

Calculation from the original experimental data of the
CIE 1931 RGB standard observer spectral chromaticity
co-ordinates and color matching functions

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Summary

This paper describes all the steps in the calculations of the CIE 1931 RGB spectral chromaticity co-ordinates and color matching functions starting from the initial experimental data of W. D. Wright and J. Guild. Sufficient information is given to allow the reader to reproduce and verify the results obtained at each stage of the calculations and to critically analyze the procedures used. In some instances, the available literature only provides limited descriptions of the actual steps in the calculations and, in others, important data were not published. Nevertheless, it has been possible to more or less reproduce the entire sequence of calculations. All the tables of numerical data are given in the accompanying computer worksheet file CIE1931_RGB.xls.

In the late 1920's, W. D. Wright working at Imperial College in London, and J. Guild at the National Physical Laboratory (NPL) near London, independently compiled the experimental colour matching data that eventually led to the specification of the CIE 1931 standard colorimetric observer. Both workers determined the compositions of additive mixtures of red, green and blue primary lights that matched each of the different colours of the visible spectrum of known wavelengths, but using different experimental arrangements. The technique of trichromatic color matching using a visual colorimeter that they used is described in the file *trichromatic color matching.doc*.

Wright published his results first in two papers and a report¹⁻³. Guild⁴ actually completed his measurements before Wright but did not publish his data until 1931. It was after the publication of Wright's second paper² that Guild realized that the two sets of measurements were almost identical once they had both been transformed into a system based on the same primary lights with the same units.

This paper starts with the independent experimental measurements of Wright and of Guild and shows the methods they used for manipulation of the two sets of data and their combination leading to the specification of the CIE 1931 RGB colorimetric system. A critical analysis shows where weaknesses exist in the data and in the published accounts. The reader can follow the detailed numerical procedures by consulting the accompanying worksheet file *CIE1931_RGB.xls*.

Description of the data obtained by Wright

Wright's initial measurements

Wright carried out trichromatic color matching experiments with ten observers having normal color vision. A square visual field was used that subtended an angle of approximately 2° at the eye so that the light seen by the observer fell on the foveal region which has only cone receptors. The visual field was divided horizontally into two equal rectangles, one illuminated by the monochromatic spectral light to be matched and the other by a mixture of the three monochromatic red (650 nm), green (530 nm) and blue (460 nm) primary lights. The amounts of each primary in the mixture could be independently varied and recorded once a match of the test light was achieved. These

amounts are the tristimulus values for the spectral colour and were normalized to give the chromaticity co-ordinates whose sum is unity. The units of the primaries were such that a mixture of equal amounts of red and green matched a yellow monochromatic light (582 nm, desaturated with a little blue). Also, a mixture of equal amounts of green and blue matched a blue-green monochromatic light (494 nm, desaturated with a little red). With this system of units, each observer generated a set of spectral chromaticity co-ordinates. These exhibited only small variations of comparable values for different observers. Each observer also determined the chromaticity co-ordinates of the NPL standard white light using a mixture of the three primaries. These values varied considerably among the different observers, a consequence of different degrees of yellow pigmentation of the macula in the foveal region of their eyes.

The averaged results were presented in the usual manner (equation 1) for matching one trichromatic unit of the spectral light C of wavelength λ .

$$C(\lambda) \equiv r_1(\lambda) R(650) + g_1(\lambda) G(530) + b_1(\lambda) B(460) \quad (1)$$

where $r_1(\lambda)$, $g_1(\lambda)$ and $b_1(\lambda)$ are the scalar chromaticity co-ordinates, whose sum is de facto unity, and $R(650)$, $G(530)$ and $B(460)$ represent respectively one trichromatic unit of each of the monochromatic primaries (red, green and blue) of given wavelength. Wright only published a graph of this original set of spectral chromaticity co-ordinates and did not provide any tabulated data. Table 1 in the worksheet *CIE1931_RGB.xls, Sheet 1, Cell A3* was derived by working back from the data discussed in the next section, for which Wright did provide a table of values. The values of $r_1(\lambda)$, $g_1(\lambda)$ and $b_1(\lambda)$ in Table 1 in the worksheet match Wright's published Figure 5¹.

With the system of units used by Wright, the average values of the chromaticity co-ordinates for the NPL standard white light were given by

$$\text{NPL white} \equiv 0.243 R(650) + 0.410 G(530) + 0.347 B(460) \quad (2)$$

The values for different observers exhibited wide variations. Figure 6 on p. 151 of Wright's first paper¹ shows that the chromaticity co-ordinates for the NPL standard white varied from 0.15 to 0.35 for the red co-ordinate, and from 0.32 to 0.54 for the green co-ordinate. He used an empirical graphical technique to obtain the average values given in equation 2 above. The precision of the averaged chromaticity co-ordinates for the NPL

standard white is questionable given the wide variations observed and the graphical method used.

