IMGS-261 Solutions to Homework #10

In the following expressions, L_0 and Δx are real-valued parameters and the *COMB* function is defined:

$$COMB[x] \equiv \sum_{n=-\infty}^{+\infty} \delta[x-n]$$

A COMB function scaled by the width b_0 may be evaluated as:

$$COMB\left[\frac{x}{b_0}\right] = \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x}{b_0}-n\right] = \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x-n\cdot b_0}{b_0}\right] = |b_0| \cdot \sum_{n=-\infty}^{+\infty} \delta[x-n\cdot b_0]$$

The spectrum of the *COMB* function is:

$$\mathcal{F}_1\{COMB[x]\} = COMB[\xi]$$

Use these definitions to sketch these functions and find expressions for and sketch their Fourier transforms:

1.
$$g[x] = \frac{1}{L_0} COMB\left[\frac{x}{L_0}\right] * RECT\left[\frac{x}{L_0/2}\right]$$

The function is the convolution of a COMB function (with separations equal to integer multiples of L_0 and where each element has unit area) and a rectangle function whose width is half of the separation of the elements of the COMB. The result is a "square wave" where the amplitudes are 0 and 1 in equal amounts (a "50% square wave"). The spectrum is the product of the spectrum of the COMB and of the RECT, and so is the product of a COMB with separations of L_0^{-1} and a SINC of width $2L_0^{-1}$.

$$\begin{aligned}
\mathcal{F}\{g[x]\} &= G[\xi] = \mathcal{F}\left\{\frac{1}{L_0}COMB\left[\frac{x}{L_0}\right]\right\} \cdot \mathcal{F}\left\{RECT\left[\frac{x}{L_0/2}\right]\right\} \\
&= COMB[L_0\xi] \cdot \frac{L_0}{2} \cdot SINC\left[\frac{L_0}{2}\cdot\xi\right] \\
&= COMB\left[\frac{\xi}{\left(\frac{1}{L_0}\right)}\right] \cdot \frac{L_0}{2} \cdot SINC\left[\frac{\xi}{\left(\frac{2}{L_0}\right)}\right]
\end{aligned}$$

Several of you assumed that the areas of the Dirac delta functions in $COMB[L_0\xi]$ are all unity, which they are not! You need to find the expression for the COMB as a summation of delta functions:

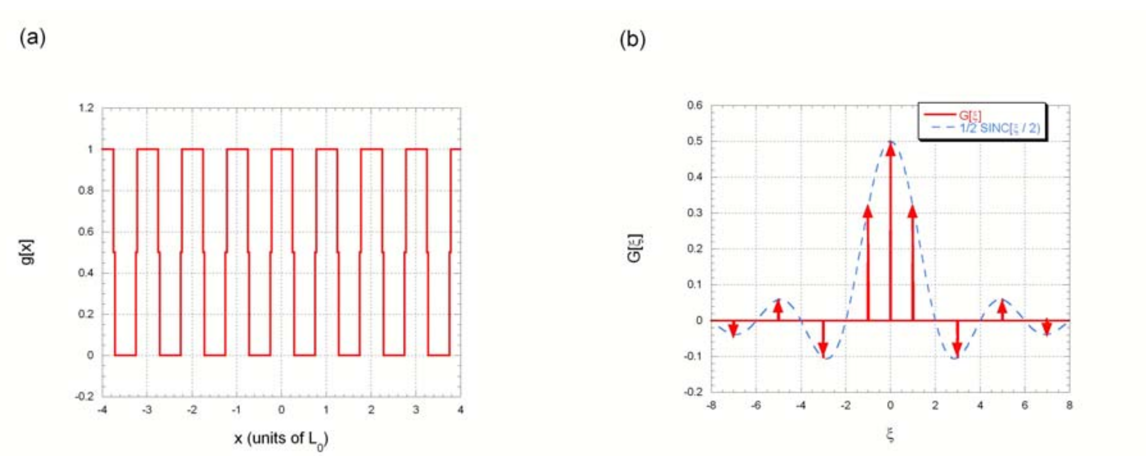
$$\begin{aligned}
COMB\left[\frac{\xi}{\left(\frac{1}{L_0}\right)}\right] &= COMB[L_0\xi] = \sum_{k=-\infty}^{+\infty} \delta[L_0\xi - k] \\
&= \sum_{k=-\infty}^{+\infty} \delta\left[L_0\cdot\left(\xi - \frac{k}{L_0}\right)\right] = \frac{1}{|L_0|}\sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right]
\end{aligned}$$

