SIMG-261-20142 Solutions to EXAM #2

Statistics (out of 60 points possible):

Mean	35.2
Std. Dev.	10.9
Median	36
Maximum	59
Minimum	15

Problems:

Problem ->	1	2	3	4	5	6	7	8
N	10	29	16	26	24	28	26	29
Mean (out of 10)	4.9	6.5	5.7	5.3	5.4	4.9	5.6	8.1
StdDev	3.4	2,7	3.8	2.7	4.0	2.4	2.5	2.5
Median	5	7	5	6	4	5	6	10
Maximum	10	10	10	10	10	10	10	10
Minimum	0	2	0	0	0	2	0	2

1. We defined the concepts of a linear operator and a shift-invariant operator; a linear operator satisfies the condition:

$$\text{if } \mathcal{O}\{f_n[x]\} = g_n[x]\,,\ \text{then } \mathcal{O}\left\{\sum_{n=1}^{N} \alpha_n \cdot f_n[x]\right\} = \sum_{n=1}^{N} \alpha_n \cdot g_n[x]$$

where $\{\alpha_n\}$ are numerical constants that are generally complex valued. A shift-invariant operator satisfies the condition

$$\text{if } \mathcal{O}\{f[x]\} = g[x]\,,\ \text{then } \mathcal{O}\{f[x-x_0]\} = g[x-x_0]$$

From these definitions, determine whether the Fourier transform operator $\mathcal{F}\{f[x]\}$ satisfies the properties of linearity and of shift invariance. You need to show this, not just write it down.

Solution: *The equation for the Fourier transform of a one-dimensional function is:*

$$\mathcal{F}_1\{f[x]\} = \int_{x=-\infty}^{x=+\infty} f[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \equiv F[\xi]$$

To test for linearity, you can do a simple test where the input amplitude is doubled:

$$\begin{aligned}
\mathcal{F}_1\{2 \cdot f[x]\} &= \int_{x=-\infty}^{x=+\infty} 2 \cdot f[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= 2 \cdot \int_{x=-\infty}^{x=+\infty} f[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= 2 \cdot F[\xi]
\end{aligned}$$

In general, the transform of a sum of functions is:

$$\begin{aligned}
\mathcal{F}_1\left\{\sum_{n=1}^{N} \alpha_n \cdot f_n[x]\right\} &= \int_{x=-\infty}^{x=+\infty} \left(\sum_{n=1}^{N} \alpha_n \cdot f_n[x]\right) \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= \sum_{n=1}^{N} \int_{x=-\infty}^{x=+\infty} \alpha_n \cdot f_n[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= \sum_{n=1}^{N} \alpha_n \cdot \int_{x=-\infty}^{x=+\infty} f_n[x] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= \sum_{n=1}^{N} \alpha_n \cdot F_n[\xi]
\end{aligned}$$

so $\boxed{\textit{the Fourier transform passes the test for linearity.}}$

To test for shift invariance, evaluate the Fourier transform of a translated function:

$$\begin{aligned}
\mathcal{F}\{f[x-x_0]\} &= \int_{x=-\infty}^{x=+\infty} f[x-x_0] \cdot \exp[-i \cdot 2\pi \cdot \xi x]\ dx \\
&= F[\xi] \cdot \exp[-i \cdot 2\pi \cdot \xi x_0] \neq F[\xi - x_0]
\end{aligned}$$

$\boxed{\textit{The Fourier transform FAILS the test for shift invariance.}}$ *This "should have been obvious" because the output variable ξ and the translation variable x_0 have different dimensions ("cycles per millimeter" vs. millimeters), and so cannot be subtracted.*

2. We defined the rectangle function by the expression:

$$RECT[x] = \begin{cases} 0 & \text{if} \quad |x| > \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad |x| = \frac{1}{2} \\ 1 & \text{if} \quad |x| < \frac{1}{2} \end{cases}$$

