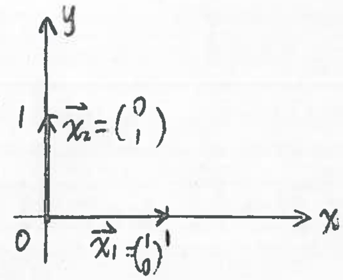
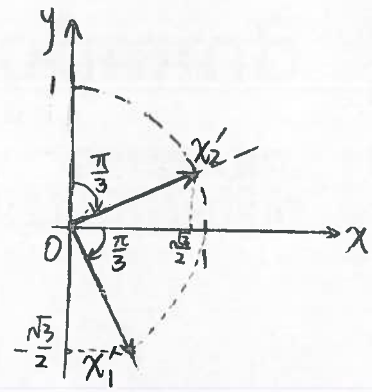
Solution for HW#2

1.(8')

(a) 2'



(b) 2'



(c) 2'

$$\vec{x_1'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{x_1} = \begin{pmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\vec{x_2'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{x_2} = \begin{pmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin(-\frac{\pi}{3}) \\ \cos(-\frac{\pi}{3}) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

(d) 2'

$$\vec{x_1'} \cdot \vec{x_2'} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = 0 \quad \therefore \vec{x_1'} \text{ and } \vec{x_2'} \text{ are orthogonal}$$

2.(4')

$$b_1 = \vec{a} \cdot \vec{x_1'} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2} a_1 - \frac{\sqrt{3}}{2} a_2$$

$$b_2 = \vec{a} \cdot \vec{x_2'} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{\sqrt{3}}{2} a_1 + \frac{1}{2} a_2$$

$$\vec{a} = b_1 \vec{x_1} + b_2 \vec{x_2}$$

3.(10')

a) 3'

$$A_1 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} \Rightarrow A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) 3'

$$A_2 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c \\ a \\ d \\ b \end{bmatrix} \Rightarrow A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \xrightarrow{\text{rotated by } \frac{\pi}{2}} \begin{bmatrix} c & d \\ a & b \end{bmatrix} \right.$$

c) 4'

$$A_3 \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \end{bmatrix} \Rightarrow A_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

d) +4'

A_n^{-1} do inverse operations so that $\vec{b_n}$ are swapped back to $\vec{x}$

$$A_1^{-1} \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A$$

$$A_2^{-1} \begin{bmatrix} c \\ a \\ d \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Rightarrow A_2^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_3^{-1} \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

as we can't use the linear combination of elements in $\vec{b_4}$ to get any element in $\vec{x}$. A_3^{-1} doesn't exist

or $$|A_3| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 0 \Rightarrow A_3^{-1} \text{ doesn't exist.}$$

4. (13')

$$z_1 = 1-2i\ ,\ z_2 = 3+i\ ,\ z_1^* = 1+2i\ ,\ z_2^* = 3-i\ ,$$

$$z_1^{-1} = \frac{1}{1-2i} = \frac{1+2i}{5} = \frac{1}{5}+\frac{2}{5}i\ ,\quad z_2^{-1} = \frac{1}{3+i} = \frac{3-i}{10} = \frac{3}{10}-\frac{1}{10}i\ ,$$

$$z_1+z_2 = 4-i\ ,\quad z_1-z_2 = -2-3i\ ,\quad z_1 z_1^* = (1-2i)(1+2i) = 1-4i^2 = 5\ ,$$

$$z_1 \cdot z_2^* = (1-2i)\cdot(3-i) = 3+2i^2-7i = 1-7i\ ,$$

$$\frac{z_2}{z_1} = \frac{3+i}{1-2i} = \frac{(3+i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+2i^2+7i}{5} = \frac{1+7i}{5} = \frac{1}{5}+\frac{7}{5}i\ ,$$

2' (b) $|z_1| = \sqrt{z_1 \cdot z_1^*} = \sqrt{5}$. $|z_2| = \sqrt{z_2 \cdot z_2^*} = \sqrt{10}$

$$\phi = \tan^{-1}\left(\frac{-2}{1}\right) = -63.4349^\circ \qquad \phi_2 = \tan^{-1}\left(\frac{1}{3}\right) = 18.4349^\circ$$

(a) (c) next page.

