

1 **APPENDIX: Derivation of Eq.(11-12) and Eq.(11-14)**

We wish to evaluate an expression for the integral of the even powers of the cosine over one cycle of oscillation. Start with the definition of the gamma function from eq.(6-37):

\[
\Gamma [n] = \int_{-\infty}^{+\infty} \text{STEP}[z] \ e^{-z} z^{n-1} \, dz
\]

Substitute \(x^2 = z\), so that \(dz = 2x \cdot dx\). The limits of integration remain unchanged:

\[
\Gamma [n] = \int_{0}^{+\infty} e^{-x^2} x^{2(n-1)} 2x \, dx
\]

Multiply by an equivalent formulation for \(\Gamma [m]\) with a different integration variable:

\[
\Gamma [n] \cdot \Gamma [m] = 2 \int_{0}^{+\infty} e^{-x^2} x^{2n-1} \, dx \cdot 2 \int_{0}^{+\infty} e^{-y^2} y^{2m-1} \, dy
\]

This double integral may be recast into polar coordinates by substituting \(x = r \cos \phi\) and \(y = r \sin \phi\), so that \(r^2 = x^2 + y^2\) and \(dx \, dy = r \, dr \, d\phi\). Because the domains of \(x\) and \(y\) are restricted to the first quadrant, the domain of the azimuth integral is \(0 \leq \phi < +\frac{\pi}{2}\).

\[
\Gamma [n] \cdot \Gamma [m] = 4 \int_{r=0}^{+\pi} \int_{\phi=0}^{+\pi} e^{-r^2} r^{2n-1} \cos^2 \phi^{2n-1} \sin^{2m-1} \phi^{2m-1} \, r \, dr \, d\phi
\]

where the integral over \(r\) has been identified as \(\frac{1}{2} \Gamma [n + m]\) by substitution of \(u = r^2\). After dividing both sides by \(2 \Gamma [n + m]\), we obtain:

\[
\frac{1}{2} \Gamma [n] \cdot \Gamma [m] = \int_{\phi=0}^{+\pi} (\cos \phi)^{2n-1} (\sin \phi)^{2m-1} \, d\phi
\]

To obtain the required integral, substitute \(m = \frac{1}{2}\) and \(n = \ell + \frac{1}{2}\), where \(\ell\) is a nonnegative integer, and use the symmetry of the cosine:

\[
\frac{1}{2} \Gamma \left[\ell + \frac{1}{2}\right] \cdot \Gamma \left[\frac{1}{2}\right] = \int_{\phi=0}^{+\pi} (\cos \phi)^{2\ell} \, d\phi
\]

\[
= \frac{1}{4} \int_{-\pi}^{+\pi} (\cos \phi)^{2\ell} \, d\phi
\]
Therefore, the integral of interest is:

\[
\int_{-\pi}^{+\pi} (\cos[\phi])^{2\ell} d\phi = 2 \frac{\Gamma\left[\frac{\ell+1}{2}\right] \cdot \Gamma\left[\frac{1}{2}\right]}{\Gamma[\ell+1]}
\]

\[
= 2 \frac{\Gamma\left[\frac{2\ell+1}{2}\right] \cdot \Gamma\left[\frac{1}{2}\right]}{\ell!}
\]

(A11-7)

where eq.(6-43) and the fact that \(\ell\) is an integer was used to identify that \(\Gamma[\ell+1] = \ell!\).

It may be useful to further simplify this relation by evaluating the gamma functions in the numerator. We know from eq.(6-49) that:

\[
\Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}
\]

(A11-8)

and from eq.(6-42):

\[
\Gamma\left[\ell + \frac{1}{2}\right] = \left(\ell - \frac{1}{2}\right) \Gamma\left[\ell - \frac{1}{2}\right]
\]

\[
= \left(\ell - \frac{1}{2}\right) \left(\ell - \frac{3}{2}\right) \Gamma\left[\ell - \frac{3}{2}\right]
\]

\[
= \left(\ell - \frac{1}{2}\right) \left(\ell - \frac{3}{2}\right) \left(\ell - \frac{5}{2}\right) \cdots \left(\frac{1}{2}\right) \Gamma\left[\frac{1}{2}\right]
\]

\[
= \left(\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \left(\ell - \frac{1}{2}\right)\right) \sqrt{\pi}
\]

\[
= \frac{1 \cdot 3 \cdot 5 \cdots (2\ell - 1)}{2^\ell} \cdot \sqrt{\pi}
\]

\[
= \frac{(2\ell - 1)!!}{2^\ell} \cdot \sqrt{\pi}
\]

