Notes from Today's Class (Th, 2015-02-19)

4×4 general circulant matrix

$$\underline{\mathbf{A}} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{bmatrix}$$

Claim: The four vectors representing the four different frequency oscillations are eigenvectors of any such matrix:

Constant vector of unit length:

$$\hat{\underline{\mathbf{x}}}_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|\hat{\underline{\mathbf{x}}}_0| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$$

$$\hat{\underline{\mathbf{x}}}_1 = \frac{1}{2} \begin{bmatrix} +1 \\ +i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix}$$

$$|\hat{\underline{\mathbf{x}}}_1| = \sqrt{\left|\frac{1}{2}\right|^2 + \left|\frac{i}{2}\right|^2 + \left|-\frac{1}{2}\right|^2 + \left|-\frac{i}{2}\right|^2} = 1$$

$$\hat{\underline{\mathbf{x}}}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$|\hat{\underline{\mathbf{x}}}_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$$

$$\hat{\underline{\mathbf{x}}}_3 = \frac{1}{2} \begin{bmatrix} +1 \\ -i \\ -1 \\ +i \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix}$$

$$|\hat{\underline{\mathbf{x}}}_3| = \sqrt{\left|\frac{1}{2}\right|^2 + \left|-\frac{i}{2}\right|^2 + \left|-\frac{1}{2}\right|^2 + \left|+\frac{i}{2}\right|^2} = 1$$

Test by applying these four vectors to three different examples of circulant matrices:

Translation Matrix:

$$\underline{\mathbf{A}}_1 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix}$$

which means that the elements of an input vector are shifted one pixel "to the left. Apply the four vectors to this matrix one at a time:

$$\underline{\mathbf{A}}_1 \hat{\underline{\mathbf{x}}}_0 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Longrightarrow \lambda_0 = 1 = 1 + 0i = 1 \exp\left[+i \cdot 0\right]$$

$$\underline{\mathbf{A}}_1 \hat{\underline{\mathbf{x}}}_1 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}i \\ -\frac{1}{2} \\ -\frac{1}{2}i \\ \frac{1}{2} \end{bmatrix} = i \cdot \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = i \cdot \hat{\underline{\mathbf{x}}}_1$$

$$\Longrightarrow \lambda_1 = +i = 0 + 1i = 1 \exp\left[+i \cdot \frac{\pi}{2}\right]$$

$$\underline{\mathbf{A}}_1 \hat{\underline{\mathbf{x}}}_2 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = -1 \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\Longrightarrow \lambda_2 = -1 = -1 + 0i = 1 \exp\left[+i \cdot (\pm\pi)\right]$$

$$\underline{\mathbf{A}}_1 \hat{\underline{\mathbf{x}}}_3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i \\ -\frac{1}{2} \\ \frac{1}{2}i \\ \frac{1}{2} \end{bmatrix} = -i \cdot \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = -i \cdot \hat{\underline{\mathbf{x}}}_3$$

$$\Longrightarrow \lambda_3 = -i = 0 - 1i = 1 \cdot \exp\left[+i \cdot \frac{3\pi}{2}\right] = 1 \cdot \exp\left[+i \cdot \frac{-\pi}{2}\right]$$

So for the translation matrix $\underline{\mathbf{A}}_1$, the four eigenvalues corresponding to the four unit-length eigenvectors are:

$$\underline{\mathbf{A}}_1 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \lambda_0 = +1, \lambda_1 = +i, \lambda_2 = -1, \lambda_3 = -i$$

All four vectors are passed with unchanged lengths

Differencing Matrix (discrete derivative):

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix}$$

Apply the four vectors to this matrix one at a time:

$$\underline{\mathbf{A}}_2 \hat{\underline{\mathbf{x}}}_0 = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} = 0 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Longrightarrow \lambda_0 = 0 = 0 + 0i = 0 \cdot \exp\left[+i \cdot 0\right]$$

$$\underline{\mathbf{A}}_2 \hat{\underline{\mathbf{x}}}_1 = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i \end{bmatrix} = (-1+i) \cdot \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = (-1+i) \cdot \hat{\underline{\mathbf{x}}}_1$$

$$\Longrightarrow \lambda_1 = -1 + i = -1 + 1i = \sqrt{2} \exp\left[+i \cdot \frac{3\pi}{4}\right]$$

