IMGS-261 HW#3 Solution

1. Suppose that $z_1 = a_1 + ib_1$ $\quad z_2 = a_2 + ib_2$

(2')

then $(z_1 + z_2)^* = (a_1 + ib_1 + a_2 + ib_2)^* = (a_1 + a_2) - i(b_1 + b_2)$

$z_1^* = a_1 - ib_1$, $z_2^* = a_2 - ib_2 \Rightarrow z_1^* + z_2^* = (a_1 + a_2) - i(b_1 + b_2)$

$\therefore (z_1 + z_2)^* = z_1^* + z_2^*$

2.

(a) $(z_1 \cdot z_2)^* = \left[(a_1 + ib_1)(a_2 + ib_2)\right]^* = \left[a_1a_2 + b_1b_2 + i(a_1b_2 + a_2b_1)\right]^*$

(2')

$= (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$

$= (a_1 - ib_1)(a_2 - ib_2) = z_1^* \cdot z_2^*$

(b) $\left(\frac{z_1}{z_2}\right)^* = \left(z_1 \cdot \frac{1}{z_2}\right)^* \overset{(a)}{=} z_1^* \cdot \left(\frac{1}{z_2}\right)^*$

(2')

$\because \left(\frac{1}{z_2}\right)^* = \left(\frac{1}{a_2 + ib_2}\right)^* = \left(\frac{a_2 - ib_2}{a_2^2 + b_2^2}\right)^* = \left(\frac{a_2 + ib_2}{a_2^2 + b_2^2}\right) = \frac{1}{a_2 - ib_2} = \frac{1}{z_2^*}$

$\therefore \left(\frac{z_1}{z_2}\right)^* = z_1^* \cdot \frac{1}{z_2^*} = \frac{z_1^*}{z_2^*}$

4. (3'+3')

According to Euler relation, $e^{i3\theta} = \cos[3\theta] + i\sin[3\theta]$

$\therefore$ $\cos[3\theta]$ is the real part of $e^{i3\theta}$ and $\sin[3\theta]$ is its imaginary part.

Then $e^{i3\theta} = (e^{i\theta})^3$, and according to Euler relation: $e^{i\theta} = \cos[\theta] + i\sin[\theta]$

$\therefore e^{i3\theta} = (\cos[\theta] + i\sin[\theta])^3$

$= \cos^3[\theta] + (i\sin[\theta])^3 + 3\cos^2[\theta] \cdot i\sin[\theta] + 3\cos[\theta] \cdot (i\sin[\theta])^2$

$= \cos^3[\theta] - i\sin^3[\theta] + i\,3\cos^2[\theta]\sin[\theta] - 3\cos[\theta]\sin^2[\theta]$

$= \underbrace{\cos^3[\theta] - 3\cos[\theta]\sin^2[\theta]}_{\cos[3\theta]} + i\underbrace{\left(3\cos^2[\theta]\sin[\theta] - \sin^3[\theta]\right)}_{\sin[3\theta]}$

Continued ...

4. (continued)

$$\therefore \cos[3\theta] = \cos^3[\theta] - 3\cos[\theta]\sin^2[\theta]$$
$$= 4\cos^3[\theta] - 3\cos[\theta]\cos^2[\theta] - 3\cos[\theta]\sin^2[\theta]$$
$$= 4\cos^3[\theta] - 3\cos[\theta]\underbrace{(\cos^2[\theta] + \sin^2[\theta])}_{=1}$$
$$= 4\cos^3[\theta] - 3\cos[\theta]$$

$$\sin[3\theta] = 3\cos^2[\theta]\sin[\theta] - \sin^3[\theta]$$
$$= 3\cos^2[\theta]\sin[\theta] + 3\sin^2[\theta]\sin[\theta] - 4\sin^3[\theta]$$
$$= 3\underbrace{(\cos^2[\theta] + \sin^2[\theta])}_{=1}\sin[\theta] - 4\sin^2[\theta]$$
$$= 3\sin[\theta] - 4\sin^3[\theta]$$