Now multiply the COMB by the SINC:

$$
\begin{aligned}
G[\xi] &= \left(\frac{1}{|L_0|}\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{L_0}\right]\right)\cdot\frac{|L_0|}{2}\cdot SINC\left[\frac{\xi}{\left(\frac{2}{L_0}\right)}\right] \\
&= \frac{1}{2}\cdot\left(\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{L_0}\right]\right)\cdot SINC\left[\frac{\xi}{\left(\frac{2}{L_0}\right)}\right] \\
&= \frac{1}{2}\cdot\sum_{k=-\infty}^{+\infty}\left(\delta\left[\xi-\frac{k}{L_0}\right]\cdot SINC\left[\frac{\xi}{\left(\frac{2}{L_0}\right)}\right]\right) \\
&= \frac{1}{2}\cdot\sum_{k=-\infty}^{+\infty}\left(\delta\left[\xi-\frac{k}{L_0}\right]\cdot SINC\left[\frac{\frac{k}{L_0}}{\left(\frac{2}{L_0}\right)}\right]\right)
\end{aligned}
$$

$$
\boxed{G[\xi]=\frac{1}{2}\cdot\sum_{k=-\infty}^{+\infty}\left(\delta\left[\xi-\frac{k}{L_0}\right]\cdot SINC\left[\frac{k}{2}\right]\right)}
$$

So the result is a SINC function with half-unit amplitude at the origin sampled twice per increment of the width parameter:



(a) $g[x]$, which is a 50% square wave (values of +1 and 0 each half the time); (b) spectrum of the square wave is a SINC function with half unit amplitude sampled at half the width parameter.

2.

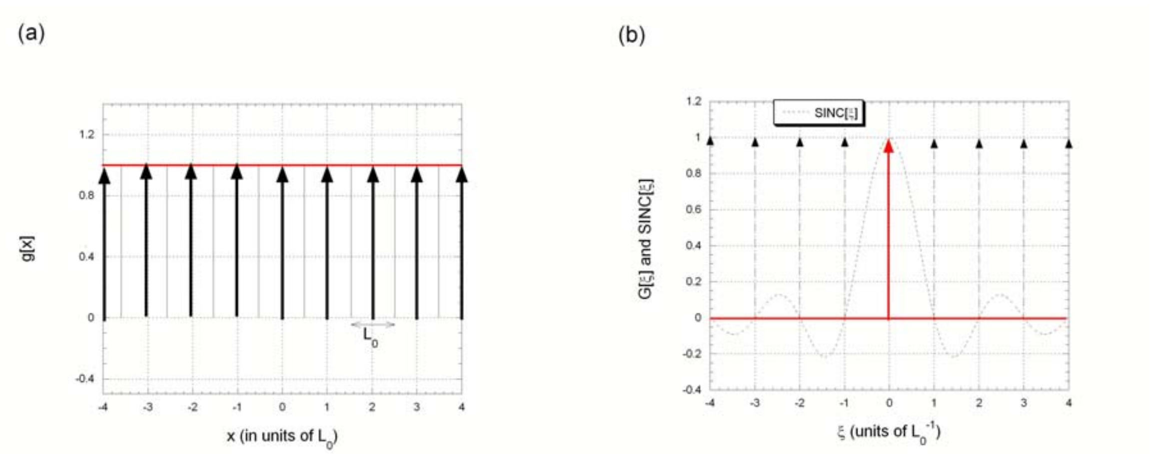
$$g[x] = \frac{1}{L_0} COMB\left[\frac{x}{L_0}\right] * RECT\left[\frac{x}{L_0}\right]$$