$$\underline{\mathbf{A}}_2 \hat{\underline{\mathbf{x}}}_2 = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = -2 \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \lambda_2 = -2 = -2 + 0i = 2 \cdot \exp\left[+i \cdot (\pm\pi)\right]$$

$$\underline{\mathbf{A}}_2 \hat{\underline{\mathbf{x}}}_3 = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ -\frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i \end{bmatrix} = (-1-i) \cdot \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = (-1-i) \cdot \hat{\underline{\mathbf{x}}}_3$$

$$\Longrightarrow \lambda_3 = -1 - i = \sqrt{2} \cdot \exp\left[+i \cdot \frac{-3\pi}{4}\right]$$

So for the differencing matrix $\underline{\mathbf{A}}_2$, the four eigenvalues corresponding to the four unit-length eigenvectors are:

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \lambda_0 = 0, \lambda_1 = \sqrt{2} \exp\left[+i \cdot \frac{3\pi}{4}\right], \lambda_2 = -2, \lambda_3 = \sqrt{2} \exp\left[-i \cdot \frac{3\pi}{4}\right]$$

Note that $\lambda_3 = \lambda_1^*$. The constant vector is blocked and the lengths of the outputs of the other three vectors are increased.

Two-Pixel Averager:

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix}$$

Apply the four vectors to this matrix one at a time:

$$\underline{\mathbf{A}}_3 \hat{\underline{\mathbf{x}}}_0 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Longrightarrow \quad \lambda_0 = 1 = 1 + 0i = 1 \cdot \exp\left[+i \cdot 0\right]$$

$$\underline{\mathbf{A}}_3 \hat{\underline{\mathbf{x}}}_1 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{1}{4}i \\ -\frac{1}{4} + \frac{1}{4}i \\ -\frac{1}{4} - \frac{1}{4}i \\ \frac{1}{4} - \frac{1}{4}i \end{bmatrix} = \frac{(+1+i)}{2} \cdot \begin{bmatrix} +\frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = \frac{(+1+i)}{2} \cdot \hat{\underline{\mathbf{x}}}_1$$

$$\Longrightarrow \quad \lambda_1 = \frac{1+i}{2} = \frac{1}{\sqrt{2}} \exp\left[+i \cdot \frac{\pi}{4}\right]$$

$$\underline{\mathbf{A}}_3 \hat{\underline{\mathbf{x}}}_2 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\mathbf{0}} = 0 \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \quad \lambda_2 = 0 = 0 + 0i = 0 \cdot \exp\left[+i \cdot 0\right]$$

:

$$\underline{\mathbf{A}}_3 \hat{\underline{\mathbf{x}}}_3 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix} \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}i \\ -\frac{1}{4} - \frac{1}{4}i \\ -\frac{1}{4} + \frac{1}{4}i \\ \frac{1}{4} + \frac{1}{4}i \end{bmatrix} = \frac{1-i}{2} \cdot \begin{bmatrix} +\frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = \frac{1-i}{2} \cdot \hat{\underline{\mathbf{x}}}_3$$

$$\Longrightarrow \quad \lambda_3 = \frac{+1-i}{2} = \frac{1}{\sqrt{2}} \cdot \exp\left[+i \cdot \frac{-\pi}{4}\right]$$

So for the two-pixel averaging matrix $\underline{\mathbf{A}}_3$, the four eigenvalues corresponding to the four unit-length eigenvectors are:

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{bmatrix} \Longrightarrow \lambda_0 = +1, \lambda_1 = \frac{1}{\sqrt{2}} \exp\left[+i \cdot \frac{\pi}{4}\right], \lambda_2 = 0, \lambda_3 = \frac{1}{\sqrt{2}} \exp\left[-i \cdot \frac{\pi}{4}\right]$$

Note again that $\lambda_3 = \lambda_1^*$. The vector with frequency $\frac{1}{2} \frac{\text{cycle}}{\text{sample}}$ is blocked, the constant vector is passed unchanged, and the lengths of the outputs of the other two vectors are decreased because the average is closer to the global mean value than the individual pixels.

The translation matrix $\underline{\mathbf{A}}_1$ is invertible, while the differencing and averaging matrices each block one eigenvector and therefore are NOT invertible.