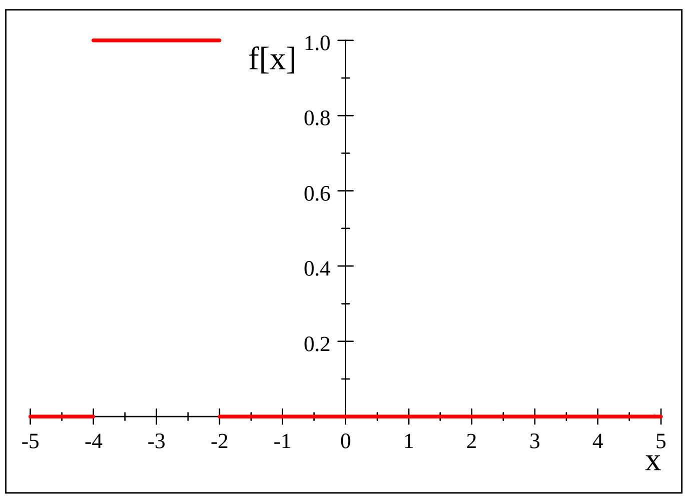
(a) Sketch $RECT\left[\frac{-x-3}{2}\right]$ and determine its area

Solution:

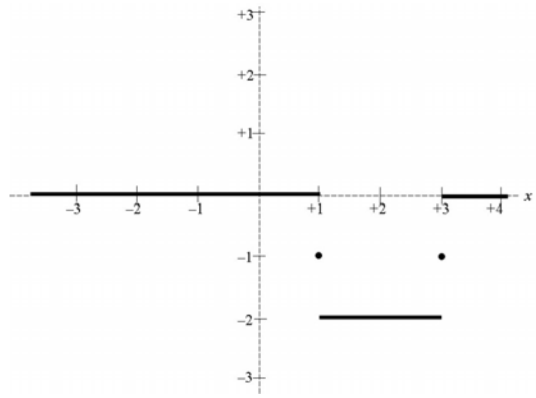
$$\begin{aligned} RECT\left[\frac{-x-3}{2}\right] &= RECT\left[-\frac{x+3}{2}\right] \\ &= RECT\left[\frac{x+3}{2}\right] \quad \textit{because } RECT[-u] = RECT[+u] \ \textit{(even function)} \end{aligned}$$

this is a rectangle of width 2 units centered at $x_0 = -3$, the area is two units.

$$RECT\left[\frac{x+3}{2}\right] = \begin{cases} 0 & \text{if} \quad \frac{x+3}{2} > \frac{1}{2} \implies x > -2 \\ \frac{1}{2} & \text{if} \quad \frac{x+3}{2} = \frac{1}{2} \implies x = -2 \\ 1 & \text{if} \quad -\frac{1}{2} < \frac{x+3}{2} < +\frac{1}{2} \implies -4 < x < -2 \\ \frac{1}{2} & \text{if} \quad \frac{x+3}{2} = -\frac{1}{2} \implies x = -4 \\ 0 & \text{if} \quad \frac{x+3}{2} < -\frac{1}{2} \implies x < -4 \end{cases}$$



(b) Determine the equation for the function with the following graph:



Solution: *"by inspection" the function is a rectangle of width two units centered about $x = +2$ and with amplitude of -2 units, so the equation is:*

$$\boxed{f[x] = -2 \cdot RECT\left[\frac{x-2}{2}\right]}$$

NOTE THAT THE AMPLITUDE (Y-AXIS POSITION) IS THE **LEADING** *FACTOR; IT IS* **NOT** *PART OF THE ARGUMENT OF THE FUNCTION. Many of you did not remember this.*

3. Simplify the expression and sketch the result:

$$g[x] = \int_{x=-\infty}^{x=+\infty} SINC[x] \cdot \delta[x-2] \; dx$$

Solution: *First apply the property of the Dirac delta function in products:*

$$f[x] \cdot \delta[x - x_0] = f[x_0] \cdot \delta[x - x_0]$$

$$\Longrightarrow SINC[x] \cdot \delta[x-2] = SINC[+2] \cdot \delta[x-2]$$

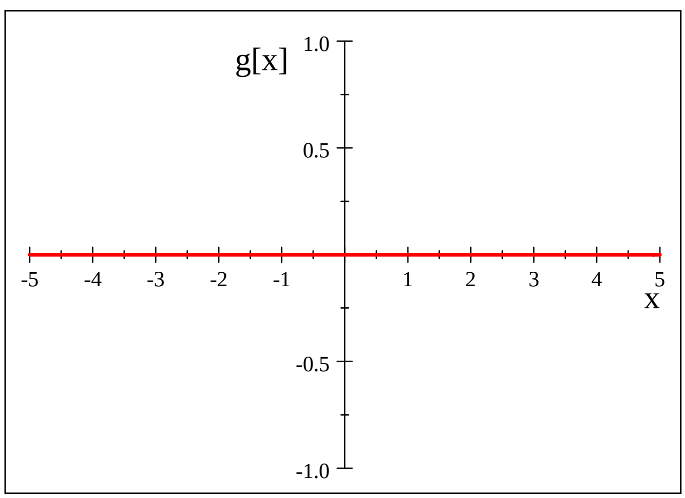
where:

