Chapter 4

Traveling Waves

4.1 Introduction

To date, we have considered oscillations, i.e., periodic, often harmonic, variations of a physical characteristic of a system. The system at one time is indistinguishable from the system observed at a later time if the time difference is an integral number of temporal periods. To maintain oscillatory behavior, the energy of the oscillator must remain within the system, i.e., there can be no losses of energy. We will now extend this picture to oscillations that travel from the source and thus transport energy away. Energy must be continually added to the system to maintain the oscillation and the transported energy can do work on other systems at a distance.

Our first task is to mathematically describe a traveling harmonic wave, i.e., denote a $y[t]$ that travels through space. A harmonic oscillation $y(t) = A_0 \cos(\omega_0 t)$, can be converted into a traveling wave by making the phase a function of both $x$ and $t$ in a very particular way. Consider the general case of an oscillatory function of space and time:

$$y[z, t] = A_0 \cos[\Phi[z, t]] = A_0 \cos[z, t].$$

We want this oscillation to move through space, e.g., toward positive $z$. In other words, if a point of constant phase on the wave (e.g., a peak of the cosine created at a particular time $\tau$) is at a point $x_0$ in space at a time $t_0$, the same point of constant phase must move to $z_1 > z_0$ at time $t_1 > t_0$.

“Snapshots” of sinusoidal wave at two different times $t_0$ and $t_1 > t_0$, showing motion of the peak originally at the origin at $t_0$. The wave is traveling towards $z = +\infty$ at velocity $v_\phi$. The phase of the first wave at the origin is 0 radians, but that of the second is negative.

Since the wave at location $z_1$ and time $t_1$ has the same phase as the wave at location $z_0$ and time
t_0\), we can say that:

\[ \Phi [z_0, t_0] = \Phi [z_1, t_1] \implies \cos [z_0, t_0] = \cos [z_1, t_1] \implies y [z_0, t_0] = y [z_1, t_1]. \]

In addition, for the wave to maintain its shape, the phase \( \Phi [x, t] \) must be a linear function of \( x \) and \( t \); otherwise the wave would compress or stretch out at different locations in space or time. Therefore:

\[ \Phi [z, t] = \alpha z + \beta t \]

\[ \implies \alpha z_0 + \beta t_0 = \alpha z_1 + \beta t_1. \]

As discussed, if \( t_1 > t_0 \implies z_1 > z_0 \) (i.e., wave moves toward \( z = +\infty \)), then \( \alpha \) and \( \beta \) must have opposite algebraic signs:

\[ \Phi [z, t] = |\alpha| z - |\beta| t \]

By dimensional analysis, we know that \( |\alpha| z - |\beta| t \) has “dimensions” of angle [radians]. We have already identified \( \beta = \omega_0 \), the angular frequency of the oscillation. Similarly, if \( [z] = \text{mm} \) must have dimensions of radians/mm, i.e., \( \alpha \) tells how many radians of oscillation exist per unit length – the angular spatial frequency of the wave, commonly denoted by \( k \):

\[ y_+ [z, t] = A_0 \cos [kz - \omega_0 t] - \text{traveling harmonic wave toward } z = +\infty \]

By identical analysis, we can derive the equation for a harmonic wave moving toward \( x = -\infty \)

\[ y_- [z, t] = A_0 \cos [kz + \omega_0 t] - \text{traveling harmonic wave toward } z = -\infty \]

The waves are functions of both space and time, i.e., three dimensions \([z, y, t]\) are needed to portray them. Generally we display \( y \) either as a function of \( z \) or fixed \( t \), or as a function of \( t \) for fixed \( z \):

### 4.1.1 2-D Plot of 1-D Traveling Wave

The 1-D traveling wave is a function of two variables: the position \( z \) and the time \( t \), and so may be graphed on axes with these labels. An example is shown in the figure, where \( z \) is plotted on the horizontal axis and \( t \) on the vertical axis. In this case, the point at the origin at \( t = 0 \) has a phase of 0 radians. That point moves in the positive \( z \) direction with increasing time, and so is a wave of the form

\[ y [z, t] = \cos [k_0 z - \omega_0 t] \]

The points with the same phase of 0 radians at later times are positioned along the line shown. The velocity of this point of constant phase is \( \frac{\Delta z}{\Delta t} \), and thus is the reciprocal of the slope of this line.
4.2 Notation and Dimensions for Waves in a Medium

Trigonometric Notation:

\[ y [z,t] - y_0 = A \cos \{ \Phi [z,t] \} = A \cos (kz \pm \omega t + \phi_0) \]

