exp[i(\frac{\pi}{4} - \theta_2 + \pi)]$

$\Rightarrow \begin{cases} r_1 = r_2 \\ -\theta_1 - \frac{\pi}{3} = \pi - \theta_2 + \frac{\pi}{4} \end{cases} \Rightarrow \begin{cases} r_1 = r_2 \\ \theta_2 = \theta_1 + \frac{19}{12}\pi \end{cases}$

Choose a possible solution: $\begin{cases} r_1 = r_2 = 1 \\ \theta_1 = 0 \\ \theta_2 = \frac{19}{12}\pi \end{cases}$

$\therefore \underline{X} = \begin{bmatrix} 1 \\ \exp[+i\frac{19}{12}\pi] \end{bmatrix}$ $\hat{\underline{X}} = \frac{X}{|X|} = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ \exp[+i\frac{19}{12}\pi] \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \exp[+i\frac{19}{12}\pi] \end{bmatrix}$

3(a)(2') $\hat{\underline{a}} \cdot \underline{x} = \frac{\underline{a}}{|\underline{a}|} \cdot \underline{x} = \frac{1}{\sqrt{0+1+1}}(0+1+0) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

(b)(2') $\hat{\underline{a}} \cdot \underline{x} = \frac{\underline{a}}{|\underline{a}|} \cdot \underline{x} = \frac{1}{\sqrt{0+1+1}} \cdot (0+(-i)\cdot(+i)+0) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

(c)(2') $\hat{\underline{a}} \cdot \underline{x} = \frac{\underline{a}}{|\underline{a}|} \cdot \underline{x} = \frac{1}{\sqrt{0+1+1}} \cdot (0+(+i)\cdot(+i)+0) = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

(d)(2') $\hat{\underline{a}} \cdot \underline{x} = \frac{\underline{a}}{|\underline{a}|} \cdot \underline{x} = \frac{1}{\sqrt{1+(1+i)(1-i)+(1-i)(1+i)}} \cdot (1\cdot(1+i)+(1-i)\cdot 1+(1+i)\cdot(1-i))$

$= \frac{1}{\sqrt{1+2+2}} \cdot (1+i+1-i+1+1) = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$

$* \quad \hat{\underline{a}} \cdot \underline{x} = \sum_{n=1}^{N} (a_n^*) \cdot x_n$

Projection of $\underline{x}$ onto $\underline{a}$ $= \hat{\underline{a}} \cdot \underline{x} = \frac{\underline{a} \cdot \underline{x}}{|\underline{a}|}$

$|\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{\sum_{n=1}^{N} (a_n)^* a_n}$

4. $\underline{A_1} = \begin{bmatrix} -2 & 0 \\ 0 & +\frac{1}{2} \end{bmatrix}$ $\underline{A_1}\underline{x} = \begin{bmatrix} -2 & 0 \\ 0 & +\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ \frac{x_2}{2} \end{bmatrix}$

(a) $\underline{A_1}$ scales the first component of $\underline{x}$ by -2 and scales
(1') the second component by $\frac{1}{2}$

(b) $|\underline{A_1}| = -1 \neq 0$ $\therefore$ $\underline{A_1}$ is invertible.
(1')

(c) $\because$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
(1')

$\therefore$ $\underline{A_1}^{-1} = -1\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$

$\underline{A_2} = \begin{bmatrix} +1 & +1 \\ -1 & +1 \end{bmatrix}$, $\underline{A_2}\,\underline{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ x_2-x_1 \end{bmatrix}$

(a) $\underline{A_2}$ evaluates the sum of the input vector's components and the
(1') difference of them.

(b) $|\underline{A_2}| = 1-(-1) = 2 \neq 0$ $\therefore$ $\underline{A_2}$ is invertible
(1')

(c) $\underline{A_2}^{-1} = \frac{1}{|\underline{A_2}|}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
(1')

$\underline{A_3} = \begin{bmatrix} +1 & +i \\ -i & +1 \end{bmatrix}$, $\underline{A_3}\underline{x} = \begin{bmatrix} +1 & +i \\ -i & +1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + i x_2 \\ x_2 - i x_1 \end{bmatrix}$

(a) (1') $\underline{A_3}$ generates a complex vector from the two components of input vectors

(b) (1') $|\underline{A_3}| = 1 - (-i)(+i) = 0$ $\therefore$ $\underline{A_3}$ is not invertible

$\underline{A_4} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\underline{A_4}\underline{x} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 0 \end{bmatrix}$

(a) (1') $\underline{A_4}$ scales the first component of the input vector by 2 and blocks the secon

(b) (1') $|\underline{A_4}| = 2\times 0 - 0 = 0$ $\therefore$ $\underline{A_4}$ is not invertible

5. (a) $\underline{\hat{a}_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\underline{\hat{a}_1}\cdot\underline{x} = \frac{1}{2}\exp[i\frac{\pi}{2}] + \frac{1}{2}\cdot 2\cdot\exp[+i\frac{3\pi}{4}] + \frac{\sqrt{2}}{2}\exp[+i\pi] + \frac{1}{2}\exp[-i\frac{\pi}{2}$