3. (4')

suppose $z_0 = |z_0| e^{i\theta}$

$$z_0^* = z_0^3 \Rightarrow |z_0| e^{-i\theta} = |z_0|^3 e^{i3\theta}$$

$$\Rightarrow \begin{cases} |z_0| = |z_0|^3 \\ e^{-i\theta} = e^{i3\theta} \end{cases} \Rightarrow \begin{cases} |z_0| = \pm 1, 0 \\ 3\theta = -\theta \pm 2k\pi, \ k = 0, \pm 1, \pm 2 \cdots \end{cases}$$

$$\Rightarrow \begin{cases} |z_0| = \pm 1, 0 \\ \theta = \pm \frac{k}{2}\pi, \ k = 0, \pm 1, \pm 2 \end{cases}$$

$\therefore$ The set of complex numbers that satisfy $z_0^* = z_0^3$ is

0 and $\pm e^{\pm i \frac{k}{2}\pi}$, $k = 0, \pm 1, \pm 2 \cdots$

which is $\{0, 1, -1, i, -i\}$

5.

Let $Z^n = r_0 e^{+i\theta} = r_0(\cos\theta + i\sin\theta)$

According to DeMoivre's theorem

$$Z^{\frac{1}{n}} = r_0^{\frac{1}{n}}\left[\cos\left(\frac{\theta+k\cdot 2\pi}{n}\right) + i\sin\left(\frac{\theta+k\cdot 2\pi}{n}\right)\right], \quad k = 0, 1, 2 \cdots n-1$$

a) $Z^2 - 1 = 0 \Rightarrow Z^2 = 1 \Rightarrow r_0 = 1,\ \theta = 0$

(2')

$\therefore Z = 1^{\frac{1}{2}}\left[\cos\left(\frac{0+k2\pi}{2}\right) + i\sin\left(\frac{0+k\cdot 2\pi}{2}\right)\right], \quad k = 0, 1$

if $k=0$, $Z = e^{i\cdot 0}$ or 1;

if $k=1$, $Z = e^{i\pi}$ or $\cos\pi + i\sin\pi = -1 + i0$.

b) $Z^3 - 1 = 0 \Rightarrow Z^3 = 1 \Rightarrow r_0 = 1,\ \theta = 0$

(2')

$\therefore Z = 1^{\frac{1}{3}}\left[\cos\left(\frac{0+k\cdot 2\pi}{3}\right) + i\sin\left(\frac{0+k\cdot 2\pi}{3}\right)\right], \quad k = 0, 1, 2$,

if $k=0$, $Z = e^{i\cdot 0}$ or 1;

if $k=1$, $Z = e^{i\frac{2\pi}{3}}$ or $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$;

if $k=2$, $Z = e^{i\frac{4\pi}{3}}$ or $\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$;

c) (2') $Z^4 = 1 \Rightarrow r_0 = 1,\ \theta = 0$

$\therefore Z = 1^{\frac{1}{4}}\left[\cos\left(\frac{0+k\cdot 2\pi}{4}\right) + i\sin\left(\frac{0+k\cdot 2\pi}{4}\right)\right], \quad k = 0, 1, 2, 3$

if $k=0$, $Z = e^{i\cdot 0}$ or 1;

if $k=1$, $Z = e^{i\frac{\pi}{2}}$ or $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$;

if $k=2$, $Z = e^{i\pi}$ or $\cos(\pi) + i\sin(\pi) = -1$;

if $k=3$, $Z = e^{i\frac{3\pi}{2}}$ or $\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$.

d) $z^8 = +1 \Rightarrow r_0 = 1,\ \theta = 0$

(2')

$\therefore\ z = 1^{\frac{1}{8}}\left[\cos\left(\frac{0+k\cdot 2\pi}{8}\right) + i\sin\left(\frac{0+k\cdot 2\pi}{8}\right)\right],\ k = 0,1,2,3,\cdots,7$

if $k=0$, $z = e^{i\cdot 0}$ or 1 ;

if $k=1$, $z = e^{i\frac{\pi}{4}}$ or $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$;

if $k=2$, $z = e^{i\cdot\frac{\pi}{2}}$ or i ;

if $k=3$, $z = e^{i\cdot\frac{3\pi}{4}}$ or $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$;

if $k=4$, $z = e^{i\cdot\pi}$ or -1 ;

if $k=5$, $z = e^{i\frac{5\pi}{4}}$ or $-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$;

if $k=6$, $z = e^{i\frac{3\pi}{2}}$ or $-i$;

if $k=7$, $z = e^{i\frac{7\pi}{4}}$ or $\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

e) $z^2 - 4i = 4 \Rightarrow z^2 = 4 + 4i \Rightarrow r_0 = \sqrt{4^2+4^2} = 4\sqrt{2}$. $\theta = \arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$