The next step is the calculation of the spectral chromaticity co-ordinates based on the same primary lights but using units such that the standard NPL white would be matched by mixing equal amounts of each primary. Thus,

$$\text{NPL white} \equiv 0.243 R(650) + 0.410 G(530) + 0.347 B(460) \quad (2)$$

$$\equiv 0.333 R'(650) + 0.333 G'(530) + 0.333 B'(460) \quad (3)$$

where $R'(650)$, $G'(530)$ and $B'(460)$ represent respectively one trichromatic unit of each of the same primaries but in the new system of units.

The chromaticity co-ordinates for a spectral light in terms of Wright's initial system of units is given by equation 1 above, but will be given by

$$C(\lambda) \equiv r_2(\lambda) R'(650) + g_2(\lambda) G'(530) + b_2(\lambda) B'(460) \quad (4)$$

for the new unit system where equal amounts of each primary match the NPL standard white light. Equations 1 and 4 describe the same spectral light, and equations 2 and 3 the same standard white light. The ratios of the amounts of any primary in these equations must be related so that

$$\frac{r_2}{r_1} = \frac{0.333}{0.243} \quad \frac{g_2}{g_1} = \frac{0.333}{0.410} \quad \frac{b_2}{b_1} = \frac{0.333}{0.347} \quad (5)$$

Thus, the normalized spectral chromaticity co-ordinates in terms of these primaries with the new units are

$$\begin{aligned} r_2 &= \frac{(0.333/0.243) r_1}{(0.333/0.243) r_1 + (0.333/0.410) g_1 + (0.333/0.347) b_1} \\ g_2 &= \frac{(0.333/0.410) g_1}{(0.333/0.243) r_1 + (0.333/0.410) g_1 + (0.333/0.347) b_1} \\ b_2 &= \frac{(0.333/0.347) b_1}{(0.333/0.243) r_1 + (0.333/0.410) g_1 + (0.333/0.347) b_1} \end{aligned} \quad (6)$$

The values in Table 1 in the worksheet *CIE1931_RGB.xls, Sheet 1, Cell A3* can thus be transformed into those of Table 2 (*Cell A44*) using equation 6. The new values in Table 2 are identical with those published by Wright².

Conversion of Wright's spectral chromaticity co-ordinates into a set based on the NPL standard primaries

The NPL primaries were monochromatic spectral lights of wavelengths 700 nm (red hydrogen emission line), 546.1 nm (green mercury emission) and 435.8 nm (blue mercury emission). To convert Wright's spectral chromaticity co-ordinates into a set based on the NPL standard primaries, we must know the values of Wright's chromaticity co-ordinates at wavelengths of 700, 546.1 and 435.8 nm. Unfortunately, Wright did not publish this information. I initially derived these numbers by graphical interpolation at the exact wavelengths of the NPL primaries using 4th order polynomials describing $r_2(\lambda)$, $g_2(\lambda)$ or $b_2(\lambda)$ as a function of wavelength (Figure 1 in worksheet *CIE1931_RGB.xls*, Sheet 1, Cell A91). Later, the equations were derived by the reverse transformation from the published results². The values obtained were then adjusted slightly to minimize the differences between the calculated and Wright's published values² of the spectral chromaticity co-ordinates (the maximum adjustment was within 0.001). The appropriate transformation equations are:

$$\begin{aligned} R(700) &= 1.0510 R'(650) - 0.0150 G'(530) \\ G(546) &= 0.1938 R'(650) + 0.8337 G'(530) - 0.0275 B'(460) \\ B(436) &= 0.0407 R'(650) - 0.0412 G'(530) + 1.0005 B'(460) \end{aligned} \quad (7)$$

Inversion of the 3 x 3 matrix of the coefficients from equation 7 gives

$$\begin{aligned} R'(650) &= 0.9818 R(700) + 0.0177 G(546) + 0.0005 B(436) \\ G'(530) &= -0.2299 R(700) + 1.1970 G(546) + 0.0329 B(436) \\ B'(460) &= -0.0494 R(700) + 0.0486 G(546) + 1.0008 B(436) \end{aligned} \quad (8)$$

This is shown in worksheet *CIE1931_RGB.xls*, Sheet 1, Cells C131 to C144.

