IMPORTANT INSTRUCTIONS

You must complete two (2) of the three (3) questions given for each of the core graduate classes. The answer to each question should begin on a new piece of paper. While you are free to use as much paper as you would wish to answer each question, please only write on one side of each sheet of paper that you use. Be sure to write your provided identification letter (located below), the question number, and the page number for each answer in the upper right-hand corner of each sheet of paper that you use. When you hand in your exam answers, be certain to write your name and your provided identification letter on the supplied 5” x 8” note card and place this in the envelope located with the proctor.

ONLY HAND IN THE ANSWERS TO THE QUESTIONS THAT YOU WOULD LIKE EVALUATED

Identification Letter: _______________

THIS EXAM QUESTION SHEET MUST BE HANDED BACK TO THE PROCTOR UPON COMPLETION OF THE EXAM PERIOD
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THIS EXAM QUESTION SHEET MUST BE HANDED BACK TO THE PROCTOR UPON COMPLETION OF THE EXAM PERIOD
1. Fourier Methods in Imaging

You know the definitions of the summation and multiplication operators:

\[ \sum_{n=0}^{N} f[n] \equiv f[0] + f[1] + \cdots + f[N-1] + f[N] \]
\[ \prod_{n=0}^{N} f[n] \equiv f[0] \cdot f[1] \cdot \cdots \cdot f[N-1] \cdot f[N] \]

Now consider a similar operator based on convolution that is denoted by the (invented) symbol \( \circledast \) that is applied over the specified range of indices.

\[ \circledast_{n=0}^{N} (f_n[x]) \equiv f_0[x] \ast f_1[x] \ast \cdots \ast f_N[x] \]

(a) Show your work to evaluate the result of

\[ g[x] = \circledast_{n=1}^{N} \left( \text{SINC} \left( \frac{x}{b_0} \right) \right) \]

where \( N \) is a positive integer and \( b_0 \) is a real-valued nonzero constant.

(b) Extend the result of (a) it to evaluate:

\[ g[x] = \circledast_{n=1}^{N} \left( \frac{1}{n} \cdot \text{SINC} \left( \frac{x}{n} \right) \right) \]

where \( N \) is an integer such that \( N < \infty \).

(c) What can you say about the result of (b) in the limit where \( N \to +\infty \)?

(d) Find a function \( f[x] \) that satisfies:

\[ g[x] = \circledast_{n=1}^{N} (f[x]) = f[x] \]

2. Fourier Methods in Imaging

This problem considers the M-C-M chirp Fourier transform:

(a) If \( \mathcal{F} \{ f[x] \} = F[\xi] \), use the theorems of the Fourier transform to evaluate:

\[ \mathcal{F} \left\{ F \left[ -\frac{x}{\alpha_0^2} \right] \right\} \]

in terms of \( f \) where \( \alpha_0 \) is a real-valued constant.
(b) The cascade of multiplication, convolution, and multiplication of a 1-D function $f[x]$ by quadratic-phase functions in the space domain yields:

$$
\left( \left( f[x] \cdot \exp \left[ -i \pi \left( \frac{x}{\alpha_0} \right)^2 \right] \right) \ast \exp \left[ +i \pi \left( \frac{x}{\alpha_0} \right)^2 \right] \right) \cdot \exp \left[ -i \pi \left( \frac{x}{\alpha_0} \right)^2 \right] = F \left[ \frac{x}{\alpha_0^2} \right]
$$

where $\alpha_0$ is the “chirp rate” of the quadratic-phase terms and $F[\xi]$ is the 1-D Fourier transform of $f[x]$. For obvious reasons, this often is called the “M-C-M chirp Fourier transform. Note that the output coordinate $\frac{x}{\alpha_0^2}$ is in the space domain but has dimensions of spatial frequency. Substitute $F \left[ \frac{x}{\alpha_0^2} \right]$ from part (a) for $f[x]$ in the left-hand side of this expression and $\mathcal{F}\{F \left[ \frac{x}{\alpha_0^2} \right]\}$ in the right-hand side.

(c) Use the theorems of the Fourier transform to evaluate in the frequency domain.

$$
\mathcal{F} \left( \left( F \left[ \frac{x}{\alpha_0^2} \right] \cdot \exp \left[ -i \pi \left( \frac{x}{\alpha_0} \right)^2 \right] \right) \ast \exp \left[ +i \pi \left( \frac{x}{\alpha_0} \right)^2 \right] \right) \cdot \exp \left[ -i \pi \left( \frac{x}{\alpha_0} \right)^2 \right]
$$

3. Fourier Methods in Imaging

The “equivalent width” of a function $f[x]$ is denoted by $\Delta x_f$ and is defined to be the ratio of the area and the central ordinate. A corresponding expression $\Delta \xi_f$ may be defined for the spectrum.