$= \cancel{\frac{1}{2}i} - \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \cancel{\frac{1}{2}i}$ (2')

$= \boxed{-\sqrt{2} + i\frac{\sqrt{2}}{2}}$

$\underline{\hat{a}_2} = \frac{1}{2}\begin{bmatrix} \exp(i\cdot 2\pi\cdot\frac{1}{4}\times 0) \\ \exp(i\cdot 2\pi\cdot\frac{1}{4}\times 1) \\ \exp(i\cdot 2\pi\cdot\frac{1}{4}\times 2) \\ \exp(i\cdot 2\pi\cdot\frac{1}{4}\times 3) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$, $\boxed{\underline{x} = \begin{bmatrix} i \\ -\sqrt{2}+\sqrt{2}i \\ -\sqrt{2} \\ -i \end{bmatrix}}$

$\hat{a}_2\cdot x = \frac{1}{2}[i + (-i)\cdot(-\sqrt{2}+\sqrt{2}i) + (-1)\cdot(-\sqrt{2}) + i(-i)]$

$= \frac{1}{2}[i + \sqrt{2}i + \sqrt{2} + \sqrt{2} + 1] = \boxed{\frac{2\sqrt{2}+1}{2} + \frac{\sqrt{2}+1}{2}i}$ (2')

$$\hat{\underline{a}}_3 = \frac{1}{2}\begin{bmatrix} \exp(+i\cdot 2\pi\cdot 0) \\ \exp(+i\cdot 2\pi\cdot \frac{1}{2}) \\ \exp(+i\cdot 2\pi\cdot 1) \\ \exp(+i\cdot 2\pi\cdot \frac{3}{2}) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\hat{\underline{a}}_3 \cdot \underline{x} = \frac{1}{2}\left[i-(-\sqrt{2}+\sqrt{2}i)-\sqrt{2}-(-i)\right]$$
$$= \boxed{\frac{2-\sqrt{2}}{2}i} \qquad (2')$$

$$\hat{\underline{a}}_4 = \frac{1}{2}\begin{bmatrix} \exp(+i\cdot 2\pi\cdot 0) \\ \exp(+i\cdot 2\pi\cdot \frac{3}{4}) \\ \exp(+i\cdot 2\pi\cdot \frac{6}{4}) \\ \exp(+i\cdot 2\pi\cdot \frac{9}{4}) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

$$\hat{\underline{a}}_4 \cdot \underline{x} = \frac{1}{2}\left[1\cdot i + i\cdot(-\sqrt{2}+\sqrt{2}i) - 1\cdot(-\sqrt{2}) - i(-i)\right]$$
$$= \frac{1}{2}\left[i - \sqrt{2}i - \cancel{\sqrt{2}} + \cancel{\sqrt{2}} - 1\right] \qquad (2')$$
$$= \boxed{-\frac{1}{2} + \frac{1-\sqrt{2}}{2}i}$$

(b) $|\hat{\underline{a}}_1 \cdot \underline{x}| = \sqrt{2+\frac{1}{2}} = \frac{\sqrt{10}}{2}$ $\quad \phi_1 = \arctan\left(\frac{\frac{\sqrt{2}}{2}}{-\sqrt{2}}\right) = \arctan(-\frac{1}{2}) = 2.6779$ rad or $153.43°$

(1') $\therefore \hat{\underline{a}}_1 \cdot \underline{x} = \frac{\sqrt{10}}{2}\cdot\exp(+i\phi_1)$

$$|\hat{\underline{a}}_2 \cdot \underline{x}| = \sqrt{\left(\frac{2\sqrt{2}+1}{2}\right)^2 + \left(\frac{\sqrt{2}+1}{2}\right)^2} = \sqrt{\frac{8+1+4\sqrt{2}}{4} + \frac{2+1+2\sqrt{2}}{4}} = \sqrt{3+\frac{3\sqrt{2}}{2}} \approx 2.2630$$

(1') $\phi_2 = \arctan\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right) = 32.2356°$ or 0.5626 rad $\therefore \hat{\underline{a}}_2 \cdot \underline{x} = 2.2630\cdot\exp(+i\phi_2)$

(1') $|\hat{\underline{a}}_3 \cdot \underline{x}| = \frac{2-\sqrt{2}}{2}$, $\quad \phi_3 = \frac{\pi}{2}$. $\quad \hat{\underline{a}}_3 \cdot \underline{x} = \frac{2-\sqrt{2}}{2}\exp(+i\frac{\pi}{2})$

$|\hat{\underline{a}}_4 \cdot \underline{x}| = \sqrt{(-\frac{1}{2})^2 + (\frac{1-\sqrt{2}}{2})^2} = \sqrt{1-\frac{\sqrt{2}}{2}} \approx 0.5412$ $\quad \phi_4 = \arctan\left(\frac{1-\sqrt{2}}{-1}\right) \approx -2.7489$ or $-157.5°$

(1') $\therefore \hat{\underline{a}}_4 \cdot \underline{x} = 0.5412\cdot\exp(+i\phi_4)$