For establishing the corresponding units of each set of primaries, the sum of equal amounts of $R'(650)$, $G'(530)$ and $B'(460)$ must match the NPL standard white light, as must the sum of equal amounts of $R(700)$, $G(546)$ and $B(436)$. Adding together the three equations in 8 gives

$$R'(650) + G'(530) + B'(460) = 0.7026 R(700) + 1.2632 G(546) + 1.0342 B(436) \quad (9)$$

In order for a mixture of equal amounts of the primaries $R(700)$, $G(546)$ and $B(436)$ to match the standard NPL white, their coefficients in equation 9 must all be

unity. Therefore, their coefficients in equation 8 must be normalized by dividing the red values by 0.7026, the green values by 1.2632 and the blue values by 1.0342. This gives from equation 8

$$\begin{aligned} R'(650) &= 1.3975 R(700) + 0.0140 G(546) + 0.0005 B(436) \\ G'(530) &= - 0.3272 R(700) + 0.9475 G(546) + 0.0318 B(436) \\ B'(460) &= - 0.0703 R(700) + 0.0384 G(546) + 0.9677 B(436) \end{aligned} \quad (10)$$

Adding the three equations 10 gives

$$R'(650) + G'(530) + B'(460) = R(700) + G(546) + B(436) \quad (11)$$

as required for mixtures equal amounts of each primary to match the standard NPL white for both sets of primaries.

The equation

$$C(\lambda) = r_3(\lambda) R(700) + g_3(\lambda) G(546) + b_3(\lambda) B(436) \quad (12)$$

is equivalent to equation 4 but expressed in terms of the NPL standard primaries.

Substitution of equation 10 into equation 4 gives

$$\begin{aligned} C(\lambda) &= r_2(\lambda) [1.3975 R(700) + 0.0140 G(546) + 0.0005 B(436)] \\ &+ g_2(\lambda) [- 0.3272 R(700) + 0.9475 G(546) + 0.0318 B(436)] \\ &+ b_2(\lambda) [- 0.0703 R(700) + 0.0384 G(546) + 0.9677 B(436)] \end{aligned} \quad (13)$$

Equations 12 and 13 then allow calculation of the spectral chromaticity co-ordinates $r_3(\lambda)$, $g_3(\lambda)$ and $b_3(\lambda)$ based on the NPL primaries.

$$\begin{aligned} r_3(\lambda) &= \frac{1.3975 r_2(\lambda) - 0.3272 g_2(\lambda) - 0.0703 b_2(\lambda)}{1.4120 r_2(\lambda) + 0.6522 g_2(\lambda) + 0.9358 b_2(\lambda)} \\ g_3(\lambda) &= \frac{0.0140 r_2(\lambda) + 0.9475 g_2(\lambda) + 0.0384 b_2(\lambda)}{1.4120 r_2(\lambda) + 0.6522 g_2(\lambda) + 0.9358 b_2(\lambda)} \\ b_3(\lambda) &= \frac{0.0005 r_2(\lambda) + 0.0318 g_2(\lambda) + 0.9677 b_2(\lambda)}{1.4120 r_2(\lambda) + 0.6522 g_2(\lambda) + 0.9358 b_2(\lambda)} \end{aligned} \quad (14)$$

The calculated values are given in Table 3 in worksheet *CIE1931_RGB.xls, Sheet 1, Cell A169* and differ only slightly from those published by Wright², with only 4 of 93 values differing by more than 0.002 (see Table 4 *CIE1931_RGB.xls, Sheet 1, Cell A212*). The deviations arise from our ignorance of the actual numerical values of the original

experimental spectral chromaticity co-ordinates, of the actual equations used by Wright for the transformation to the NPL primaries, and of the precision of his manual matrix calculation. Smith and Guild⁵ also suggest that there is an error in one of Wright's coefficients but this was not identified.

Wright's measurements of the relative luminance values of his primaries were not particularly reliable, by his own admission. Despite this, Wright² calculated the relative luminance of one trichromatic unit of each of the NPL primaries as $L_R = 1.00$ (700 nm), $L_G = 4.33$ (546 nm) and $L_B = 0.047$ (436 nm). The values were quite close to those established by Guild⁴. In addition, the values Wright used for the luminous flux of the NPL standard white source were based on an incorrect black body temperature of 4800 K and do not agree with data given later by Guild, particularly between 540 and 590 nm. Guild⁴ points out that Wright did not have access to the correct information for this standard source.

Description of the data obtained by Guild

Guild actually carried out trichromatic color matching experiments with seven observers more than a year before Wright began his studies. He used a visual colorimeter based on the same principles as Wright's instrument but with three primary lights obtained using red, green and blue filters in front of white light sources. The units of his primaries were such that an additive mixture of equal amounts matched the color of the NPL standard white light. The results were tabulated as spectral chromaticity co-ordinates $r_4(\lambda)$, $g_4(\lambda)$ and $b_4(\lambda)$ and are given in the worksheet *CIE1931_RGB.xls, Sheet 2, Cell A2* (Table 5). It is important to note that Guild's tabulated data⁴ (eventually from 380 to 780 nm at 5 nm intervals) were based on actual measurements at only 36 wavelengths, the other values being obtained by interpolation and extrapolation using a graphical technique.