(a) For functions where the two equivalent widths are defined, derive the expression for their product.

(b) Demonstrate the validity of this expression for $f[x] = 4 \cdot RECT[2x]$

(c) Consider the same expression for the space-domain function:

$$
f[x] = RECT \left[ \frac{x - b_0}{b_0} \right]
$$

where $b_0$ is some nonzero real number. What does the result mean for the expression derived in (a)?

4. Radiometry

You are trying to determine the SNR of your camera for the given setup. You determine that the output of the 0.0201 $m^2$ source you are using is $L_s = 60 [W m^{-2} sr^{-1}]$ with a mean radiating wavelength of 600 nm. It is positioned 20 degrees from the normal of a 50% reflecting surface at a distance of $r = 40$ cm away. The camera you are using has square pixels that are 10 $\mu m$ on a side with a QE of 90%. From a spec sheet you see that the lens transmission is 95% and the dark noise value (i.e., noise from all electronic sources) is 579 electrons. What is the SNR for the given setup if you set the camera to f/4 and use a shutter speed of $\frac{1}{100}$ seconds? How does this SNR compare to the shot-limited SNR? (Note: $h = 6.63 \times 10^{-34} [J \cdot s]$ and $c = 3 \times 10^8 [m \cdot s^{-1}]$).
5. **Radiometry**

I am flying my personal 747 plane in the early evening over Rochester, at an altitude of 1000 meters, to cross-check some earlier measurements of residential heat loss with my new camera. A week earlier, my colleague, Dr. Who, tells me that he collected data on two specific buildings with temperatures of 311 K (100 F) and 305 K (90 F) with corresponding radiances of $7.5 \times 10^{-5} \text{[Wm}^{-2}\text{sr}^{-1}]$ and $5 \times 10^{-5} \text{[Wm}^{-2}\text{sr}^{-1}]$, respectively. He tells me that the band pass was centered at 10 $\mu$m (i.e., mean wavelength) with a bandwidth of 1.5 $\mu$m and that he imaged the buildings off-axis at 20 degrees.

I set out to repeat the experiment with my new camera that has 50 $\mu$m on a side, square detectors, a lens transmission of 80% and a fall-off factor of 3. My camera spec sheet tells me the noise equivalent power metric for my camera is $1 \times 10^{-11}$ watts. While flying, Dr. Who makes a measurement of the atmospheric absorption coefficient for me which is $2 \times 10^{-4} \text{[m}^{-1}]$ (the same as when he performed the experiment). If I set my camera to f/4 will I be able to tell the different buildings apart (in the absence of all other noise sources)?
6. Radiometry

I'm in the ocean taking pictures of sharks. My colleague, sitting on his newly renovated Viking boat, shines a 300 [W sr\(^{-1}\)] spot light in the direction of the shark from \(H_1 = 1.5\) m above the water at an angle of \(\theta = 40\) deg. I know that the extinction in this part of the atmosphere above the water is \(\beta_{\text{ext}} = 5.7 \times 10^{-3} [m^{-1}]\).

The 30\% reflecting gray shark is in \(H_2 = 2\) meters of water where I know the absorption coefficient of water (at 600 nm) is \(\beta_{\alpha} = 0.2229 [m^{-1}]\). I set my underwater camera with 98\% transmissive optics to f/4 and a shutter speed of \(\frac{1}{60}\) seconds. Ignoring water surface reflections and the fact that the shark could bite me, how close can I get to the shark to get the perfect on-axis exposure of \(H_0 = 1.5 \times 10^{-3} [J \cdot m^2]\)? State any assumptions you have made.
7. Human Visual System

An audience of 100 college students (50 male, 50 female) is provided red/cyan anaglyph 3D glasses to view a projected “stereo” movie. Most in the audience have normal color vision; they are trichromats with a normal distribution of S, M, and L cones. Some in the audience are anomalous trichromats, some are dichromats.

(a) Explain how the red/cyan glasses work for the normal observers.

(b) Describe the difference in color vision between the normal observers, the anomalous trichromats, and the dichromats. Include in your description the likely cause of the differences, the effects of the differences, and the approximate number of anomalous trichromats and dichromats you would expect to find in the audience of 100 if they represent a random sample of the overall population.