Complex Notation:

\[ y [z,t] = A e^{i\Phi [z,t]} = \text{Re} \left\{ e^{i(kz \pm \omega t + \phi_0)} \right\} \]

\( y = \text{position of the characteristic of the medium, e.g., } [y] = \text{angle, voltage, ... ;} \)
\( y_0 = \text{equilibrium value of the characteristic;} \)
\( A = \text{amplitude of the wave, i.e., maximum displacement from equilibrium, } [A] = [y]; \)
\( z,t = \text{spatial and temporal coordinates, } [z] = \text{length (e.g., mm), } [t] = \text{s;} \)
\( T = \text{period of the wave, } [T] = \text{s}, T = \frac{2\pi}{\nu}; \)
\( \lambda = \text{wavelength, } [\lambda] = \text{mm} \)
\( \omega = \text{angular temporal frequency of the wave, } [\omega] = \text{radians per s}; \)
\( k = \text{angular spatial frequency of the wave, } [k] = \frac{\text{radians}}{\text{mm}}; \)
\( \nu = \text{temporal frequency of the wave, } [\nu] = \text{cycles s} = \text{Hz}, \nu = \frac{2\pi}{\lambda}; \)
\( \Phi = \text{phase angle of the wave, } [\Phi] = \text{radians, (in this case, } \Phi \text{ is linear in time and space);} \)
\( \phi_0 = \text{initial phase of the wave, i.e., phase angle } t = 0, z = 0, [\phi_0] = [\Phi] = \text{radians.} \)
\( \sigma = \text{wavenumber, } \sigma = \frac{1}{\lambda} \text{, the number of wavelengths per unit length, } [\sigma] = \text{mm}^{-1}. \)

Relations between the phase and the temporal frequencies

\[ \omega = -\frac{\partial \Phi}{\partial t} \]
\[ \nu = \frac{\omega}{2\pi} = -\frac{1}{2\pi} \frac{\partial \Phi}{\partial t} \]

4.3 Velocity of Traveling Waves

The phase velocity \( v_\phi \) of a wave is the speed of travel of a point of constant phase. A definition for phase velocity can be derived by dimensional analysis: \( [v_\phi] = \text{mm per s; } [\omega] = \text{radians per s; } [k] = \text{radians per mm} \):

\[ \Rightarrow \left[ \frac{\omega}{k} \right] = \frac{\text{radians per second}}{\text{radians per mm}} = \frac{\text{radian-mm}}{\text{radian-s}} = \frac{\text{mm}}{s} \]

Slightly more rigorously, we can find the phase velocity of a wave by taking derivatives of the equation for the wave:

\[ y [z,t] = A \cos [kz - \omega t + \phi_0], \]
\[ \frac{\partial y}{\partial t} = -(\omega)A \sin [kz - \omega t + \phi_0] = +A\omega \cdot \sin [kz - \omega t + \phi_0], \]
\[ \frac{\partial y}{\partial z} = -(k)A \sin [kz - \omega t + \phi_0] = -Ak \cdot \sin [kz - \omega t + \phi_0] \]
\[ v_\phi = \left| \frac{\partial z}{\partial t} \right| = \left( \frac{\partial y}{\partial t} \right) \left( \frac{\partial \phi}{\partial \phi} \right) = \left| -\frac{\omega}{k} \right| = \frac{\omega}{k}, \]

or by considering the point of constant phase \( b \) radians:

\[ kz - \omega t = b \Rightarrow z = \left( \frac{b}{k} \right) + \frac{\omega}{kt} = b' + \left( \frac{\omega}{k} \right) t \quad b' \equiv \frac{b}{k} \text{ is a new constant} \]

Consider the positions \( z_1 \) and \( z_2 \) of the same point of constant phase at different times \( t_1 \) and \( t_2 \):
\[
\begin{align*}
  z_1 &= b' + \left( \frac{\omega}{k} \right) t_1 \\
  z_2 &= b' + \left( \frac{\omega}{k} \right) t_2 \\
  \Rightarrow z_1 - z_2 &= \Delta z = \left( \frac{\omega}{k} \right) (t_1 - t_2) = \left( \frac{\omega}{k} \right) \Delta t \\
  v_\phi &= \frac{\Delta x}{\Delta t} = \frac{\omega}{k} = v_\phi.
\end{align*}
\]