In this case, the width parameter of the COMB is equal to the width of the rectangle, so the function $g[x]$ *is the sum of rectangles spaced by their widths; the result is a unit-amplitude constant. This can be seen by evaluating the spectrum:*

$$\begin{aligned}
G[\xi] &= COMB[L_0\xi] \cdot L_0 \cdot SINC[L_0\xi] \\
&= \left(\frac{1}{L_0}\sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right]\right) \cdot L_0 \cdot SINC[L_0\xi] \\
&= \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right] \cdot SINC[L_0\xi] \\
&= \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right] \cdot SINC\left[L_0 \cdot \frac{k}{L_0}\right] \\
&= \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right] \cdot SINC[k] = \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right] \cdot \frac{\sin[k\pi]}{k\pi} \\
&= \cdots + \delta\left[\xi - \frac{-2}{L_0}\right] \cdot SINC[-2] + \delta\left[\xi - \frac{-1}{L_0}\right] \cdot SINC[-1] \\
&\quad + \delta[\xi - 0] \cdot SINC[0] + \delta\left[\xi - \frac{+1}{L_0}\right] \cdot SINC[+1] + \cdots \\
&= \cdots + 0 + 0 + \delta[\xi] \cdot 1 + 0 + 0 + \cdots \\
&= \delta[\xi]
\end{aligned}$$

So the space-domain function is:

$$g[x] = \mathcal{F}^{-1}\{\delta[\xi]\} = 1[x]$$



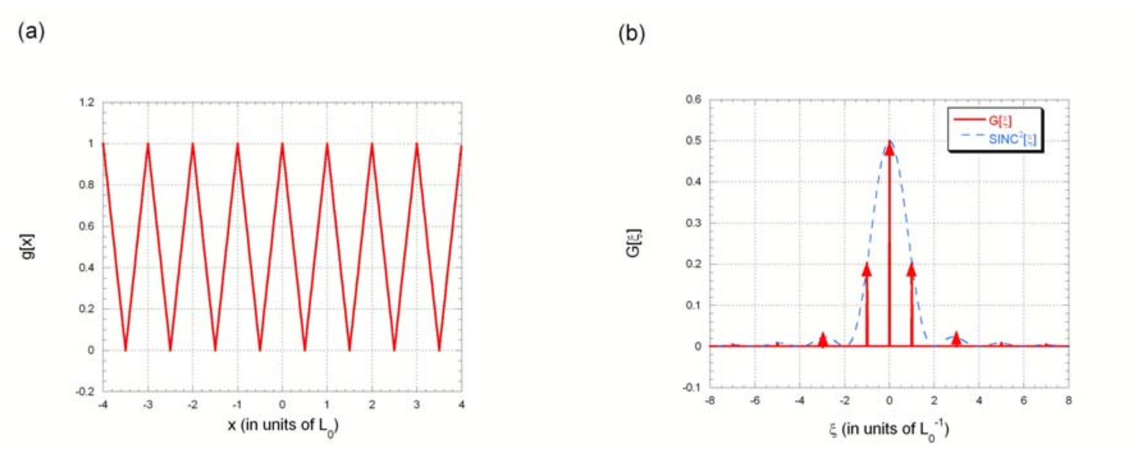
(a) plot of the delta functions (in black) which are convolved with unit-width rectangles to produce $g[x]$ *(in red), which is the unit constant. (b) the spectrum of the unit constant is* $\delta[\xi]$.

Same comment here ... many of you just assumed (and incorrectly) that you knew the form of the space-domain COMB function; you "knew" that each Dirac delta function was scaled by L_0^{-1}. *You need to substitute the expression into the definition and evaluate the weights until you really have absorbed it.*

3.

$$g[x] = \frac{1}{L_0} COMB\left[\frac{x}{L_0}\right] * TRI\left[\frac{x}{L_0/2}\right]$$

This is the convolution of the same COMB function and a TRIANGLE with width parameter equal to half; the result is an array of triangles:



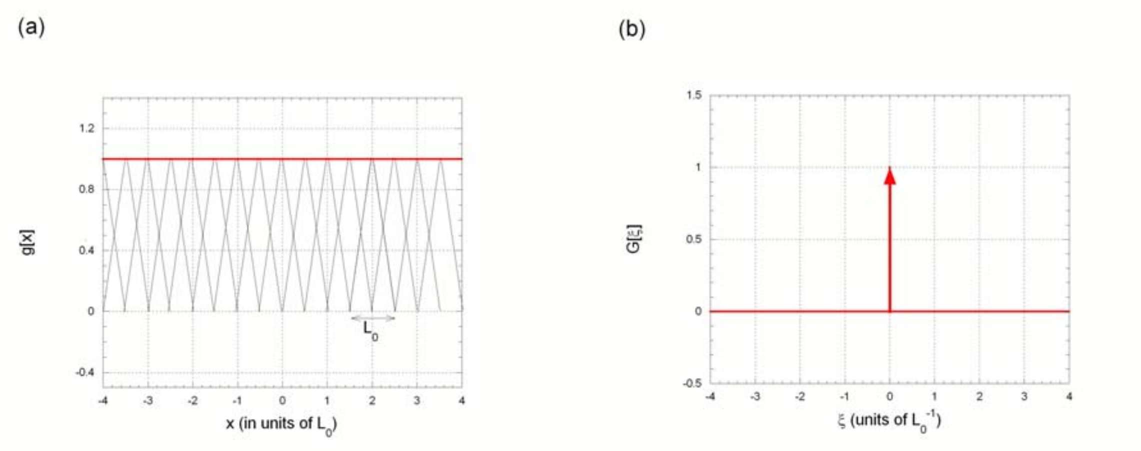
The spectrum is the product of the spectra of the COMB and of the TRI, and so is the product of a COMB and the square of a SINC:

$$\begin{aligned} G[\xi] &= \left(\frac{1}{L_0}\sum_{k=-\infty}^{+\infty}\delta\left[\xi - \frac{k}{L_0}\right]\right)\cdot\frac{L_0}{2}\cdot SINC^2\left[\frac{\xi}{\left(\frac{2}{L_0}\right)}\right] \\ &= \frac{1}{2}\cdot\sum_{k=-\infty}^{+\infty}\left(\delta\left[\xi - \frac{k}{L_0}\right]\cdot SINC^2\left[\frac{k}{2}\right]\right) \end{aligned}$$

4.

$$g[x] = \frac{1}{L_0} COMB\left[\frac{x}{L_0}\right] * TRI\left[\frac{x}{L_0}\right]$$

This is analogous to #2, where the width parameters of the rectangle and the comb function were equal; here the width parameters of the triangle and the comb are equal, so the space-domain function is the sum of equally spaced triangle functions:



$$\begin{aligned} g[x] &= \frac{1}{L_0} COMB\left[\frac{x}{L_0}\right] * TRI\left[\frac{x}{L_0}\right] \\ &= \sum_{n=-\infty}^{+\infty} \delta[x - nL_0] * TRI\left[\frac{x}{L_0}\right] \\ &= \sum_{n=-\infty}^{+\infty} TRI\left[\frac{x - nL_0}{L_0}\right] \\ &= 1[x] \end{aligned}$$

The sum of the triangles is the unit constant, so the spectrum is a Dirac delta function:

$$\begin{aligned} G[\xi] &= \left(\frac{1}{L_0} \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{L_0}\right]\right) \cdot L_0 \cdot SINC^2\left[\frac{\xi}{\left(\frac{1}{L_0}\right)}\right] \\ &= \sum_{k=-\infty}^{+\infty} \left(\delta\left[\xi - \frac{k}{L_0}\right] \cdot SINC^2[k]\right) \\ &= \delta[\xi] \end{aligned}$$

5.

$$g[x] = \left(\frac{1}{\Delta x} \cdot COMB\left[\frac{x}{\Delta x}\right]\right) \cdot \cos\left[2\pi\left(\frac{x}{8 \cdot \Delta x}\right)\right]$$

This is a cosine function sampled at spacing Δx and the result is a cosine sampled 8 times per cycle, so its spatial frequency is:

$$\xi_0 = \frac{1}{8 \cdot \Delta x}$$

The Fourier transform is evaluated via the modulation theorem:

$$\mathcal{F}\{f[x] \cdot m[x]\} = F[\xi] * M[\xi] = \int_{\beta=-\infty}^{+\infty} F[\beta] \cdot M[\xi - \beta]\, d\beta = \int_{\beta=-\infty}^{+\infty} M[\beta] \cdot F[\xi - \beta]\, d\beta$$

The Fourier transforms of the component functions are:

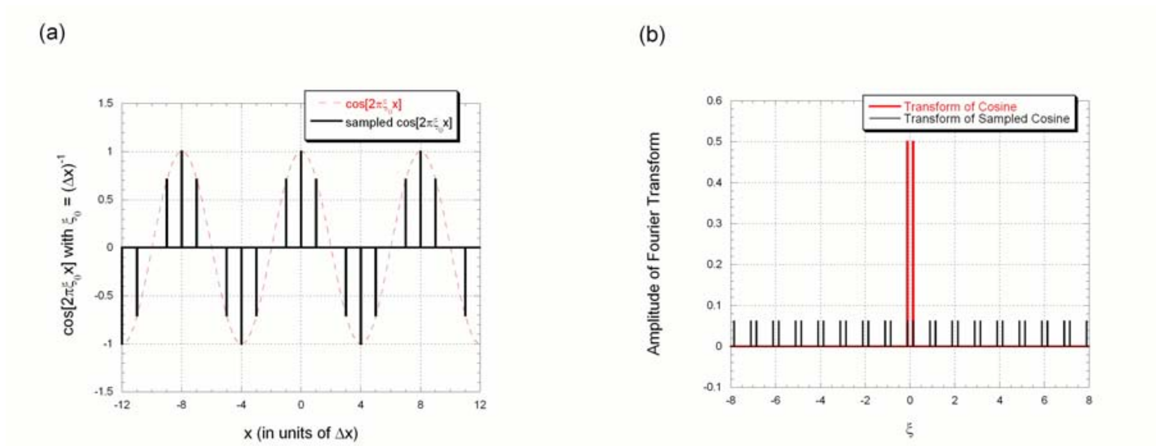
$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{\Delta x} \cdot COMB\left[\frac{x}{\Delta x}\right]\right\} &= \frac{1}{\Delta x} \cdot \mathcal{F}\left\{COMB\left[\frac{x}{\Delta x}\right]\right\} = \frac{1}{\Delta x} \cdot \Delta x \cdot COMB[\Delta x \cdot \xi] \\
&= COMB[\Delta x \cdot \xi] = COMB\left[\frac{\xi}{(\Delta x)^{-1}}\right] \\
&= \sum_{k=-\infty}^{+\infty} \delta[(\Delta x \cdot \xi) - k] \\
&= \sum_{k=-\infty}^{+\infty} \delta\left[\Delta x \cdot \left(\xi - \frac{k}{\Delta x}\right)\right] \\
&= \frac{1}{\Delta x} \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{\Delta x}\right]
\end{aligned}$$

an infinite set of Dirac delta functions located at integer multiples of $\frac{1}{\Delta x}$ each with area $\frac{1}{\Delta x}$.

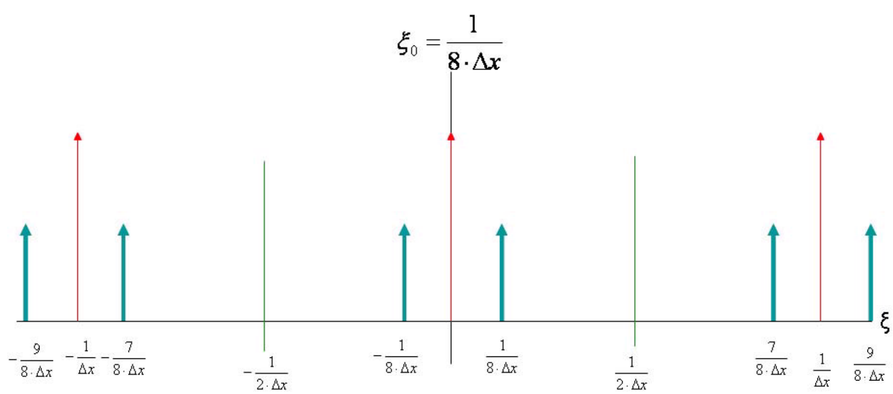
$$\begin{aligned}
\mathcal{F}\left\{\cos\left[2\pi\left(\frac{x}{8 \cdot \Delta x}\right)\right]\right\} &= \mathcal{F}\{\cos[2\pi\xi_0 x]\} \\
&= \frac{1}{2}\delta[\xi + \xi_0] + \frac{1}{2}\delta[\xi - \xi_0] \\
&= \frac{1}{2}\delta\left[\xi + \frac{1}{8 \cdot \Delta x}\right] + \frac{1}{2}\delta\left[\xi - \frac{1}{8 \cdot \Delta x}\right]
\end{aligned}$$