(c) Would the red/cyan glasses work for the anomalous trichromats and/or dichromats? Defend your answer.
8. **Human Visual System**

The Contrast Sensitivity Function (CSF) describes the response of the human visual system in the frequency domain by showing the inverse of the threshold contrast to detect extended 1D sine targets over a range of spatial frequencies.

![Contrast Sensitivity Function](image1)

The Line-Spread Function (LSF) of the human eye, as measured in Campbell and Gubisch's classical experiments, describes the spread of optical energy in the spatial domain. The figure below shows the LSF for a well-corrected eye with a 2.4mm pupil.

![Line-Spread Function](image2)

(a) Can the Contrast Sensitivity Function (CSF) be predicted from the Line-Spread Function (LSF)?

(b) Discuss the relationship (if any) between the two functions, and

(c) What else (if anything) must be understood about the structure and behavior of the visual system in the spatial domain to understand its behavior in the frequency domain.
9. **Human Visual System**

Almost all cathode-ray tube (CRT), liquid-crystal display (LCD), plasma, and organic light-emitting diode (OLED) television displays use red, green, and blue (RGB) primaries to reproduce colors. Regardless of which display technology is used, moving images are updated at least 30 times per second, and any significant variation in screen luminance is modulated at a temporal frequency of at least 60 Hz.

(a) Explain why color television systems have three primaries.

(b) Explain why color television systems use RGB primaries. (Include in your explanation an alternative set of primaries, and why RGB is a better choice than your alternative.)

(c) Explain why moving images are updated at least 30 times per second.

(d) Explain why significant luminance variation is modulated at a temporal frequency of at least 60 Hz.

10. **Digital Image Mathematics**

A 1-D sampled analog signal $f[n]$ whose amplitude $f$ lies in the range $f_{\text{min}} \leq f \leq f_{\text{max}}$. The signal is quantized to $m$ bits so that there are $2^m \equiv N$ levels in this range such that $f_{\text{min}}$ is quantized to level 0 and $f_{\text{max}}$ to level $N-1$. Assume that the quantization error is uniformly distributed over the available range. Determine the rate of increase of the signal-to-noise ratio of the quantized signal represented in logarithmic “units” as the number $m$ of quantizer bits is increased.

11. **Digital Image Mathematics** Two 8-pixel sampled images have the following gray values:

$$f_1[n] = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7$$
$$f_2[n] = 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 0$$

(a) Sketch (roughly) the 2-D histogram of these images.

(b) Calculate the means of the two images.

(c) Calculate the covariance matrix of the two images.

(d) Calculate the eigenvectors and eigenvalues of the covariance matrix.

(e) Project the data onto the eigenvectors to evaluate the gray values of the principal components.
12. Digital Image Mathematics

The probability distribution of normalized amplitudes (gray values) $f$ in the image $f[n,m]$ is shown below on the left. The goal is to transform the gray levels to produce an output image $g[n,m]$ with the same histogram as the reference image $r[n,m]$ shown below on the right. Assume that the gray values are continuous (not discrete levels) and find an expression for the transformation that will accomplish this. Explain the features of the transformation you derived, i.e., describe the reasons for its “shape” compared to the original histogram.

(left) probability distribution of gray value of input image $f[n,m]$; (right) probability distribution of reference image.

13. Optics for Imaging

A uniform coherent plane wave of electric field amplitude, $E_0$, and wavelength, $\lambda$, is incident upon two slits, separated by a distance $2a$, as shown in the diagram below. Another pair of slits, separated by a distance $2b$ is placed between two lenses of local lengths, $f$. The slits may be expressed as Dirac delta functions.
(a) Explain in words how the irradiance will be distributed along \( x' \) and \( x'' \).

(b) By direct use of the Fourier integral, determine the electric field profile, \( E(x') \), in the \( x' \)-plane. Assume that the lens diameter is infinitely large.

(c) Determine the electric field profile, \( E(x'') \), in the \( x'' \)-plane, again assuming that the lens diameter is infinitely large.

(d) For a given value of \( a, \lambda, \) and \( f \), determine the value(s) of \( b \) where \( E(x'') = 0 \).

14. Optics for Imaging

A star having a radial extent \( R \) and distance \( L \) is examined through a telescope of focal length \( f \) and aperture diameter \( D \). The image is transmitted through a filter that transmits light at wavelength \( \lambda \). Sketch the system and label it appropriately.

(a) Determine the radial extent of the image.