Since the standard primaries at the NPL in the 1920's were spectral lights with wavelengths of 700, 546.1 and 435.8 nm, Guild transformed his original chromaticity co-ordinates into a set based on these primaries with units such that a mixture of equal amounts again matched the color of the NPL standard white source. He did publish⁴ the

required transformation equations giving the chromaticity co-ordinates of the standard NPL primaries in terms of the filter primaries R_F , G_F and B_F .

$$\begin{aligned} R(700) &= 1.0604 R_F - 0.0604 G_F \\ G(546) &= 0.0600 R_F + 0.9552 G_F - 0.0152 B_F \\ B(436) &= 0.0493 R_F - 0.0639 G_F + 1.0146 B_F \end{aligned} \quad (15)$$

Note that the coefficient of 0.0600 in equation 15 is incorrectly printed in his paper⁴. Inversion of the 3 x 3 matrix of the coefficients from equation 15 gives

$$\begin{aligned} R_F &= 0.9396 R(700) + 0.0595 G(536) + 0.0009 B(436) \\ G_F &= -0.0598 R(700) + 1.0442 G(546) + 0.0156 B(436) \\ B_F &= -0.0494 R(700) + 0.0628 G(546) + 0.9866 B(436) \end{aligned} \quad (16)$$

and normalization, so that $R_F + G_F + B_F$ equals $R(700) + G(536) + B(436)$, gives the specification of Guild's experimental primaries in terms of those of the NPL standards.

$$\begin{aligned} R_F &= 1.1315 R(700) + 0.0510 G(536) + 0.0009 B(436) \\ G_F &= -0.0720 R(700) + 0.8952 G(546) + 0.0156 B(436) \\ B_F &= -0.0595 R(700) + 0.0539 G(546) + 0.9835 B(436) \end{aligned} \quad (17)$$

As before, the above matrix then allows calculation of the spectral chromaticity co-ordinates $r_5(\lambda)$, $g_5(\lambda)$ and $b_5(\lambda)$ based on the NPL primaries.

$$\begin{aligned} r_5(\lambda) &= \frac{1.1315 r_4(\lambda) - 0.0720 g_4(\lambda) - 0.0595 b_4(\lambda)}{1.1833 r_4(\lambda) + 0.8387 g_4(\lambda) + 0.9779 b_4(\lambda)} \\ g_5(\lambda) &= \frac{0.0510 r_4(\lambda) + 0.8952 g_4(\lambda) + 0.0539 b_4(\lambda)}{1.1833 r_4(\lambda) + 0.8387 g_4(\lambda) + 0.9779 b_4(\lambda)} \\ b_5(\lambda) &= \frac{0.0009 r_4(\lambda) + 0.0156 g_4(\lambda) + 0.9835 b_4(\lambda)}{1.1833 r_4(\lambda) + 0.8387 g_4(\lambda) + 0.9779 b_4(\lambda)} \end{aligned} \quad (18)$$

Transformation of the $r_4(\lambda)$, $g_4(\lambda)$ and $b_4(\lambda)$ spectral chromaticity co-ordinates in Table 5 using equation 18 gives Table 6 in the worksheet *CIE1931_RGB.xls, Sheet 2, Cell A127*. The calculated values do not agree exactly with those published by Guild⁴. Most of the differences are minor (< 0.0007) but at 600 nm two coefficients differ by 0.0026. These differences are presumably a consequence of the greater precision of the computer matrix calculation or of possible arithmetical errors in Guild's manual calculation.

It was after Wright had published his table of spectral chromaticity co-ordinates in terms of the NPL primaries² that Guild realized that they were almost identical to his own data, which he subsequently published⁴. Figure 1 below compares Wright's spectral chromaticity co-ordinates $r_3(\lambda)$, $g_3(\lambda)$ and $b_3(\lambda)$, given as points every 10 nm, with Guild's values $r_5(\lambda)$, $g_5(\lambda)$ and $b_5(\lambda)$, shown as continuous lines with data every 5 nm. The good correspondence between the two independent sets of data allowed their combined use in the development of CIE 1931 RGB standard colorimetric observer.