### 4.4 Superposition of Traveling Waves

Consider the superposition of two traveling waves with the same amplitude, different phase velocities, and different frequencies:

\[
\begin{align*}
  y_1[z, t] &= A \cos \left( k_1 z - \omega_1 t \right) \\
  y_2[z, t] &= A \cos \left( k_2 z - \omega_2 t \right).
\end{align*}
\]

We can use the same derivation developed for oscillations by defining a new frequency for both:

\[
\begin{align*}
  \Omega_1 &= \frac{k_1}{t} - \omega_1 \\
  \Omega_2 &= \frac{k_2}{t} - \omega_2.
\end{align*}
\]

\[
y[z, t] = y_1[z, t] + y_2[z, t] = A \cos \left( k_1 z - \omega_1 t \right) + \cos \left( k_2 z - \omega_2 t \right).
\]

just as before. By evaluating the sum and difference frequencies, we obtain:

\[
\begin{align*}
  \left( \frac{\Omega_1 + \Omega_2}{2} \right) t &= \left( \frac{k_1}{t} - \frac{\omega_1}{t} + \frac{k_2}{t} - \frac{\omega_2}{t} \right) t \equiv \frac{k_1 + k_2}{2} z - \frac{\omega_1 + \omega_2}{2} t \\
  \text{where } k_{\text{avg}} &= \frac{k_1 + k_2}{2}, \omega_{\text{avg}} = \frac{\omega_1 + \omega_2}{2}
\end{align*}
\]

\[
\begin{align*}
  \left( \frac{\Omega_1 - \Omega_2}{2} \right) t &= \left( \frac{k_1}{t} - \omega_1 - \frac{k_2}{t} + \omega_2 \right) t \equiv \frac{k_1 - k_2}{2} z - \frac{\omega_1 - \omega_2}{2} t \\
  \text{where } k_{\text{mod}} &= \frac{k_1 - k_2}{2}, \omega_{\text{mod}} = \frac{\omega_1 - \omega_2}{2}
\end{align*}
\]
4.5 Standing Waves

Consider the superposition of two waves with the same amplitude $A_0$, temporal frequency $\nu_0$, and wavelength $\lambda_0$, but that are traveling in opposite directions:

$$f_1 (z, t) + f_2 (z, t) = A_0 \cos \left[ k_0 z - \omega_0 t \right] + A_0 \cos \left[ k_0 z + \omega_0 t \right]$$

This is the product of a spatial wave with wavelength $\lambda_0$ and a temporal oscillation with frequency $\nu_0$.

Standing waves produced by the sum of waves traveling in opposite directions, shown as functions of the spatial coordinate at five different times. The sum is a spatial wave whose amplitude oscillates.

4.6 Anharmonic Traveling Waves, Dispersion

Thus far the only traveling waves we have considered have been harmonic, i.e., consisting of a single sinusoidal frequency. From the principle of Fourier analysis, an anharmonic traveling wave can be decomposed into a sum of traveling harmonic wave components, i.e., waves of generally differing
amplitudes over a discrete set of frequencies:

\[ y [z, t] = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} A_n \cos [k_n z - \omega_n t + \phi_n], \]

where \( A_n, k_n, \) and \( \omega_n \) are the amplitude, angular spatial frequency, and angular spatial frequency of the \( n^{th} \) wave. Therefore, we can define the phase velocity of the \( n^{th} \) wave as:

\[ (v_{\phi})_n = \frac{\omega_n}{k_n}. \]

Now suppose that a particular anharmonic oscillation is composed of two harmonic components \( y [x, t] = y_1 (x, y) + y_2 [x, t] \). If the two components have the same phase velocity, \( (v_{\phi})_1 = (v_{\phi})_2 \), then points of constant phase on the two waves will move with the same speed and maintain the same relative phase. The shape of the resultant wave is invariant over time. Such a wave is called nondispersive, because points of constant phase on the components do not separate over time.

What if the phase velocities are different, i.e., if \( (v_{\phi})_1 \neq (v_{\phi})_2 \)? In this case, points of constant phase on the two waves will move at different velocities, and therefore the distance between points of constant phase will change as a function of position or time. Therefore the shape of the superposition wave will change as a function of time; these waves are dispersive.

Note that the dispersion is a characteristic of the medium within which the waves travel, and not of the waves themselves. It is the medium that determines the velocities and thus whether the waves travel together or if they disperse with time and space.