4 a)

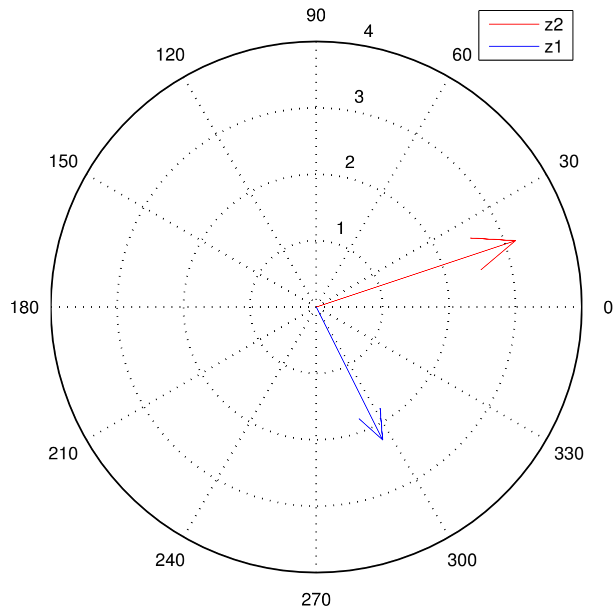
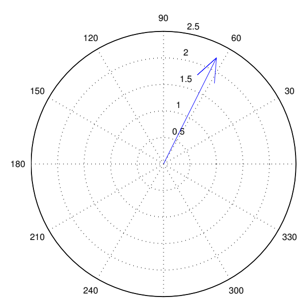
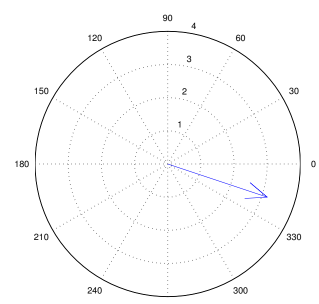


Figure 1: z_1 and z_2 on the Argand diagram of the complex plane

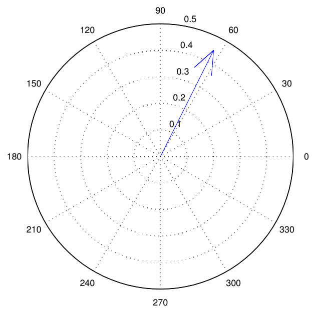
4 c)



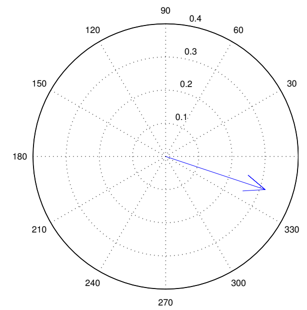
(a) $z_1^{\star}$



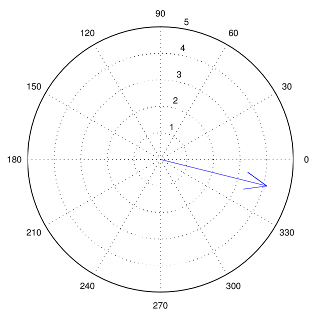
(b) $z_2^{\star}$



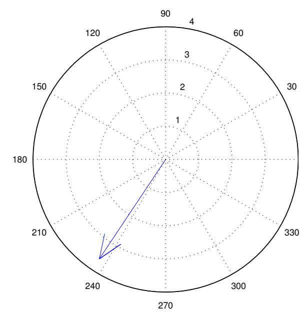
(c) z_1^{-1}



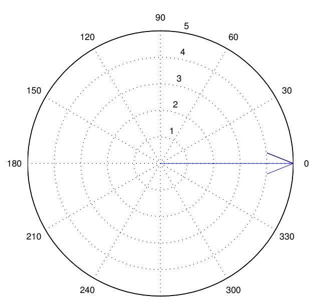
(d) z_2^{-1}



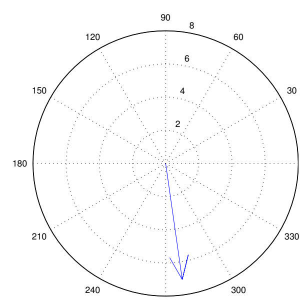
(e) $z_1 + z_2$



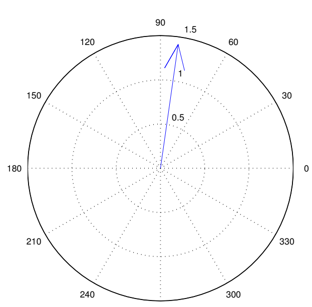
(f) $z_1 - z_2$



(g) $z_1 \cdot z_1^{\star}$



(h) $z_1 \cdot z_2^{\star}$



(i) $\frac{z_2}{z_1}$