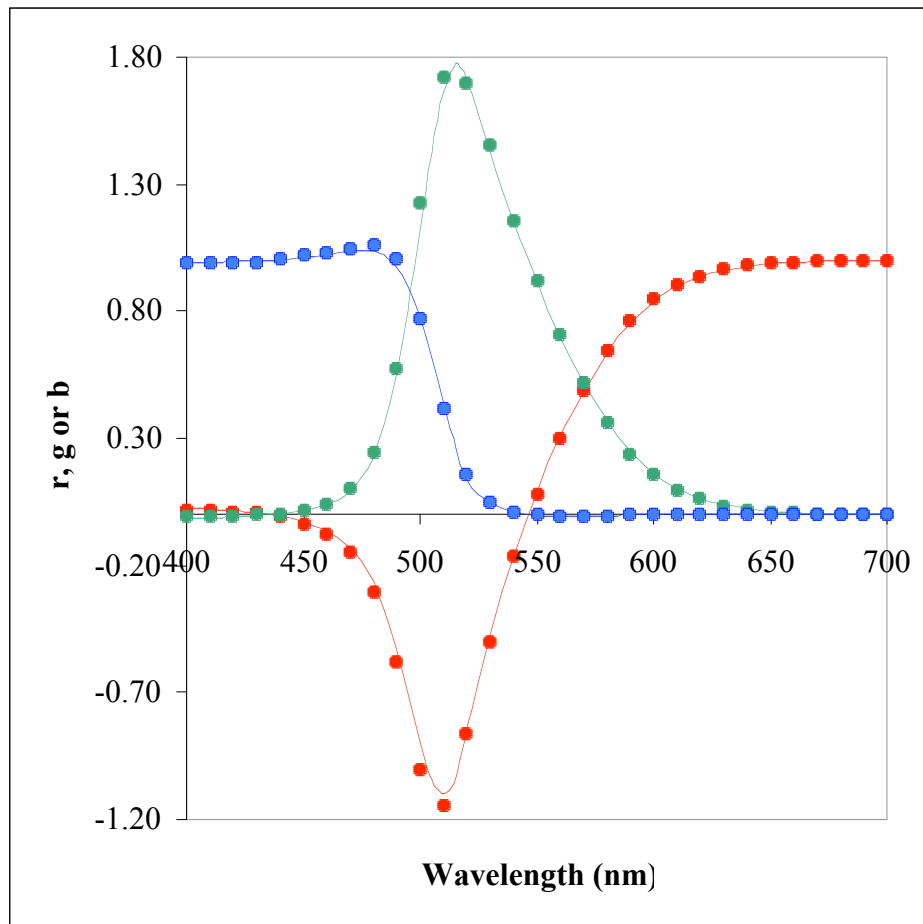


Figure 1. Comparison of Wright's (points) and Guild's (lines) spectral chromaticity co-ordinates (based on the NPL primaries) as a function of wavelength.

Units and relative luminance values

Before looking at the final transformation of Wright's and Guild's data into the CIE-RGB system, let's look at the problems of units for the primaries and the luminance of the actual monochromatic test light. Chromaticity co-ordinates for the colour of a given wavelength are specified by a colorimetric equation such as

$$C(\lambda) \equiv r(\lambda) R + g(\lambda) G + b(\lambda) B \quad (19)$$

Since the sum of $r(\lambda)$, $g(\lambda)$ and $b(\lambda)$ is unity, the spectral test color corresponds to one trichromatic unit. In the actual measurements, the test lights will have quite different intensities at different wavelengths depending upon the optical arrangement. A spectral light may actually correspond to say n trichromatic units where n will have a different value for each wavelength examined. Thus,

$$n(\lambda) C(\lambda) = n(\lambda) r(\lambda) R + n(\lambda) g(\lambda) G + n(\lambda) b(\lambda) B \quad (20)$$

where $n(\lambda) r(\lambda)$, $n(\lambda) g(\lambda)$ and $n(\lambda) b(\lambda)$ are the tristimulus values of the spectral test light. The luminance of this monochromatic light $L(\lambda)$ is given by

$$L(\lambda) = K_m E(\lambda) V(\lambda) \quad (21)$$

where K_m is a conversion constant equal to 683 lumen/W, $E(\lambda)$ is the spectral distribution of radiant power of the light seen by the observer, and $V(\lambda)$ the photopic efficacy function. Both $E(\lambda)$ and $V(\lambda)$ vary with wavelength. Thus,

$$L(\lambda) = K_m E(\lambda) V(\lambda) = n(\lambda) r(\lambda) L_R + n(\lambda) g(\lambda) L_G + n(\lambda) b(\lambda) L_B \quad (22)$$

where L_R , L_G and L_B are the luminance values of one trichromatic unit of each of the three primaries R, G and B. Thus, the number of trichromatic units of the test light is given by

$$n(\lambda) = \frac{K_m E(\lambda) V(\lambda)}{L_R r(\lambda) + L_G g(\lambda) + L_B b(\lambda)} \quad (23)$$

The monochromatic test light will be matched according to equation 20 with $n(\lambda)$ having the value given above.