$$SINC[x] \equiv \frac{\sin[\pi x]}{\pi x}$$

so that:

$$SINC[2] = \frac{\sin[2\pi]}{2\pi} = \frac{0}{2\pi} = 0$$

$$\begin{aligned} g[x] &= \int_{x=-\infty}^{x=+\infty} SINC[x] \cdot \delta[x-2] \; dx \\ &= \int_{x=-\infty}^{x=+\infty} SINC[2] \cdot \delta[x-2] \; dx \\ &= \int_{x=-\infty}^{x=+\infty} 0 \cdot \delta[x-2] \; dx \\ & \boxed{g[x] = 0[x]} \end{aligned}$$



$g[x] = \int_{x=-\infty}^{x=+\infty} SINC[x] \cdot \delta[x-2] \; dx = 0[x]$

You can also do this by evaluating the integral as a convolution:

$$\begin{aligned}
\int_{x=-\infty}^{x=+\infty} SINC[x]\cdot\delta[x-2]\ dx &= \int_{x=-\infty}^{x=+\infty} SINC[x]\cdot\delta[-(x-2)]\ dx \text{ (because } \delta[-x]=\delta[+x]\text{)}\\
&= \int_{x=-\infty}^{x=+\infty} SINC[x]\cdot\delta[2-x]\ dx\\
&= \int_{x=-\infty}^{x=+\infty} SINC[u]\cdot\delta[2-u]\ du\\
&= \left.(SINC[x]*\delta[x])\right|_{x=+2}\\
&= \left.(SINC[x])\right|_{x=+2} = SINC[+2] = 0
\end{aligned}$$

Not many did this problem, and yet it was one of the easiest on the test.

4. Evaluate the convolution of the pairs of rectangles listed:

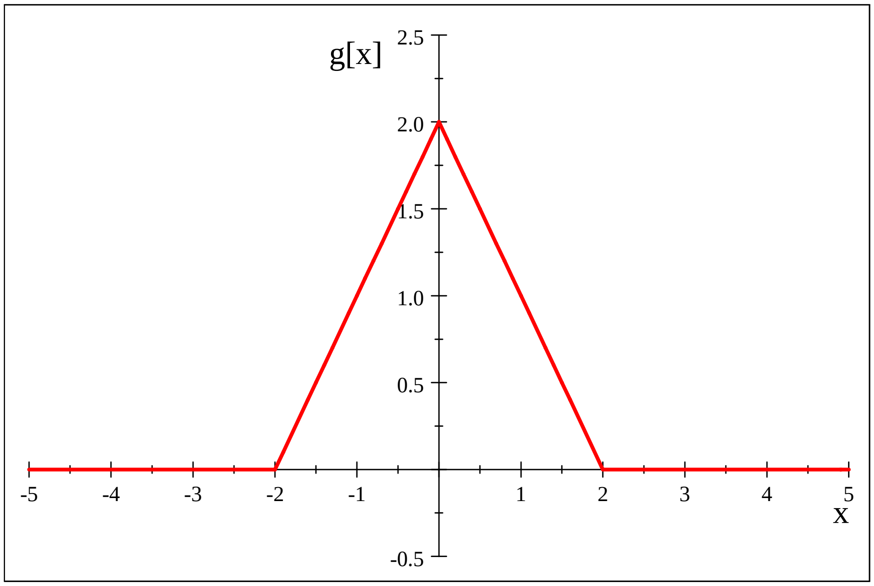
(a) $RECT\left[\frac{x}{2}\right] * RECT\left[\frac{x}{2}\right]$

$$\begin{aligned} RECT\left[\frac{x}{2}\right] * RECT\left[\frac{x}{2}\right] &= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha}{2}\right] \cdot RECT\left[\frac{x-\alpha}{2}\right] dx \\ &= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha}{2}\right] \cdot RECT\left[\frac{x}{2}-\frac{\alpha}{2}\right] dx \end{aligned}$$

