Angles

• Angle \( \theta \) is the ratio of two lengths:
  
  – \( R \): physical distance between observer and objects [km]
  
  – \( S \): physical distance along the arc between 2 objects
  
  – Lengths are measured in same “units” (e.g., kilometers)
  
  – \( \theta \) is “dimensionless” (no units), and measured in “radians” or “degrees”

\[
\frac{\theta}{R} = \frac{S}{R} = \text{physical distance along the arc between 2 objects}
\]

“Angular Size” and “Resolution”

• Astronomers usually measure sizes in terms of angles instead of lengths
  
  – because the distances are seldom well known

Trigonometry

\[
\theta = \tan^{-1}\left(\frac{Y}{R}\right)
\]

\[\theta \approx \frac{Y}{R} - \frac{Y^3}{3R^3}\]

Definitions

\[
\tan[\theta] = \frac{\text{opposite side}}{\text{adjacent side}}
\]

\[
\sin[\theta] = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{Y}{\sqrt{R^2 + Y^2}} = \frac{1}{\sqrt{1 + \frac{Y^2}{R^2}}}
\]

Angles: units of measure

• \( 2\pi (= 6.28) \) radians in a circle
  
  – 1 radian = \( 360^\circ / 2\pi \approx 57^\circ \)
  
  \( \Rightarrow \approx 206,265 \) seconds of arc per radian

• Angular degree (\( ^\circ \)) is too large to be a useful angular measure of astronomical objects
  
  – 1\(^\circ\) = 60 arc minutes
  
  – 1 arc minute = 60 arc seconds [arcsec]
  
  – 1\(^\circ\) = 3600 arcsec
  
  – 1 arcsec = (206,266)^\(-1\) = 5 \times 10^4 radians = 5 \mu radians

Number of Degrees per Radian

\[
2\pi \text{ radians per circle} = \frac{360^\circ}{2\pi} \approx 57.296^\circ
\]

\( \approx 57^\circ 17' 45'' \)
Trigonometry in Astronomy

\[ \theta \equiv \frac{S}{R} \approx \frac{Y}{R} \approx \frac{1}{\sqrt{1 + \frac{R^2}{Y^2}}} \]

Usually \( R >> S \), so \( Y \approx S \), so \( \theta \approx \tan(\theta) \approx \sin(\theta) \)

\[
\sin(\theta) \approx \tan(\theta) \approx \theta \quad \text{for} \quad \theta \approx 0
\]

Relationship of Trigonometric Functions for Small Angles

Check it!
18° = \( \frac{18 \times (2\pi \text{ radians per circle})}{360\text{° per circle}} \) = 0.1\pi radians \approx 0.314 radians

Calculated Results
\[
\tan(18\text{°}) \approx 0.32
\]
\[
\sin(18\text{°}) \approx 0.31
\]

0.314 \approx 0.32 \approx 0.31

\[ \theta \approx \tan(\theta) \approx \sin(\theta) \text{ for } |\theta| < 0.1\pi \]

Astronomical Angular “Yardsticks”

- Easy yardstick: your hand held at arms’ length
  - fist subtends angle of \( \approx 5\text° \)
  - spread between extended index finger and thumb \( \approx 15\text° \)

- Easy yardstick: the Moon
  - diameter of disk of Moon AND of Sun \( \approx \frac{1}{2}\text° \) = \( \frac{1}{2} \cdot \frac{1}{60} \text{ radian} \) = \( \frac{1}{100} \text{ radian} \) = 30 arcmin = 1800 arcsec

“Resolution” of Imaging System

- Real systems cannot “resolve” objects that are closer together than some limiting angle
  - “Resolution” = “Ability to Resolve”
- Reason: “Heisenberg Uncertainty Relation”
  - Fundamental limitation due to physics

Image of Point Source

1. Source emits “spherical waves”
2. Lens “collects” only part of the sphere and “flips” its curvature
3. “piece” of sphere converges to form image
With Smaller Lens

Lens "collects" a smaller part of sphere.
Can’t locate the equivalent position (the "image") as well
Creates a “fuzzier” image

Image of Two Point Sources

Fuzzy Images “Overlap”
and are difficult to distinguish
(this is called “DIFFRACTION”)

Image of Two Point Sources

Apparent angular separation of the stars is $\Delta\theta$

Resolution and Lens Diameter

• Larger lens:
  – collects more of the spherical wave
  – better able to “localize” the point source
  – makes “smaller” images
  – smaller $\Delta\theta$ between distinguished sources means BETTER resolution

\[ \Delta\theta \approx \frac{\lambda}{D} \]

$\lambda$ = wavelength of light
$D$ = diameter of lens

Equation for Angular Resolution

\[ \Delta\theta \approx \frac{\lambda}{D} \]

$\lambda$ = wavelength of light
$D$ = diameter of lens

• Better resolution with:
  – larger lenses
  – shorter wavelengths
• Need HUGE “lenses” at radio wavelengths to get the same resolution

Resolution of Unaided Eye

• Can distinguish shapes and shading of light of objects with angular sizes of a few arcminutes

• Rule of Thumb: angular resolution of unaided eye is 1 arcminute
Telescopes and magnification

- Telescopes *magnify* distant scenes

- Magnification = increase in *angular size*
  - (makes $\Delta \theta$ appear larger)

Simple Telescopes

- Simple refractor telescope (as used by Galileo, Kepler, and their contemporaries) has two lenses
  - Objective lens
    - Collects light and forms intermediate image
    - "positive power"
    - Diameter $D$ determines the resolution
  - Eyepiece
    - Acts as "magnifying glass"
    - Forms magnified image that appears to be infinitely far away