When the tristimulus values $n(\lambda)r(\lambda)$, $n(\lambda)g(\lambda)$ and $n(\lambda)b(\lambda)$ are calculated for matching the colors of the wavelengths comprising a standard white light source of known spectral power distribution, and with luminance proportional to $E(\lambda)V(\lambda)$ at each wavelength, they are given the symbols $\bar{r}(\lambda)$, $\bar{g}(\lambda)$ and $\bar{b}(\lambda)$ and called color matching

functions. The units of the primaries are such that a mixture of equal amounts of each primary matches the standard white light. Wright and Guild based their calculations on the NPL standard white light but the CIE color matching functions are related to the equal energy white light that has the same radiant power in each wavelength interval.

Each of the color matching functions in equation 22 is integrated over the range of visible wavelengths, and the three values obtained are then normalized so that their sum is unity. This gives the chromaticity co-ordinates of a standard white light r_W , g_W and b_W which has a spectral radiant power distribution $E(\lambda)$ constituted of the sum of all the $n(\lambda)C(\lambda)$ colors. In practice, if values of $r(\lambda)$, $g(\lambda)$ and $b(\lambda)$ are available over the visible range of wavelengths at 5 nm intervals, the integrals can be replaced by summations (equation 24) and since the values of L_R , L_G and L_B are generally relative rather than absolute values, the value of the constant K_m can be omitted in the summations.

$$\begin{aligned} r_W &= \frac{\sum n r(\lambda)}{\sum n r(\lambda) + \sum n g(\lambda) + \sum n b(\lambda)} & g_W &= \frac{\sum n g(\lambda)}{\sum n r(\lambda) + \sum n g(\lambda) + \sum n b(\lambda)} \\ b_W &= \frac{\sum n b(\lambda)}{\sum n r(\lambda) + \sum n g(\lambda) + \sum n b(\lambda)} \end{aligned} \quad (24)$$

Table 7 in the worksheet “1931CIE_RGB.xls, Sheet 3, Cell A103” shows an example of such a calculation using Guild’s spectral chromaticity co-ordinates based on the filter primaries (Table 5). The units of the filter primaries were based on matching the NPL standard white with a mixture of equal amounts of each primary. The NPL standard white consists of the sum of the separate $n(\lambda)C(\lambda)$ colors and therefore the sums of the respective color matching functions over all wavelengths are equal, leading to equal chromaticity co-ordinates for this white light. In Table 7 the relative luminance of the monochromatic light to be matched is given as $E(\lambda)V(\lambda) \times 100$ according to Guild⁴. He had established by measurement, and slight adjustments, that the relative luminance values of one trichromatic unit of each of the red, green and blue filter primaries (L_{RF} , L_{GF} and L_{BF}) were respectively 1.000, 2.858 and 0.169.

Since this initial set of spectral chromaticity co-ordinates were transformed into a set based on the NPL primaries, we must now calculate the relative luminous intensities of one trichromatic unit of each of these primaries. We have already derived the equations of Guild’s filter primaries in terms of the NPL standard primaries (equation

17), with each set of primaries having their own particular units. If these equations are added together $R_F + G_F + B_F = R(700) + G(536) + B(436)$. If equation 17 is written in terms of the relative luminance values for one trichromatic unit of each of R_F , G_F and B_F ,

$$\begin{aligned} L_{RF} &= 1.000 = 1.1315 L_R + 0.0510 L_G + 0.0009 L_B \\ L_{GF} &= 2.858 = -0.07202 L_R + 0.8952 L_G + 0.0156 L_B \\ L_{BF} &= 0.169 = -0.0595 L_R + 0.0539 L_G + 0.9835 L_B \end{aligned} \quad (25)$$

where L_R , L_G and L_B are respectively the values of the relative luminance of one trichromatic unit of each of the NPL primaries R(700), G(536) and B(436). Solution of equation 25 gives $L_R = 0.7373$, $L_G = 3.2514$ and $L_B = 0.0384$ (*1931CIE_RGB.xls, Sheet 3, Cell A197 to A233*). Since the same standard white light was used in establishing the units for both sets of primaries, note that the relative luminance of this source obtained by summation of the relative luminance values of the primaries is identical for both sets of primaries, i.e. $1.000 + 2.858 + 0.169 = 0.7373 + 3.2514 + 0.0384$. If the relative luminance values of the NPL primaries are normalized so that $L_R = 1.000$, then $L_G = 4.4102$ and $L_B = 0.0521$. The validity of these values can be verified by carrying out a calculation based on equation 23 to establish the chromaticity co-ordinates of the NPL standard white light based on the NPL monochromatic primaries. These chromaticity co-ordinates are all essentially equal to 0.333 (*Table 8, 1931CIE_RGB.xls, Sheet 4, Cell A5 and G2*).