change variable of integration:

$$u \equiv \frac{\alpha}{2} \implies d\alpha = 2 \cdot du$$

$$\begin{aligned} RECT\left[\frac{x}{2}\right] * RECT\left[\frac{x}{2}\right] &= \int_{u=-\infty}^{u=+\infty} RECT[u] \cdot RECT\left[\frac{x}{2}-u\right] \cdot 2 \cdot du \\ &= 2 \cdot \int_{u=-\infty}^{u=+\infty} RECT[u] \cdot RECT\left[\frac{x}{2}-u\right] du \\ &= 2 \cdot \left(RECT[u] * RECT[u]\right)\Big|_{u=\frac{x}{2}} \\ &= 2 \cdot TRI\left[\frac{x}{2}\right] \\ &= 2 \cdot \begin{cases} 0 & \text{if} \quad x > +2 \\ 1-\frac{x}{2} & \text{if} \quad 0 < x < 2 \\ 1+\frac{x}{2} & \text{if} \quad -2 < x < 0 \\ 0 & \text{if} \quad x < -2 \end{cases} \end{aligned}$$



(b) $RECT\left[\frac{x-2}{2}\right] * RECT\left[\frac{x+1}{2}\right]$

Solution: *This is the convolution of two rectangles each of width two but one is centered at* $x_0 = +2$ *and the other is centered at* $x_1 = -1$*; the only effect of the translations is to translate the resulting triangle:*

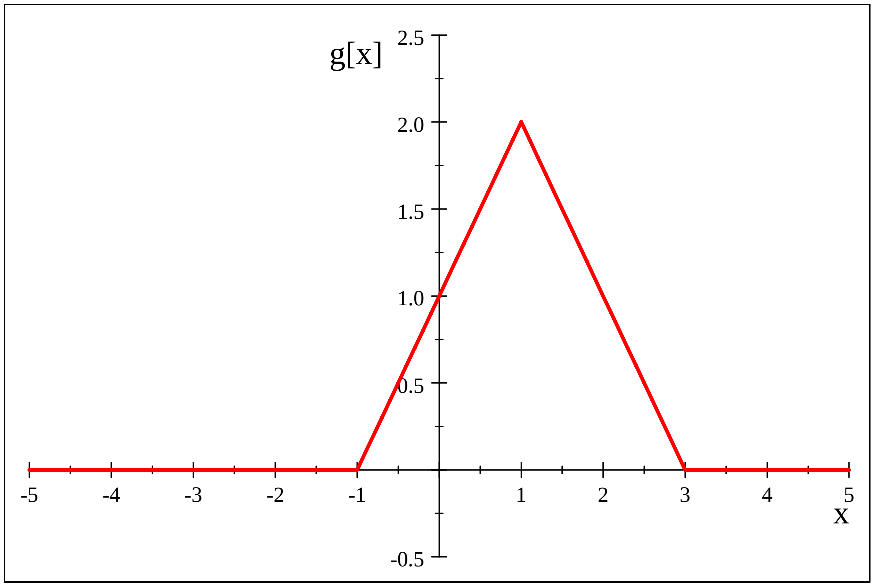
$$
\begin{aligned}
&RECT\left[\frac{x-2}{2}\right] * RECT\left[\frac{x+1}{2}\right] \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x-\alpha)+1}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x+1)-\alpha}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x+1)}{2} - \frac{\alpha}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x+1)}{2} - \frac{\alpha-2+2}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x+1)}{2} - \frac{(\alpha-2)}{2} - \frac{2}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x+1-2)}{2} - \frac{(\alpha-2)}{2}\right]\, dx \\
&= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT\left[\frac{\alpha-2}{2}\right] \cdot RECT\left[\frac{(x-1)}{2} - \frac{(\alpha-2)}{2}\right]\, dx
\end{aligned}
$$

change variable:

$$
u = \frac{\alpha-2}{2} \implies d\alpha = 2 \cdot du
$$

$$
\begin{aligned}
g[x] &= \int_{\alpha=-\infty}^{\alpha=+\infty} RECT[u] \cdot RECT\left[\frac{(x-1)}{2} - u\right]\, 2 \cdot du \\
&= 2 \cdot \int_{\alpha=-\infty}^{\alpha=+\infty} RECT[u] \cdot RECT\left[\frac{(x-1)}{2} - u\right]\, du \\
&= 2 \cdot TRI\left[\frac{x-1}{2}\right]
\end{aligned}
$$

$$
= 2 \cdot \begin{cases} 0 & \text{if} \quad x > +3 \\ 1 - \frac{x-1}{2} & \text{if} \quad 1 < x < 3 \\ 1 + \frac{x-1}{2} & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad x < -1 \end{cases}
$$


g[x]
2.5
2.0
1.5
1.0
0.5
-0.5
-5
-4
-3
-2
-1
1
2
3
4
5
x

5. Evaluate the discrete Fourier transform of the following four-sample arrays of data indexed from $n = 0$ to $N = 3$:

(a) $f[0] = f[1] = f[2] = f[3] = 1$
Solution: *This is the "constant vector" with length* $|\underline{\mathbf{x}}| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = +2$. *The easy way to evaluate the DFT is to multiply this vector by the transformation matrix* $\underline{\mathbf{D}}^{-1}$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $f[0] = f[2] = +1, \; f[1] = f[3] = -1$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

(c) $f[0] = +1, \; f[1] = 0, \; f[2] = -1, \; f[3] = 0$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

6. If the continuous Fourier transform of the arbitrary function $f[x]$ is $F[\xi]$, derive (i.e., do not just "write down") the expression for the Fourier transform of $f\left[x-\frac{3}{2}\right]$ and express it in the two ways, i.e., as real and imaginary parts and as magnitude with phase.

First, specify the forms of the real and imaginary parts of the Fourier transform:

$$\int_{x=-\infty}^{x=+\infty} f[x] \cdot (\exp[-i \cdot 2\pi\xi x])\ dx \equiv F[\xi] = \operatorname{Re}\{F[\xi]\} + i \cdot \operatorname{Im}\{F[\xi]\}$$

Now apply the shift theorem (or prove it!)

$$\int_{x=-\infty}^{x=+\infty} f\left[x-\frac{3}{2}\right] \cdot (\exp[-i \cdot 2\pi\xi x])\ dx \Longrightarrow u \equiv x - \frac{3}{2} \Longrightarrow x = u + \frac{3}{2}$$

$$\begin{aligned}
& \int_{x=-\infty}^{x=+\infty} f\left[x-\frac{3}{2}\right] \cdot (\exp[-i \cdot 2\pi\xi x])\ dx \\
&= \int_{x=-\infty}^{x=+\infty} f[u] \cdot \left(\exp\left[-i \cdot 2\pi\xi \cdot \left(u+\frac{3}{2}\right)\right]\right)\ dx \\
&= \int_{x=-\infty}^{x=+\infty} f[u] \cdot (\exp[-i \cdot 2\pi\xi \cdot u]) \cdot \left(\exp\left[-i \cdot 2\pi\xi \cdot \frac{3}{2}\right]\right)\ dx \\
&= \left(\exp\left[-i \cdot 2\pi\xi \cdot \frac{3}{2}\right]\right) \cdot \int_{x=-\infty}^{x=+\infty} f[u] \cdot (\exp[-i \cdot 2\pi\xi \cdot u])\ dx \\
\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\} &= F[\xi] \cdot \left(\exp\left[-i \cdot 2\pi\xi \cdot \frac{3}{2}\right]\right) \\
&= F[\xi] \cdot \left(\cos\left[2\pi\xi \cdot \frac{3}{2}\right] - i \cdot \sin\left[2\pi\xi \cdot \frac{3}{2}\right]\right) \\
&= (\operatorname{Re}\{F[\xi]\} + i \cdot \operatorname{Im}\{F[\xi]\}) \cdot \left(\cos\left[2\pi\xi \cdot \frac{3}{2}\right] - i \cdot \sin\left[2\pi\xi \cdot \frac{3}{2}\right]\right)
\end{aligned}$$