(A11-9)

where the notation \((2\ell - 1)!!\) for the odd factorial (the product of all odd integers less than or equal to \(2\ell - 1\)) has been used as defined in eq.(11-13). Substitution of eq.(A10-9) into eq.(A10-7) yields:

\[
\int_{\phi=-\pi}^{+\pi} (\cos[\phi])^{2\ell} d\phi = 2 \frac{\Gamma\left[\frac{2\ell+1}{2}\right] \cdot \Gamma\left[\frac{1}{2}\right]}{\Gamma[\ell+1]}
\]

\[
= 2 \frac{(2\ell - 1)!!}{2^\ell} \cdot \ell!
\]

\[
= 2\pi \frac{(2\ell - 1)!!}{2^\ell} \cdot \ell!
\]

\[
= 2\pi \frac{(2\ell - 1)!!}{2^\ell \cdot (2\ell - 1) \cdot (2\ell - 2) \cdots \cdot 1}
\]

\[
= 2\pi \frac{(2\ell - 1)!!}{2^\ell \cdot (2\ell - 2) \cdot (2\ell - 4) \cdots \cdot 2 \cdot 1}
\]

\[
= \frac{(2\ell - 1)!!}{(2\ell)!!}
\]

(A11-10)

where the notation \((2\ell)!!\) denotes the even factorial product that also was defined in eq.(11-13).
The results of eq.(A11-7) or eq.(A11-10) are identical and either may be substituted into eq.(A10-7) to obtain:

\[
\int_{\phi=-\pi}^{+\pi} (\cos [\phi])^{2\ell} d\phi = \begin{cases} 2\pi \cdot 1 = 2\pi & \text{for } \ell = 0 \\ 2\pi \cdot \frac{1}{2} = \pi & \text{for } \ell = 1 \\ 2\pi \cdot \frac{3}{8} = \frac{3\pi}{4} & \text{for } \ell = 2 \\ 2\pi \cdot \frac{15}{48} = 2\pi \cdot \frac{5}{16} = \frac{5\pi}{8} & \text{for } \ell = 3 \\ 2\pi \cdot \frac{35}{128} = \frac{35\pi}{64} & \text{for } \ell = 4 \end{cases}
\] (A11-11a)

These terms are substituted into eq.(11-14) to obtain the desired power series, which is identical to the expansion for \(J_0[2\pi r\rho]\) that was derived in eq.(B6.7):

\[
\int_{-\pi}^{+\pi} e^{-2\pi r\rho \cos[\psi]} d\psi = \int_{-\pi}^{+\pi} \cos [2\pi r\rho \cdot \cos [\phi]] d\phi
\]

\[
= \int_{-\pi}^{+\pi} \left( 1 - \left(\frac{2\pi r\rho}{2}\right)^2 \cos^2 [\phi] + \left(\frac{2\pi r\rho}{4}\right)^4 \cos^4 [\phi] - \left(\frac{2\pi r\rho}{6}\right)^6 \cos^6 [\phi] + \cdots \right) d\phi
\]

\[
= 2\pi - \left(\frac{(2\pi r\rho)^2}{2}\right) \cdot \pi + \left(\frac{(2\pi r\rho)^4}{4}\right) \cdot \frac{3\pi}{24} - \left(\frac{(2\pi r\rho)^6}{6}\right) \cdot \frac{5\pi}{720} + \left(\frac{(2\pi r\rho)^8}{8}\right) \cdot \frac{35\pi}{40,320} - \cdots
\]

\[
= 2\pi \left( 1 - \left(\frac{(2\pi r\rho)^2}{4}\right) + \left(\frac{(2\pi r\rho)^4}{64}\right) - \left(\frac{(2\pi r\rho)^6}{147,456}\right) + \cdots \right)
\]

\[
= 2\pi J_0[2\pi r\rho] \quad \text{(A11-12)}
\]