a pair of Dirac delta functions with half-unit area located at $\xi = \pm\frac{1}{8 \cdot \Delta x}$. The convolution of these two spectra produces a replica of the pair of Dirac delta function centered about each Dirac delta function in the frequency-domain COMB, so the replicas are located at $\xi = \frac{k}{\Delta x} \pm \frac{1}{8 \cdot \Delta x} = \frac{8k \pm 1}{8 \cdot \Delta x} = \pm\frac{1}{8 \cdot \Delta x}, \pm\frac{7}{8 \cdot \Delta x}, \pm\frac{9}{8 \cdot \Delta x}, \pm\frac{15}{8 \cdot \Delta x}, \pm\frac{17}{8 \cdot \Delta x}, \pm\frac{23}{8 \cdot \Delta x}, \pm\frac{25}{8 \cdot \Delta x}, \cdots$. These *are "far apart" in the frequency domain.*

$$\begin{aligned}
G[\xi] &= \left(\frac{1}{\Delta x}\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{\Delta x}\right]\right) * \left(\frac{1}{2}\delta\left[\xi+\frac{1}{8\cdot\Delta x}\right]+\frac{1}{2}\delta\left[\xi-\frac{1}{8\cdot\Delta x}\right]\right)\\
&= \left(\frac{1}{2\Delta x}\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{\Delta x}\right]\right) * \left(\delta\left[\xi+\frac{1}{8\cdot\Delta x}\right]+\delta\left[\xi-\frac{1}{8\cdot\Delta x}\right]\right)\\
&= \frac{1}{2\cdot\Delta x}\sum_{k=-\infty}^{+\infty}\delta\left[\frac{k}{\Delta x}+\frac{1}{8\cdot\Delta x}\right]+\delta\left[\frac{k}{\Delta x}-\frac{1}{8\cdot\Delta x}\right]\\
&= \frac{1}{2\cdot\Delta x}\sum_{k=-\infty}^{+\infty}\left(\delta\left[\frac{8k+1}{8\cdot\Delta x}\right]+\delta\left[\frac{8k-1}{8\cdot\Delta x}\right]\right)
\end{aligned}$$



(a) cosine sampled 8 times per cycle; (b) spectrum of cosine (in red) and of sampled cosine (in black), showing the periodicity of the latter.



Spectrum of the sampled function for $\xi_0 = \frac{1}{8\cdot\Delta x}$*, showing the elements of* $COMB\left[\frac{\xi}{\Delta x^{-1}}\right]$ *in red, of the periodic spectrum in blue, and the limits of the period of width* $\frac{1}{\Delta x}$ *in green.*

6.

$$g[x] = \left(\frac{1}{\Delta x} \cdot COMB\left[\frac{x}{\Delta x}\right]\right) \cdot \cos\left[2\pi\left(\frac{x}{4 \cdot \Delta x}\right)\right]$$

In this case, the spatial frequency of the cosine is larger by a factor of 2:

$$\xi_1 = \frac{1}{4 \cdot \Delta x} = 2 \cdot \frac{1}{8 \cdot \Delta x} = 2 \cdot \xi_0$$

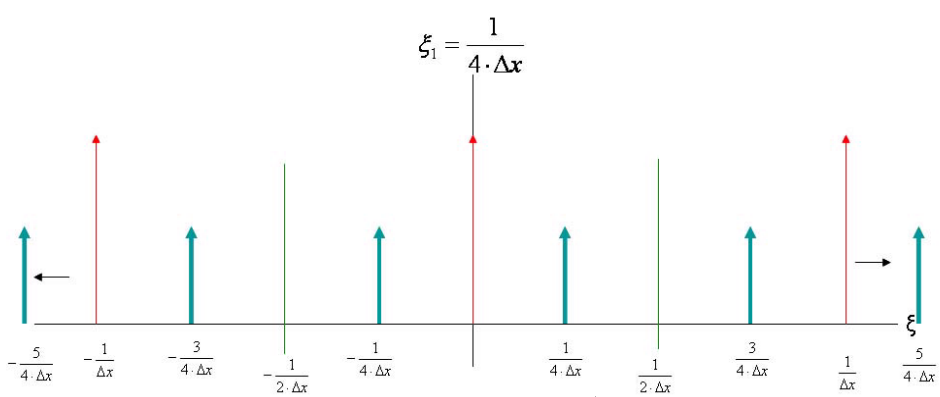
so the cosine function is sampled 4 times per cycle. The spectrum of the cosine is:

$$\mathcal{F}\left\{\cos\left[2\pi\left(\frac{x}{4 \cdot \Delta x}\right)\right]\right\} = \frac{1}{2}\delta\left[\xi + \frac{1}{4 \cdot \Delta x}\right] + \frac{1}{2}\delta\left[\xi - \frac{1}{4 \cdot \Delta x}\right]$$