Real and imaginary parts

$$\begin{aligned}
\operatorname{Re}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\} &= \operatorname{Re}\{F[\xi]\} \cdot \cos\left[2\pi\xi \cdot \frac{3}{2}\right] + \operatorname{Im}\{F[\xi]\} \cdot \sin\left[2\pi\xi \cdot \frac{3}{2}\right] \\
\operatorname{Im}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\} &= \operatorname{Im}\{F[\xi]\} \cdot \cos\left[2\pi\xi \cdot \frac{3}{2}\right] - \operatorname{Re}\{F[\xi]\} \cdot \sin\left[2\pi\xi \cdot \frac{3}{2}\right]
\end{aligned}$$

Magnitude:

$$\begin{aligned}
\left|\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right| &= \sqrt{\left(\operatorname{Re}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\}\right)^2 + \left(\operatorname{Im}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\}\right)^2} \\
\left(\operatorname{Re}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\}\right)^2 &= (\operatorname{Re}\{F[\xi]\})^2 \cdot \cos^2\left[2\pi\xi \cdot \frac{3}{2}\right] + (\operatorname{Im}\{F[\xi]\})^2 \cdot \sin^2\left[2\pi\xi \cdot \frac{3}{2}\right] \\
\left(\operatorname{Im}\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\}\right)^2 &= (\operatorname{Im}\{F[\xi]\})^2 \cdot \cos^2\left[2\pi\xi \cdot \frac{3}{2}\right] + (-\operatorname{Re}\{F[\xi]\})^2 \cdot \sin^2\left[2\pi\xi \cdot \frac{3}{2}\right]
\end{aligned}$$

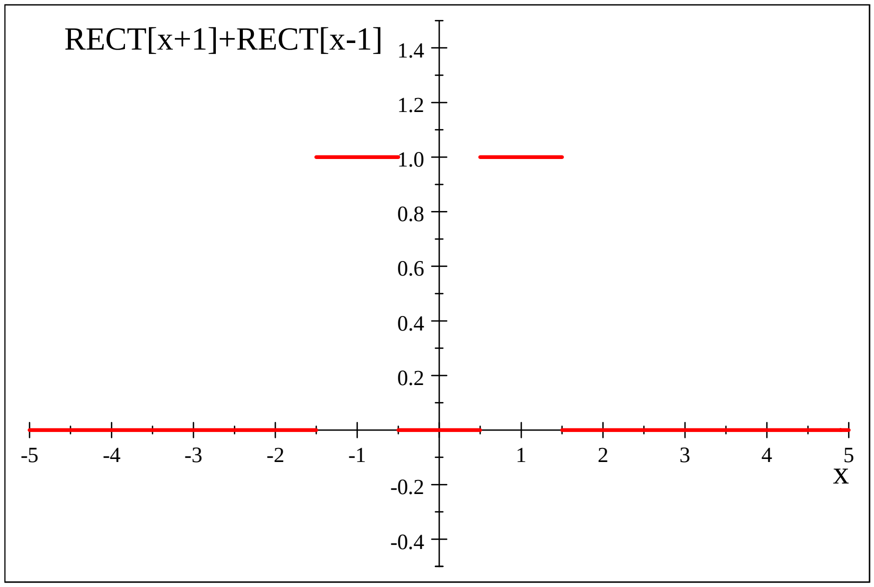
$$\left|\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right| = \left|\mathcal{F}\left\{f\left[x\right]\right\}\right| = \left|\mathcal{F}\left\{F\left[\xi\right]\right\}\right|$$

Phase:

$$\Phi\left\{\mathcal{F}\left\{f\left[x-\frac{3}{2}\right]\right\}\right\} = \tan^{-1}\left[\frac{\mathrm{Im}\left\{F\left[\xi\right]\right\}\cdot\cos\left[2\pi\xi\cdot\frac{3}{2}\right] - \mathrm{Re}\left\{F\left[\xi\right]\right\}\cdot\sin\left[2\pi\xi\cdot\frac{3}{2}\right]}{\mathrm{Re}\left\{F\left[\xi\right]\right\}\cdot\cos\left[2\pi\xi\cdot\frac{3}{2}\right] + \mathrm{Im}\left\{F\left[\xi\right]\right\}\cdot\sin\left[2\pi\xi\cdot\frac{3}{2}\right]}\right]$$