### 4.7 Average Velocity and Modulation (Group) Velocity

We added two traveling waves of different frequencies and obtained the same result we saw when adding two oscillations: the sum of two harmonic waves yields the product of two harmonic waves with modulation and average spatial and temporal frequencies. Using the new terms: \( k_{\text{avg}}, k_{\text{mod}}, \omega_{\text{avg}}, \) and \( \omega_{\text{mod}} \), we can define the phase velocities of the average and modulation waves:

\[
\begin{align*}
 v_{\text{avg}} & \equiv \frac{\omega_{\text{avg}}}{k_{\text{avg}}} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{\omega_1}{k_1} + \frac{\omega_2}{k_2}, \\
 v_{\text{mod}} & \equiv \frac{\omega_{\text{mod}}}{k_{\text{mod}}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\omega_1}{k_1} - \frac{\omega_2}{k_2}.
\end{align*}
\]

These two velocities have the same meaning as the phase velocity of the single wave, i.e., it is the velocity of a point of constant phase of the average traveling wave frequency or of the modulation wave frequency, or beats wave. The modulation velocity is also commonly called the group velocity.

#### 4.7.1 Example: Nondispersive Waves \( (v_{\phi})_1 = (v_{\phi})_2 \)

In a nondispersive medium, the phase velocity is constant over frequency (or wavelength), i.e.,

\[ (v_{\phi})_1 = \frac{\omega_1}{k_1} = (v_{\phi})_2 = \frac{\omega_2}{k_2}. \]

Note that \( \omega_1 \neq \omega_2 \) and \( k_1 \neq k_2 \) – only the ratios are equal. Now find expressions for \( v_{\text{mod}} \) and \( v_{\text{avg}} \):

\[
\begin{align*}
 v_{\text{avg}} &= \frac{\omega_{\text{avg}}}{k_{\text{avg}}} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{\omega_1 + \omega_2}{k_1 + k_2}, \\
 &= \frac{\omega_1 \left( 1 + \frac{\omega_2}{\omega_1} \right)}{k_1 \left( 1 + \frac{k_2}{k_1} \right)}.
\end{align*}
\]

Since \( \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} \) for nondispersive waves \( \Rightarrow \frac{\omega_2}{\omega_1} = \frac{k_2}{k_1} \) and:

\[
\begin{align*}
 v_{\text{avg}} &= \frac{\omega_1 \left( 1 + \frac{k_2}{k_1} \right)}{k_1 \left( 1 + \frac{k_2}{k_1} \right)} = \frac{\omega_1}{k_1} = v_1 = v_2 = v_{\text{avg}}.
\end{align*}
\]
Similarly for the velocity of the modulation wave:

$$v_{\text{mod}} = \frac{\omega_{\text{mod}}}{k_{\text{mod}}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\omega_1}{k_1} \left( 1 - \frac{k_2}{k_1} \right)$$

Since $$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$$ for nondispersive waves, then $$\frac{\omega_1}{k_1} = \frac{k_2}{k_1}$$ and:

$$v_{\text{mod}} = \frac{\omega_1}{k_1} \left( 1 - \frac{k_2}{k_1} \right) = \frac{\omega_1}{k_1} = \frac{v_1}{v_2} = v_{\text{mod}} = v_{\text{avg}}.$$

Note also that $$\omega_{\text{mod}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta \omega}{\Delta k} \implies \frac{\Delta \omega}{\Delta k} = \omega_{\text{mod}}.$$

In a nondispersive medium, all waves (all spatial and temporal frequencies and all modulation and average waves) travel at the same velocity.

### 4.8 Dispersion Relation for Nondispersive Traveling Waves

Waves are nondispersive in some important physical cases: e.g., light propagation in a vacuum and audible sound in air. Since $$\frac{\omega}{k} = v_\phi$$, we can easily express the temporal angular frequency $$\omega$$ in terms of the angular wavenumber $$k$$:

$$\omega = \omega(k) = (v_\phi) \cdot k$$ where $$v_\phi$$ is constant, so that $$\omega \propto k$$.

The expression of $$\omega$$ in terms of $$k$$ is called a dispersion relation. We can plot $$\omega[k]$$ vs. $$k$$, giving a straight line in the nondispersive case.