Galilean Telescope

Ray incident "above" the optical axis emerges "above" the axis
image is "upright"

$\theta' > \theta$
Larger ray angle $\Rightarrow$ angular magnification

Keplerian Telescope

Ray incident "above" the optical axis emerges "below" the axis
image is "inverted"

$|\theta'| > \theta$
Larger ray angle $\Rightarrow$ angular magnification
### Telescopes and magnification

- Ray trace for refractor telescope demonstrates how the increase in magnification is achieved
  - *Seeing the Light*, pp. 169-170, p. 422
- From similar triangles in ray trace, can show that
  \[
  \text{magnification} = \frac{f_{\text{eyelens}}}{f_{\text{objective}}}
  \]
  - \(f_{\text{objective}}\) = focal length of objective lens
  - \(f_{\text{eyelens}}\) = focal length of eyelens
- magnification is negative ⇒ image is inverted

### Magnification: Requirements

- To increase apparent angular size of Moon from “actual” to angular size of “fist” requires magnification of:
  \[
  \frac{5}{0.5} = 10 \times
  \]
- Typical Binocular Magnification
  - with binoculars, can easily see shapes/shading on Moon’s surface (angular sizes of 10′s of arcseconds)
- To see further detail you can use small telescope w/ magnification of 100-300
  - can distinguish large craters w/ small telescope
  - angular sizes of a few arcseconds

### Ways to Specify Astronomical Distances

- **Astronomical Unit (AU)**
  - distance from Earth to Sun
  - 1 AU ≈ 93,000,000 miles ≈ 1.5 × 10^9 km
- light year = distance light travels in 1 year
  \[
  1 \text{ light year} = 60 \text{ sec/min} \times 60 \text{ min/hr} \times 24 \text{ hrs/day} \times 365.25 \text{ days/year} \times (3 \times 10^5) \text{ km/sec}
  \]
  \[= 9.5 \times 10^{12} \text{ km} = 5.9 \times 10^{12} \text{ miles} \approx 6 \text{ trillion miles} \]

### Aside: parallax and distance

- Only direct measure of distance astronomers have for objects beyond solar system is **parallax**
  - Parallax: apparent motion of nearby stars against background of very distant stars as Earth orbits the Sun
  - Requires images of the same star at two different times of year separated by 6 months

### Parallax as Measure of Distance

- \(P\) is the “parallax”
- typically measured in arcseconds
- Gives measure of distance from Earth to nearby star (distant stars assumed to be an “infinite” distance away)

### Definition of Astronomical Parallax

- “half-angle” of triangle to foreground star is 1′
  - Recall that 1 radian = 206,265′
  - 1′ = (206,265)′ radians = 5 × 10^-6 radians = 5 μradians
- \(R = 206,265 \text{ AU} = 2 \times 10^9 \text{ AU} = 3 \times 10^{13} \text{ km} \)
  - 1 parsec = 3×10^13 km = 20 trillion miles = 3.26 light years

### Ways to Specify Astronomical Distances

- **Astronomical Unit (AU)**
  - distance from Earth to Sun
  - 1 AU ≈ 93,000,000 miles ≈ 1.5 × 10^9 km
Parallax as Measure of Distance

- \( R = \frac{1}{P} \)
  - \( R \) is the distance (measured in pc) and \( P \) is parallax (in arcsec)
  - Star with parallax (half angle!)
    - \( \frac{1}{2}'' \) is at distance of 2 pc \( \approx 6.5 \) light years
    - Star with parallax of 0.1'' is at distance of 10 pc \( \approx 32 \) light years
  - **SMALLER PARALLAX MEANS FURTHER AWAY**

Limitations to Magnification

- Can you use a telescope to increase angular size of nearest star to match that of the Sun?
  - Nearest star is \( \alpha \) Cen (alpha Centauri)
  - Diameter is similar to Sun’s
  - Distance is 1.3 pc
    - 1.3 pc = 4.3 light years = 1.5 \( \times 10^{13} \) km from Earth
    - Sun is 1.5 \( \times 10^{8} \) km from Earth
    - \( \Rightarrow \) would require angular magnification of 100,000 = \( 10^5 \)
  - \( \Rightarrow f_{\text{objective}} = f_{\text{eyelens}} \times 10^5 \)

- **BUT**: you can’t magnify images by arbitrarily large factors!
  - **Remember diffraction!**
    - Diffraction is the **unavoidable** propensity of light to change direction of propagation, i.e., to “bend”
    - Cannot focus light from a point source to an arbitrarily small “spot”
  - Increasing magnification involves “spreading light out” over a larger imaging (detector) surface
  - **Diffraction Limit** of a telescope
    \[
    \Delta \theta \approx \frac{\lambda}{D}
    \]

Magnification: limitations

- **BUT**: atmospheric effects typically dominate diffraction effects
  - Most telescopes are limited by “seeing”: image “smearing” due to atmospheric turbulence
- Rule of Thumb:
  - Limiting resolution for visible light through atmosphere is equivalent to that obtained by a telescope with \( D = 3.5'' \) (\( \approx 90 \) mm)
    \[
    \Delta \theta = \frac{\lambda}{D} \quad \text{at} \quad \lambda = 500\text{nm} \quad (\text{Green light})
    \]
    \[
    = \frac{500 \times 10^{-9} \text{m}}{0.09 \text{m}} = 5.6 \times 10^{-5} \text{ radians} \approx 5.6 \mu \text{rad}
    \]
    \[
    = 1.2 \text{ arcsec} \approx 1/50 \text{ of eye's limit}
    \]