Establishing the data for the 1931 CIE RGB system

Wright's and Guild's values of the spectral chromaticity co-ordinates were based on matching monochromatic spectral lights with the three NPL primaries. Their units were based on equal amounts of each primary matching the NPL standard white light. Guild calculated the average of his and Wright's experimental and interpolated chromaticity co-ordinates and finally tabulated data from 380 to 780 nm at 5 nm intervals. For the values obtained by interpolation and extrapolation, Guild used a graphical technique⁴ involving the r-g and g-b chromaticity diagrams. Values of chromaticity co-ordinates were interpolated at 1 nm intervals. On large-scale chromaticity plots, the best curve was

drawn through the experimental points. Then any deviant interpolated points that were not on this curve were laterally displaced onto the curve. Finally, the spacing of the points with respect to varying wavelength was regularized.

The averaged Guild-Wright chromaticity co-ordinates taken from Guild's paper⁴ are given in Table 9 (*1931CIE_RGB.xls, Sheet 5, Cell A7*). The published values differ slightly from the averages calculated from Tables 4 and 6 (*1931CIE_RGB.xls, Sheet 2, Cell A127*) but the details of the smoothing and averaging procedures used by Guild were never adequately described. A slight modification of the values of the relative luminance of one trichromatic unit of each primary was also required so as to achieve equal values of the three chromaticity co-ordinates of the NPL standard white light ($L_R = 1.000$, $L_G = 4.390$ and $L_B = 0.048$). The calculation of the color matching functions based on these values and on equation 23 is given in Table 10 (*1931CIE_RGB.xls, Sheet 5, Cell J7*).

During the discussions prior to the 1931 CIE meeting it was decided that the units of the NPL primaries should be based on a mixture of equal quantities of each primary matching the colour of the equal energy white. This is a theoretical spectral power distribution having the same power in every wavelength interval. In order to change the units, the values of the chromaticity coefficients for matching this white with the NPL primaries are required. They are calculated using equation 23 and the values of $L_R = 1.000$, $L_G = 4.390$ and $L_B = 0.048$ but with $E(\lambda)$ equal to 100 at all wavelengths. The results of this calculation are $r_W = 0.3011$, $g_W = 0.3148$ and $b_W = 0.3841$ (See Table 11, *1931CIE_RGB.xls, Sheet 5, Cell R6*).

If R, G and B represent the NPL primaries in units such that equal amounts match the equal energy white, and R', G' and B' the same primaries such that equal amounts match the NPL standard white light, the matching of the equal energy white (EEW) is given by the equations

$$\begin{aligned} \text{EEW} &= 0.3333 R + 0.3333 G + 0.3333B \\ &= 0.3011 R' + 0.3148 G' + 0.3841 B' \end{aligned} \quad (26)$$

Thus, one trichromatic unit of the red primary R' in the old units equals $0.3333 / 0.3011$ units of the red primary R in the new units. For converting Table 9 into a set of chromaticity co-ordinates based on the new units related to the equal energy white standard, the conversion factors are therefore

$$r = \frac{0.3333}{0.3011} r' \quad g = \frac{0.3333}{0.3148} g' \quad b = \frac{0.3333}{0.3841} b' \quad (27)$$

The calculated values of r, g and b will require normalization so that their sum is unity.

These conversion factors for changing the units of the NPL primaries should allow calculation of the spectral chromaticity co-ordinates of the CIE 1931 RGB system from the averaged Wright-Guild values. Unfortunately, the calculation gives values with small but significant deviations from the CIE data. This is illustrated in the worksheet (*1931CIE_RGB.xls, Sheet 6 in Table 12, Cell J8 and in Table 13, Cell R8*). In addition, the chromaticity co-ordinates for the equal energy white do not agree with the values calculated from the equations for transformation of the Wright-Guild averaged data into the CIE 1931RGB system as given by Smith and Guild⁵. From equation 26,

$$\begin{aligned} R' &= \frac{0.3333}{0.3011} R = 1.10698 R & G' &= \frac{0.3333}{0.3142} G = 1.05894 G \\ B' &= \frac{0.3333}{0.3850} B = 0.86781 B \end{aligned}$$

Then for $B' = B$,

$$R' = \frac{1.10698}{0.86781} R = 1.27560 R \quad G' = \frac{1.05894}{0.86781} G = 1.22025 G \quad B' = B \quad (28)$$

Smith and Guild⁵ give $B' = B$, $R' = 1.27663 R$ and $G' = 1.22460 G$, values which are slightly different. These authors⁵ described how the CIE 1931 RGB system data were calculated in a long paper in 1932. A key sentence appears on page 79 of their paper. “Alterations [*to the averaged Wright-Guild spectral chromaticity co-ordinates*] were called for owing to a numerical error in one of Wright’s figures, and to other irregularities present in the unsmoothed mean of Guild’s and Wright’s results”. The error in Wright’s data was not identified but is presumably at or around 500 nm. Unfortunately, a table of the “smoothed” Wright-Guild chromaticity co-ordinates that were converted into the CIE data was never published. The “smoothed average” spectral chromaticity co-ordinates of Table 14 are compared with the averages of the values of Guild and Wright (Table 9) in the worksheet *1931CIE_RGB.xls, Sheet 7, Cell A7 and G10*.