**Dispersion Relation for Nondispersive Waves, Two types of wave with different velocities**

$$(v_\phi)_1 > (v_\phi)_2.$$
for light traveling in a medium such as glass. The common specification of the phase velocity of light in medium is the refractive index \( n \):

\[
n = \frac{c}{v_\phi}
\]

where \( v_\phi \) is the phase velocity of light in the medium. In a dispersive medium, we can interpret group velocity in another way:

\[
\omega(k) = k \cdot v_\phi \\
\implies v_{\text{mod}} = \frac{d\omega}{dk} = \frac{d}{dk} (k \cdot v_\phi) \\
= \left( \frac{dk}{dk} \right) v_\phi + k \cdot \left( \frac{dv_\phi}{dk} \right) = v_\phi + k \left( \frac{dv_\phi}{dk} \right).
\]

In other words, the group velocity is the sum of the phase velocity \( v_\phi \) and a term proportional to \( \frac{dv_\phi}{dk} \), which is the change in phase velocity with wavenumber:

\[
\frac{dv_\phi}{dk} > 0 \implies v_{\text{mod}} > v_\phi \\
\frac{dv_\phi}{dk} < 0 \implies v_{\text{mod}} < v_\phi.
\]

As the phase velocity varies, the refractive index varies inversely (faster velocity \( \implies \) smaller index). Variation of the refractive index implies a change in the refractive angle of light entering or exiting the medium (via Snell’s law). Variation of refractive index with wavelength implies that different frequencies will refract at different angles. This is the principle of the dispersing prism.

### 4.9.1 Example: Dispersive Traveling Waves

Consider a medium with dispersion relation of the form of a power law:

\[
\omega(k) = \alpha k^\ell
\]

where \( \ell \) is a real number. The average and modulation velocities are:

\[
v_{\text{avg}} = \frac{\omega}{k} = \frac{\alpha (k^\ell)}{k} = \alpha k^{\ell-1} \\
v_{\text{mod}} = \frac{d\omega}{dk} = \frac{d}{dk} (\alpha k^\ell) = \ell (\alpha k^{\ell-1}) = \ell \cdot v_{\text{avg}}.
\]

So if \( \ell > 1 \), then \( v_{\text{mod}} > v_{\text{avg}} \), and if \( \ell < 1 \), \( v_{\text{mod}} < v_{\text{avg}} \). The first relation corresponds to anomalous dispersion and the second to normal dispersion. The dispersion relation for normal dispersion is nonlinear and concave down, while that for anomalous dispersion is nonlinear and concave up. Of course, for nondispersive waves the dispersion relation is linear.
Phase and modulation (group) velocities on the dispersion plot $\omega[k]$. The phase velocity at wavenumber $k_1$ is $\omega_1/k_1$, while the velocity of the modulation wave is the slope of the dispersion curve evaluated at $k_1$, $v_{\text{mod}} = \frac{d\omega}{dk}|_{k=k_1}$.

In a medium with normal dispersion, the refractive index $n$ increases with frequency $\nu(\omega)$ and decreases with wavelength $\lambda$. Therefore $n$ decreases as the wavenumber $k$ increases, i.e., $\frac{dn}{dk} > 0$. Thus in real media, the average waves travel faster than the modulation.

Refractive index $n$ vs. wavelength $\lambda$ for several media, demonstrating the decrease in index (and thus increase in phase velocity) of light with increasing wavelength.

4.9.2 Propagation of the Superposition of Two Waves in Media with Normal and Anomalous Dispersion

Recall that an anharmonic, though periodic, oscillation can be expressed as a sum of harmonic terms of different frequencies, i.e., as a Fourier series. We can therefore find the effect of dispersion on an anharmonic traveling wave by decomposing it into its Fourier series of harmonic terms and propagating each separately at its own velocity. The resultant is found by resumming the resulting
components. For example, if:

\[ f(z, t) = A_1 \sin(k_1 z - \omega_1 t) + \frac{A_1}{3} \sin(k_2 z - 3\omega_1 t) + \frac{A_1}{5} \sin(k_3 z - 5\omega_1 t) \]

As we’ve already seen, \( f(z, t) \) is the sum of the first three terms of a square wave. In the nondispersive case, \( k_2 = 3k_1 \) and \( k_3 = 5k_1 \), and \( v_1 = v_2 = v_3 \). The wave at the source is shown below:

In dispersive media, \( v_1 \neq v_2 \neq v_3 \), and the relative phase of the three components will vary as the wave travels through space. Therefore the resultant wave will become increasingly distorted.

Normal Dispersion

“Snapshots” of sums of two traveling waves with different frequencies at five different times in a medium with normal dispersion, so the wave with longer wavelength travels faster than that with shorter wavelength. The “modulation” wave moves more slowly than the “average” wave.
4.9. DISPERSIVE TRAVELING WAVES

Anomalous Dispersion

“Snapshots” of sums in a medium with anomalous dispersion, so the wave with shorter wavelength travels faster than that with longer wavelength and the “modulation” wave moves faster than the “average” wave.