So the frequency-domain convolution is:

$$\begin{aligned} G[\xi] &= \left(\frac{1}{\Delta x}\sum_{k=-\infty}^{+\infty}\delta\left[\xi - \frac{k}{\Delta x}\right]\right) * \left(\frac{1}{2}\delta\left[\xi + \frac{1}{4 \cdot \Delta x}\right] + \frac{1}{2}\delta\left[\xi - \frac{1}{4 \cdot \Delta x}\right]\right) \\ &= \frac{1}{2 \cdot \Delta x}\sum_{k=-\infty}^{+\infty}\left(\delta\left[\frac{4k+1}{4 \cdot \Delta x}\right] + \delta\left[\frac{4k-1}{4 \cdot \Delta x}\right]\right) \end{aligned}$$

So the locations of the Dirac delta functions are $\xi = \frac{k}{\Delta x} \pm \frac{1}{4 \cdot \Delta x} = \frac{4k \pm 1}{4 \cdot \Delta x} = \pm\frac{1}{4 \cdot \Delta x}, \pm\frac{3}{4 \cdot \Delta x}, \pm\frac{5}{4 \cdot \Delta x}, \pm\frac{7}{4 \cdot \Delta x}, \pm\frac{9}{4 \cdot \Delta x}, \cdots$, *and so are "not quite so far apart."*

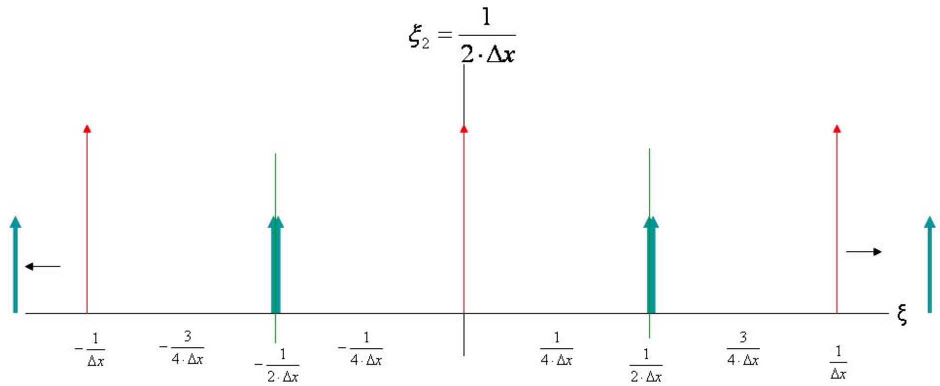


Spectrum of the sampled function for $\xi_0 = \frac{1}{4 \cdot \Delta x}$, *showing the elements of* $COMB\left[\frac{\xi}{\Delta x^{-1}}\right]$ *in red, of the periodic spectrum in blue, and the limits of the period of width* $\frac{1}{\Delta x}$ *in green. The spectrum of the cosine still may be extracted from the sampled spectrum by multiplying by* $RECT\left[\frac{\xi}{\Delta x^{-1}}\right]$

7.

$$g[x] = \left(\frac{1}{\Delta x} \cdot COMB\left[\frac{x}{\Delta x}\right]\right) \cdot \cos\left[2\pi\left(\frac{x}{2 \cdot \Delta x}\right)\right]$$

sampled two times per cycle, so the pairs of Dirac delta functions from each side line up exactly