Since equal amounts of each set of primaries, in their own units, should match the equal energy white, the chromaticity co-ordinates of the equal energy white in the old

units based on the NPL white are calculated from the Guild-Smith published transformation equations based on reversing equation 28.

$$\begin{aligned}
 R + G + B &= \frac{1}{1.27663} R' + \frac{1}{1.22460} G' + B' \\
 &= 0.3013 R' + 0.3141 G' + 0.3846 B'
 \end{aligned}
 \tag{29}$$

This gives the chromaticity co-ordinates of the equal energy white, based on matching with the NPL primaries with units such that a mixture of equal amounts matches the NPL standard white light. The values differ slightly from the values calculated in Table 11 (*1931CIE_RGB.xls, Sheet 5, Cell R6*) based on Guild’s published values of the averaged chromaticity co-ordinates.

Smith and Guild⁵ did not tabulate the set of “smoothed” chromaticity co-ordinates (based on matching with the NPL white) and they did not specify the relative luminance values stating that “the assumed values were in good agreement with the experimental determinations”. I derived a set of spectral chromaticity co-ordinates corresponding to Smith and Guild’s “smoothed” values and the related relative luminance values by working backwards from the CIE 1931 RGB specification. The values are given in Table 14 (*1931CIE_RGB.xls, Sheet 7, Cell A11*). The calculated values of the relative luminance of one trichromatic unit of each of the primaries are $L_R = 1.000$, $L_G = 4.4036$ and $L_B = 0.0471$. When these relative luminance values are used with the “smoothed” chromaticity co-ordinates from Table 14 and equation 23 to calculate the chromaticity co-ordinates of the equal energy white (with units based on matching the NPL standard white), the values obtained were $r_w = 0.3013$, $g_w = 0.3140$ and $b_w = 0.3847$. These values are in good agreement with those calculated using the transformation equations of Smith and Guild (See Tables 15 and 16, *1931CIE_RGB.xls, Sheet 7, Cell P11 and V10*).

Few details of the calculation of the 1931 CIE RGB data were published but it has been possible to clarify the main steps that were undertaken. It is at this stage of the calculations that the available information is weakest and often absent. I assume that this was a consequence of the haste with which the data were being assembled just prior to the 1931 CIE meeting⁶.

The “smoothed” set of chromaticity co-ordinates in Table 14 in the worksheet *1931CIE_RGB.xls, Sheet 8, Cell A6* were transformed into the CIE RGB data⁷ using

equation 27 but with chromaticity co-ordinates of the equal energy white given by $r_w = 0.3013$, $g_w = 0.3140$ and $b_w = 0.3847$ for matching the the NPL primaries with units such that equal amounts of a mixture of the three primaries matches the NPL standard white. The relative luminance values of one trichromatic unit of each of the NPL primaries, $L_R = 1.000$, then $L_G = 4.4036$ and $L_B = 0.0471$ differ somewhat from those published by Guild but are consistent. The values of the calculated CIE 1931 RGB chromaticity co-ordinates are given in Table 15 (*1931CIE_RGB.xls, Sheet 8, Cell J7*) and compared with the actual CIE values⁷ in Table 16 (*1931CIE_RGB.xls, Sheet 8, Cell O7*).

The relative luminance values of the primaries for the CIE-RGB system were calculated from the chromaticity co-ordinates of the EEW using equation 28 $r_w = 0.3013$, $g_w = 0.3140$ and $b_w = 0.3847$ (*1931CIE_RGB.xls, Sheet 8, Cell P97*). Values obtained were $L_R = 1.000$, then $L_G = 4.5907$ and $L_B = 0.0601$ as given by the CIE⁷. The calculated and tabulated CIE 1931 RGB color matching functions are compared in Tables 17 and 18 (*1931CIE_RGB.xls, Sheet 8, Cells V4 and AC6*) and are essentially identical. Note that the color matching functions in Table 17 are 100 times greater than those in Table 18.

Conclusion

Published accounts of the calculations used to derive the CIE 1931 RGB chromaticity co-ordinates and color matching functions are at best sketchy and on occasions data has not been published. This paper, and the accompanying worksheet *1931CIE_RGB.xls*, present in some detail the sequence of steps leading from the original experimental measurements of Wright and of Guild to the CIE 1931 RGB colorimetric system. It is not an easy story to unravel but the data presented here is probably a close representation of the actual calculations that were carried out by Guild just prior to the CIE meeting in Cambridge in 1931 when the CIE 1931 standard observer was first specified. The further calculations transforming the RGB data into the XYZ of the CIE 1931 standard observer have been described by Fairman, Brill and Hemmendinger⁸.

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