7. For the following functions, sketch them and evaluate their Fourier transforms by direct integraion:

(a) $RECT\left[x+1\right]+RECT\left[x-1\right]$



$$F\left[\xi\right]=\int_{x=-\infty}^{x=+\infty}\left(RECT\left[x+1\right]+RECT\left[x-1\right]\right)\cdot\left(\exp\left[-i\cdot 2\pi\cdot\xi x\right]\right)\,dx$$
$$=\int_{x=-\infty}^{x=+\infty}RECT\left[x+1\right]\cdot\left(\exp\left[-i\cdot 2\pi\cdot\xi x\right]\right)\,dx$$
$$+\int_{x=-\infty}^{x=+\infty}RECT\left[x-1\right]\cdot\left(\exp\left[-i\cdot 2\pi\cdot\xi x\right]\right)\,dx$$

$$
\begin{aligned}
& \int_{x=-\infty}^{x=+\infty} RECT\,[x+1] \cdot (\exp\,[-i \cdot 2\pi \cdot \xi x])\;\; dx \\
= & \int_{x=-\frac{3}{2}}^{x=-\frac{1}{2}} 1 \cdot (\exp\,[-i \cdot 2\pi \cdot \xi x])\;\; dx \\
& = \left. \frac{\exp\,[-i \cdot 2\pi \cdot \xi x]}{-i \cdot 2\pi \cdot \xi} \right|_{x=-\frac{3}{2}}^{x=-\frac{1}{2}} \\
= & \frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot -\frac{1}{2}\right]}{-i \cdot 2\pi \cdot \xi} - \frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot -\frac{3}{2}\right]}{-i \cdot 2\pi \cdot \xi} \\
= & \frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot \left(-\frac{1}{2} - 1 + 1\right)\right]}{-i \cdot 2\pi \cdot \xi} - \frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot \left(-\frac{3}{2} - 1 + 1\right)\right]}{-i \cdot 2\pi \cdot \xi} \\
= & \frac{\exp\,[-i \cdot 2\pi \cdot \xi \cdot -1]}{-i \cdot 2\pi \cdot \xi} \cdot \left(\frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot \left(+\frac{1}{2}\right)\right]}{-i \cdot 2\pi \cdot \xi} - \frac{\exp\left[-i \cdot 2\pi \cdot \xi \cdot \left(-\frac{1}{2}\right)\right]}{-i \cdot 2\pi \cdot \xi} \right) \\
= & \exp\,[-i \cdot 2\pi \cdot \xi \cdot -1] \cdot \frac{\sin\left[2\pi \cdot \xi \cdot \left(+\frac{1}{2}\right)\right]}{\pi\xi} \\
= & \exp\,[-i \cdot 2\pi \cdot \xi \cdot -1] \cdot SINC\,[\xi] \\
= & \exp\,[+i \cdot 2\pi \cdot \xi] \cdot SINC\,[\xi]
\end{aligned}
$$

By the same method:

$$
\int_{x=-\infty}^{x=+\infty} RECT\,[x-1] \cdot (\exp\,[-i \cdot 2\pi \cdot \xi x])\;\; dx = \exp\,[-i \cdot 2\pi \cdot \xi] \cdot SINC\,[\xi]
$$

so the output is:

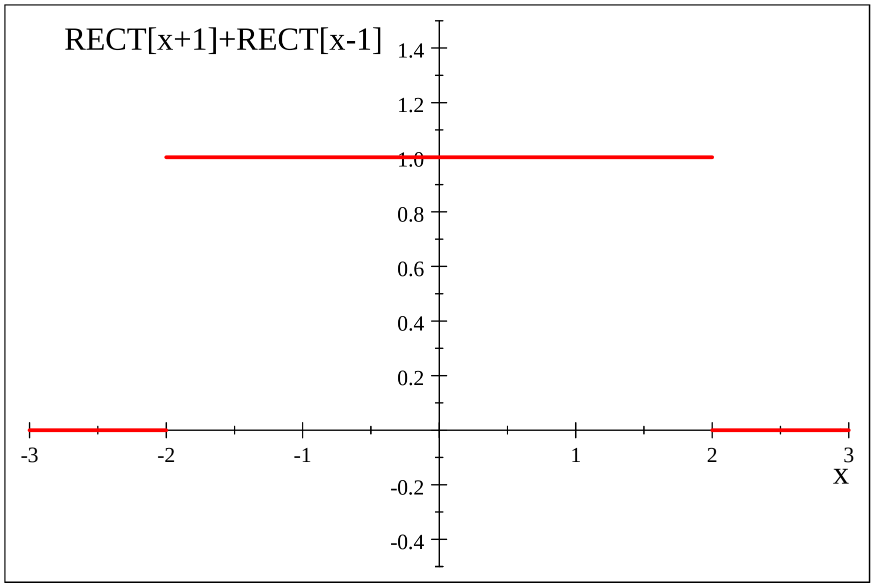
$$
\begin{aligned}
g\,[x] &= (\exp\,[+i \cdot 2\pi \cdot \xi] + \exp\,[-i \cdot 2\pi \cdot \xi]) \cdot SINC\,[\xi] \\
&= 2 \cdot \cos\,[2\pi\xi] \cdot SINC\,[\xi]
\end{aligned}
$$

