Random Variables

Lecture 2

Spring Quarter, 2002
Definition

Let $E$ be an experiment whose outcomes are a sample space $\mathcal{U}$ for which the probability $P(e)$ is defined for each outcome $e \in \mathcal{U}$. A random variable is a function $X(e)$ that associates a number with each outcome $e \in \mathcal{U}$. 

\[ U \quad \begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
e_1 \\
\downarrow \\
C_1, C_2, C_3, C_4 \\
\downarrow \\
\mathbb{R}
\end{array} \]
Intervals

- Every interval on $\mathcal{R}$ corresponds to a set of outcomes in $\mathcal{U}$.
- Let $I \in \mathcal{R}$ be an interval. Then $\mathcal{A} = \{e \in \mathcal{U} : X(e) = x, x \in I\}$ is an event.
- $P(\mathcal{A})$ can be calculated, and $P(I) = P(\mathcal{A})$
- Every interval of $\mathcal{R}$ is an event whose probability can be calculated.
- For the figure below, $\mathcal{A} = \{e_2, e_4, e_5\}$ and $P(I) = P(\mathcal{A}) = P(e_2) + P(e_4) + P(e_5)$
Consider the semi-infinite interval $I_x = \{s : s \leq x\}$ be the interval to the left of $x$.

Let $X(e)$ be a random variable.

Let $A(x)$ be the event that $X \in I(x)$ Then $A(x) = \{e : X(e) \leq x\}$.

The probability $P(X \in I) = P(X \leq x) = P(A(x))$ is well defined for every $x$.

The probability $P(X \leq x)$ is a special function of $x$ called the probability distribution function.

$$F_X(x) = P(X \leq x)$$
Probability Distribution Function

\[ \lim_{x \to -\infty} F_X(x) = 0 \]
\[ \lim_{x \to \infty} F_X(x) = 1 \]
\[ P(a < x \leq b) = F_X(b) - F_X(a) \]
Discrete Distribution

A discrete distribution function has a finite number of discontinuities. The random variable has a nonzero probability only at the points of discontinuity.

The distribution function for a discrete random variable is a staircase that increases from left to right.
Continuous Distribution

Suppose that \( F_X(x) \) is continuous for all \( x \). Then

\[
\lim_{\varepsilon \to 0} F_X(x) - F_X(x - \varepsilon) = 0
\]

so that \( P(X = x) = 0 \) for all \( x \).

The derivative is well-defined where \( F_X(x) \) is continuous.

\[
\frac{dF_X(x)}{dx} = \lim_{\varepsilon \to 0} \frac{F_X(x) - F_X(x - \varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{P(x - \varepsilon < X \leq x)}{\varepsilon}
\]

The slope of the probability distribution function is equivalent to the density of probability.

\[
f_X(x) = \frac{dF_X(x)}{dx}
\]
Continuous Distribution

The distribution function (a) for a continuous random variable and (b) its probability density function. Note that the probability density function is highest where the slope of the distribution function is greatest.
Continuous Distribution

\[ f_X(x) = \frac{dF_X(x)}{dx} \]

\[ F_X(x) = \int_{-\infty}^{x} f_X(u)du \]

\[ P(a < X \leq b) = F_X(b) - F_X(a) \]

\[ = \int_{a}^{b} f_X(u)du \]

The probability \( P(a < X \leq b) \) is related to the change in height of the distribution and to the area shown in the probability density function.
Mixed Distribution

The range of a mixed distribution contains isolated points and points in a continuum. The distribution function is a smooth curve except at one or more points where there are finite steps.

\[ f_X(x) = c(x) + \sum_k P(X = x_k)\delta(x - x_k) \]

\[ c(x) = dF_X/dx \]

where \( F(x) \) is continuous.
Random Vector

Let $\mathbf{E}$ be an experiment whose outcomes are a sample space $\mathcal{U}$ for which the probability $P(e)$ is defined for each outcome $e \in \mathcal{U}$.

A random vector is a function $\mathbf{X}(e) = [X_1(e), X_2(e), \ldots, X_r(e)]$ where $X_i(e)$, $i = 1, 2, \ldots, r$ are random variables defined over the space $\mathcal{U}$. 
Joint Probability Distribution Function

\[ F_{X_1X_2}(x_1, x_2) = P(X_1 \leq x_1) \cap (X_2 \leq x_2) \]

1. \[ F_{X_1X_2}(-\infty, -\infty) = 0 \]
2. \[ F_{X_1X_2}(-\infty, x_2) = 0 \] for any \( x_2 \)
3. \[ F_{X_1X_2}(x_1, -\infty) = 0 \] for any \( x_1 \)
4. \[ F_{X_1X_2}(+\infty, +\infty) = 1 \]
5. \[ F_{X_1X_2}(+\infty, x_2) = F_{X_2}(x_2) \] for any \( x_2 \)
6. \[ F_{X_1X_2}(x_1, +\infty) = F_{X_1}(x_1) \] for any \( x_1 \)
Joint Probability Distribution Function

The probability that an experiment produces a pair \((X_1, X_2)\) that falls in a rectangular region with lower left corner \((a, c)\) and upper right corner \((b, d)\) is

\[
P[(a < X_1 \leq b) \cap (c < X_2 \leq d)] = F_{X_1X_2}(b, d) - F_{X_1X_2}(a, d) - F_{X_1X_2}(b, c) + F_{X_1X_2}(a, c)
\]
Joint Probability Density Function

\[ f_{X_1X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1X_2}(x_1, x_2)}{\partial x_1 \partial x_2} \]

\[ f_{U,V}(u, v) \geq 0 \]

\[ F_{U,V}(u, v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{U,V}(\xi, \eta) d\xi d\eta \]
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{U,V}(\xi, \eta) d\xi d\eta = 1 \]

\[ F_U(u) = \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{U,V}(\xi, \eta) d\xi d\eta \]

\[ F_V(v) = \int_{-\infty}^{\infty} \int_{-\infty}^{v} f_{U,V}(\xi, \eta) d\xi d\eta \]

\[ f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, \eta) d\eta \]

\[ f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(\xi, v) d\xi \]
Die Tossing Example

Mapping of the outcomes of the die tossing experiment onto points in a plane by a particular pair of random variables.

Each outcome maps into a pair of random variables.