(2')

$\therefore\ z = (4\sqrt{2})^{\frac{1}{2}}\left[\cos\left(\frac{\frac{\pi}{4}+k\cdot 2\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{4}+k\cdot 2\pi}{2}\right)\right],\ k = 0,1$

if $k=0$, $z = 2\sqrt[4]{2}\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right] = 2\cdot\sqrt[4]{2}\cdot e^{i\frac{\pi}{8}}$ or $2.1974 + 0.9102i$

if $k=1$, $z = 2\cdot\sqrt[4]{2}\left[\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right] = 2\sqrt[4]{2}\cdot e^{i\frac{9\pi}{8}}$ or $-2.1974 - 0.9102i$

6 $(5')$

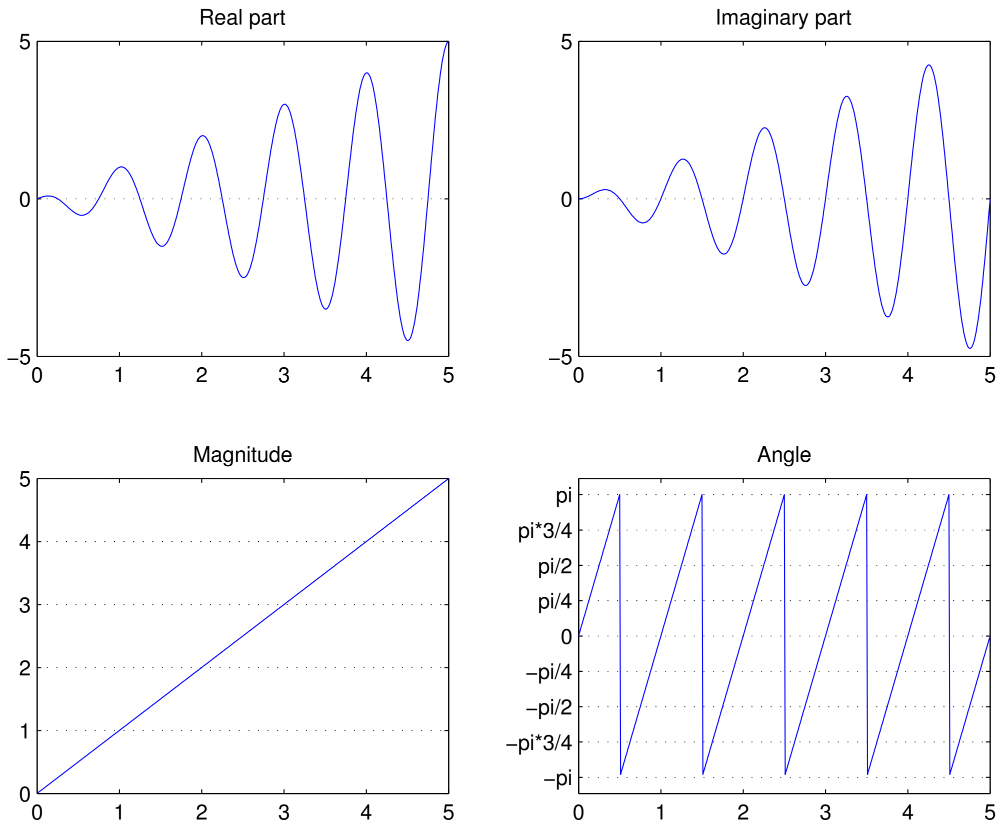

Figure 1: The real part, imaginary part, magnitude, and phase of $f_1(x)$ for $x \in [0, 5]$