(b) Determine an expression for the distance, \( L_0 \), that would cause the image to appear as a single point source (that is, its extent is unresolvable). Discuss your reasoning.

(c) Determine whether the diffracted starlight (over the distance \( L_0 \)) satisfies the far-field diffraction (Rayleigh range) condition.

(d) Treat the star as a point source on the optical axis at the distance \( L_0 \) from the input face of the telescope aperture. The star is placed at the origin of a coordinate system. When the starlight reaches the aperture, the phase at the edge \( (L_0, D/2) \) is different than that at the center \( (L_0, 0) \). Determine the phase difference.
15. **Optics for Imaging**

The diagram below shows an optical system that is characterized by an ABCD ray transfer matrix. The system produces an image on the right side of the “black box” when an object is placed on the left side of the black box. Each ray height and angle is represented by the matrix expression:

\[
\begin{bmatrix}
y' \\
\theta'
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
y \\
\theta
\end{bmatrix}.
\]

(a) Using the object rays, 1 and 2, along with the points \(\xi\) and \(\zeta\), sketch the corresponding image rays and points on the right side of the system. Note: \(\zeta\) is on the optical axis. Justify your results.

(b) The system has a lateral magnification \(M\) and an angular magnification \(m\). Use the rays and points in the illustration to determine the values of \(A\), \(B\), and \(D\) in terms of \(M\) and \(m\). Justify your results.

(c) If the illustration represents a simple one-lens imaging system with object distance \(d\), image distance \(d'\), and focal length \(f\), determine values of \(A\), \(B\), \(C\), and \(D\) by use of the matrix multiplication of translation and lens ABCD matrices.

16. **Digital Image Processing**

Construct a fully populated approximation pyramid and corresponding prediction residual pyramid for the image

\[
f(x, y) =
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix}.
\]

Use 2 x 2 block neighborhood averaging for the approximation filter in the following block diagram, and assume the interpolation filter implements pixel replication.
17. Digital Image Processing

(a) Can variable-length coding procedures be used to compress a histogram equalized image with \(2^n\) intensity levels? Explain.

(b) Can such an image contain spatial or temporal redundancies that could be exploited for data compression?

18. Digital Image Processing

Design an invisible watermarking system based on the discrete Fourier transform.

19. Probability, Noise, and System Modeling

A photon stream has an average rate of \(\lambda = 8\) photons per second. Let \(X\) represent the number of photons that arrive in the interval \((0, 10)\) and \(Y\) represent the number that arrive in the interval \((7, 17)\). Also let \(U\), \(V\), and \(W\) be defined as the number of photons that arrive in the intervals \((0, 7)\), \((7, 10)\), and \((10, 17)\), respectively. Clearly, \(X = U + V\) and \(Y = V + W\).
Calculate the covariance matrix and the correlation coefficient for the random variables $X$ and $Y$.

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20. **Probability, Noise, and System Modeling**

Let $X$ be a random process and let $x(n)$ be a member function such as that shown in (a) below. It has been proposed that statistical information such as the mean value, variance, maximum and minimum values and average power can be determined by constructing a curve $g(T)$ that plots the fraction of the counts for which $x < T$ where $T$ is a variable threshold. An example plot of $g(T)$ vs $T$ is shown in (b). This curve corresponds to the fraction of the axis associated with the shaded part of the waveform as illustrated in (a).

(a) Example of an ensemble member function.

(b) Plot of $g(T)$ vs $T$

(a) For a particular example of $x(n)$ it has been found that

$$g(T) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{T - 0.09}{0.27 + 2}} e^{-t^2} dt$$

Use this information to estimate $E[X]$.

(b) Estimate the probability density function for samples of $x(n)$ for the above example.

(c) What assumptions must be satisfied for the above results to be representative of the random process $X$. 


A photon field with an average intensity of $\lambda$ photons/ms/mm$^2$ is observed by a sensor. The detector has an effective collection area $A = 1$ mm$^2$ is exposed for 4 ms. The detector output $X$ is an electron stream with an electron produced with probability 0.8 by each photon. The electrons are then amplified in a device with a constant gain of 5. The observed system output $S$ is the sum of the amplified electrons plus random noise $Z$ that is uncorrelated with the data and has a normal $N(\mu = 0, \sigma^2 = 225)$ distribution.

(a) A value $S = 80$ was observed. What is the estimated value of $\lambda$ based upon this observation?

(b) Compute the DQE of the sensor system for an operating level of $\lambda = 5$ as the average input intensity.