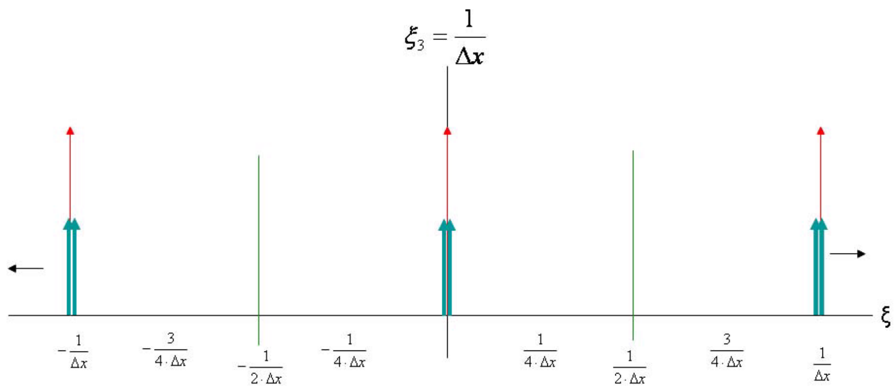


Spectrum of the sampled function for $\xi_0 = \frac{1}{2 \cdot \Delta x}$, *showing the elements of* $COMB\left[\frac{\xi}{\Delta x^{-1}}\right]$ *in red, of the periodic spectrum in blue, and the limits of the period of width* $\frac{1}{\Delta x}$ *in green. The elements of the spectrum of the cosine now "overlap" but still may be extracted from the sampled spectrum by multiplying by* $RECT\left[\frac{\xi}{\Delta x^{-1}}\right]$ *(just barely!)*

8.

$$g[x] = \left(\frac{1}{\Delta x} \cdot COMB\left[\frac{x}{\Delta x}\right]\right) \cdot \cos\left[2\pi\left(\frac{x}{1 \cdot \Delta x}\right)\right]$$

sampled one time per cycle, so all samples are identically unit-weighted Dirac delta functions, which is the same set of samples that would be obtained from a unit constant. The interpolated function will be the unit constant instead of the cosine.



Spectrum of the sampled function for $\xi_0 = \frac{1}{1 \cdot \Delta x}$*, showing the elements of* $COMB\left[\frac{\xi}{\Delta x^{-1}}\right]$ *in red, of the periodic spectrum in blue, and the limits of the period of width* $\frac{1}{\Delta x}$ *in green. The elements of the spectrum of the cosine now "overlap" at the integer multiples of* $(\Delta x)^{-1}$ *and may NOT be extracted from the sampled spectrum by multiplying by* $RECT\left[\frac{\xi}{\Delta x^{-1}}\right]$

9. For numbers 5-8, determine if the cosine function may be recovered by convolution with a SINC-function interpolator $h[x] = A_0 \cdot SINC\left[\frac{x}{b_0}\right]$ and find the allowed ranges of the parameters A_0 and b_0.

From the graphs of the spectrum for #6, it is apparent that the rectangle in the frequency domain may have width in the range from as narrow as $\frac{1}{4\cdot\Delta x}$ *to as wide as* $\frac{7}{4\cdot\Delta x}$

$$\mathcal{F}^{-1}\left\{RECT\left[\frac{1}{(4\cdot\Delta x)^{-1}}\right]\right\} \propto SINC\left[\frac{x}{4\cdot\Delta x}\right]$$
$$\mathcal{F}^{-1}\left\{RECT\left[\frac{1}{\left(\frac{7}{4\cdot\Delta x}\right)}\right]\right\} \propto SINC\left[\frac{x}{\frac{4\cdot\Delta x}{7}}\right]$$

Note that the "normal" width of the rectangle is:

$$RECT\left[\frac{\xi}{(\Delta x)^{-1}}\right] \Longrightarrow \mathcal{F}\left\{RECT\left[\frac{\xi}{(\Delta x)^{-1}}\right]\right\} \propto SINC\left[\frac{x}{\Delta x}\right]$$

In #7, the width of the frequency-domain rectangle may be as narrow as $\frac{1}{2\cdot\Delta x}$ *to as wide as* $\frac{3}{2\cdot\Delta x}$

$$\mathcal{F}^{-1}\left\{RECT\left[\frac{1}{(2\cdot\Delta x)^{-1}}\right]\right\} \propto SINC\left[\frac{x}{2\cdot\Delta x}\right]$$
$$\mathcal{F}^{-1}\left\{RECT\left[\frac{1}{\left(\frac{3}{2\cdot\Delta x}\right)}\right]\right\} \propto SINC\left[\frac{x}{\frac{2\cdot\Delta x}{3}}\right]$$

In #8, the width of the frequency-domain rectangle must be $\frac{1}{\Delta x}$

$$\mathcal{F}^{-1}\left\{RECT\left[\frac{1}{(\Delta x)^{-1}}\right]\right\} \propto SINC\left[\frac{x}{\Delta x}\right]$$

which is the "normal" ideal interpolator.

In #9, the samples are identical to the samples that would have been obtained from a constant function $f[x] = 1[x]$, *and so the cosine may not be recovered.*