Homework 2

1. Consider the experiment of tossing three fair coins—a penny, a nickel and a dime. Construct a visualization of the sample space $\mathcal{U}$. Let $X =$Number of Tails. Show the set in $\mathcal{U}$ that corresponds to each value of $X$. Calculate and plot the distribution function $F_X(x)$.

2. Plot the function $D(x) = F_X(x) - F_X(x-1)$ for the distribution function in Exercise 1. Interpret the result in terms of probabilities.

3. Consider the function $d_\Delta(x) = (F_X(x) - F_X(x-\Delta))/\Delta$. Plot this function using the distribution function of problem 1 for $\Delta = 0.5$ and $\Delta = 0.1$. Give an interpretation of the results. What happens as $\Delta \to 0$?

4. Consider the experiment of rolling a fair die as described in Example ??

   List the following events and compute their probabilities
   
   (a) $X_1 + X_2 = 1$
   (b) $X_1 = X_2$
   (c) $X_1 \neq X_2$

5. Consider the experiment of rolling a pair of fair dice. Let $X_1$ be the number showing on the first face and $X_2$ be the number showing on the second. Let $X_3 = X_1 + X_2$. Compute the following probabilities.

   (a) $P[X_3 = 5]$
   (b) $P[X_3 = 5 \mid X_1 = 2]$ by using the definition of conditional probability.
   (c) $P[X_3 = k]$ for $k = 1, 2, \ldots, 12$.
   (d) $P[X_3 = k \mid X_1 = 2]$ for $k = 1, 2, \ldots, 12$.

6. Show that the two-dimensional probability density function is non-negative by making use of the properties of the 2-D probability distribution function.

7. Let $\mathbf{W} = (U, V)$ be a two-dimensional random vector with the joint probability density function

   \[ f_{U,V}(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left[ -\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)} \right] \]

   where $|\rho| \leq 1$. We will learn that this is the joint pdf of correlated normal random variables with correlation coefficient $\rho$. Show that

   \[ f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \]
and

\[ f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \]

for any value of \( \rho \), and determine the value of \( \rho \) for which the random variables are statistically independent.

8. Suppose that \( U \) is uniformly distributed over \([-1, 1]\) and that \( V = U^2 \). Find \( F_V(v) \) and \( f_V(v) \). Notice that the transformation is not single-valued so that care must be taken in the analysis.