From the sketch, you could also write the function as the difference of two rectangles:

$$
\begin{aligned}
RECT\,[x+1] + RECT\,[x-1] &= RECT\left[\frac{x}{3}\right] - RECT\left[\frac{x}{1}\right] \\
F\,[\xi] &= 3 \cdot SINC\left[\frac{\xi}{\left(\frac{1}{3}\right)}\right] - SINC\,[\xi]
\end{aligned}
$$

(b) $RECT\left[\frac{x+1}{2}\right] + RECT\left[\frac{x-1}{2}\right]$

Solution: *note that these two rectangles just touch at $x = 0$, so the result is a single rectangle $RECT\left[\frac{x}{6}\right]$*



We can just use the scaling theorem:

$$\mathcal{F}\left\{RECT\left[\frac{x}{4}\right]\right\} = 4 \cdot SINC\left[\frac{x}{\left(\frac{1}{4}\right)}\right]$$

or we can evaluate the transforms of the two

$$\begin{aligned} g[x] &= 2 \cdot 2 \cdot \cos[2\pi \cdot 2\xi] \cdot SINC[2\xi] \\ &= 4 \cdot \cos[4\pi\xi] \cdot SINC\left[\frac{\xi}{\left(\frac{1}{2}\right)}\right] \end{aligned}$$

so these two expressions are equivalent:

$$4 \cdot SINC\left[\frac{x}{\left(\frac{1}{4}\right)}\right] = 4 \cdot \cos[4\pi\xi] \cdot SINC\left[\frac{\xi}{\left(\frac{1}{2}\right)}\right]$$

of course, you didn't have to make the last observation.

Hint: you may sum the transforms of the component functions.

8. We defined Fourier analysis of the continuous function $f[x]$ to be

$$\int_{x=-\infty}^{x=+\infty} f[x] \cdot (\exp[+i \cdot 2\pi\xi x])^* \; dx \equiv F[\xi]$$

(a) Write down the corresponding expression for Fourier synthesis of the function $F[\xi]$

$$f[x] = \int_{\xi=-\infty}^{\xi=+\infty} F[\xi] \cdot \exp[+i \cdot 2\pi\xi x] \; d\xi$$

(b) Evaluate the expression you wrote down in part (a) for $F[\xi] = RECT[\xi]$ to find the corresponding space-domain function $f[x]$

$$\begin{aligned}
\int_{\xi=-\infty}^{\xi=+\infty} RECT[\xi] \cdot \exp[+i \cdot 2\pi\xi x] \; d\xi &= \int_{\xi=-\frac{1}{2}}^{\xi=+\frac{1}{2}} 1 \cdot \exp[+i \cdot 2\pi\xi x] \; d\xi \\
&= \left. \frac{\exp[+i \cdot 2\pi\xi x]}{+i \cdot 2\pi x} \right|_{\xi=-\frac{1}{2}}^{\xi=\frac{1}{2}} \\
&= \frac{\exp\left[+i \cdot 2\pi \cdot \frac{1}{2} \cdot x\right]}{+i \cdot 2\pi x} - \frac{\exp\left[+i \cdot 2\pi \cdot \left(-\frac{1}{2}\right) \cdot x\right]}{+i \cdot 2\pi x} \\
&= \frac{\sin[\pi x]}{\pi x} \equiv SINC[x]
\end{aligned}$$

$$\boxed{\mathcal{F}^{-1}\{RECT[\xi]\} = SINC[x] \implies \mathcal{F}\{SINC[x]\} = RECT[\xi]}$$