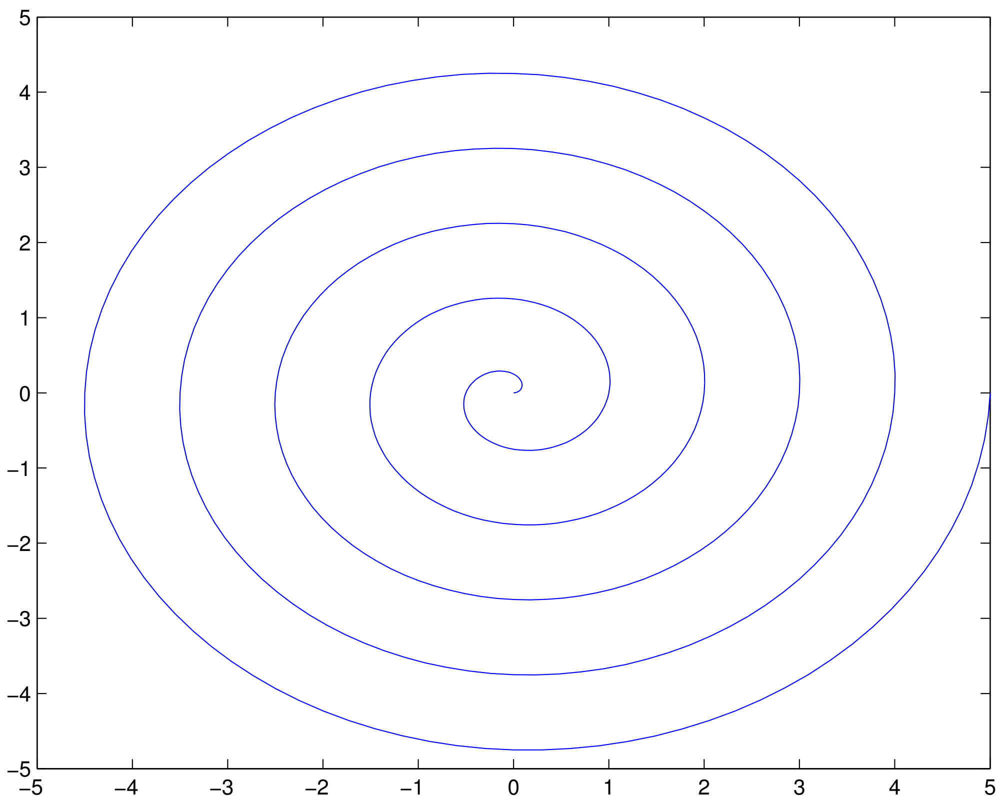

Figure 2: The Argand diagram of $f_1(x)$ for $x \in [0, 5]$

7. $\sum_{n=1}^{N} \cos[n\theta] = Re\{\sum_{n=1}^{N} e^{in\theta}\}$, $\sum_{n=1}^{N} \sin[n\theta] = Im\{\sum_{n=1}^{N} e^{in\theta}\}$

Bonus

$$\sum_{n=1}^{N} e^{in\theta} = \sum_{n=1}^{N} (e^{i\theta})^n = \frac{e^{i\theta}[1-(e^{i\theta})^N]}{1-e^{i\theta}}$$

$$= \frac{e^{i\frac{\theta}{2}} \cdot e^{i\frac{\theta}{2}} \cdot e^{i\frac{N\theta}{2}} (e^{-i\frac{N\theta}{2}} - e^{i\frac{N\theta}{2}})}{e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}})}$$

$\because \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$

$$= \frac{e^{i\frac{\theta}{2}} \cdot e^{i\frac{N\theta}{2}} (-2\sin[\frac{N\theta}{2}])}{-2\sin[\frac{\theta}{2}]}$$

$$= e^{i\frac{N+1}{2}\theta} \cdot \frac{\sin[\frac{N\theta}{2}]}{\sin[\frac{\theta}{2}]}$$

$$= \left(\cos[\tfrac{N+1}{2}\theta] + i\sin[\tfrac{N+1}{2}\theta]\right) \cdot \frac{\sin[\frac{N\theta}{2}]}{\sin[\frac{\theta}{2}]}$$

$\therefore$ $\sum_{n=1}^{N} \cos[n\theta] = Re\{\sum_{n=1}^{N} e^{in\theta}\} = \cos[\frac{N+1}{2}\theta] \cdot \frac{\sin[\frac{N\theta}{2}]}{\sin[\frac{\theta}{2}]}$ (+4')

$\sum_{n=1}^{N} \sin[n\theta] = Im\{\sum_{n=1}^{N} e^{in\theta}\} = \sin[\frac{N+1}{2}\theta] \cdot \frac{\sin[\frac{N\theta}{2}]}{\sin[\frac{\theta}{2}]}$ (+4')