1051-320 – Linear Mathematics for Imaging

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Office Hours TBD, and by appointment

Webpages:  http://www.cis.rit.edu/class/simg320/index.html
http://www.cis.rit.edu/people/faculty/easton/LinearMath/index.html

Meeting Rooms/Times:  TTh, 2:00PM – 3:50PM, Room 76-1230
Additional OPTIONAL times for group problem sessions.

Prerequisites:
Calculus I - IV

Details:
Homework will be assigned, and is to be handed in on time (adjustments will be considered in advance). Problems will be graded and solutions handed out.

Homework – 30% (Assignments usually given Mondays, due 1 week later in class.)
Midterm Exam (anticipated date: W 1/15/2006) – 30%
Final Exam (cumulative) – 40%

My philosophy on exams: they are intended to test understanding of material, which is the ability to assimilate concepts and synthesize useful results for applications. This is not the same as the ability to parrot discussions of concepts or replicate the solutions to homework problems. I particularly like problems that appear to be difficult but actually are easy if you see the connection. My exams seem to have a reputation among students.

Subject:
This course introduces mathematical tools for describing imaging systems. In other words, the course develops mathematical models of imaging systems and applies them to problems relevant in imaging.

Text materials:
Many books are available that touch on aspects of mathematical models of imaging systems and parameters, though the appropriate single book which encompasses all (or even most) aspects at the appropriate level has yet to be published. For now, the text materials come from the manuscript Linear Systems with Applications to Imaging, which have been copied and will be sold at cost. Of course, many other text resources are available, including:


Unfortunately, Gaskill’s book does not give sufficient consideration to the derivation and application of discrete (sampled) systems. This subject is covered to some extent in my book and other resources are listed on subsequent pages.
Mathematical Foundations of Linear Systems:
1. For review, Schaum's Outlines on Calculus, Linear Algebra, Vector Analysis, Matrices, and Complex Variables and Schaum's Mathematical Handbook
4. Any of several books on mathematical physics, e.g., Kreysig, Arfken, Byron and Fuller, ...

Fourier Transforms in Mathematics:

Fourier Transforms in Physics/Engineering:


**Discrete Fourier Transforms:**

**Linear Systems and Optical Imaging:**

**Image Recovery:**

**Useful References from Magazines and Journals:**

Other books containing useful discussions of imaging subjects:
2. *Image Reconstruction in Radiology*, J. Anthony Parker, CRC Press, 1990, (of much more general application than the title indicates; written for medical students and radiologists, does not require a "high" level of mathematical knowledge, useful intuitive discussions of imaging principles) RC78.7.D53 P36.

Computing Resources:

Many computational software packages are available that would be helpful when learning the material in this class. CIS has selected **IDL™** from RSI as its "standard" package. It is installed on the UNIX workstations in the Center, and also is available for purchase from CIS; the price for a full working copy is $200 (or thereabouts), vs. the list price of »$1500. Other packages exist, including **Mathematica™** (available on RIT VAX), **MathCad™**, **Matlab™**, and **Scientific Workplace™**. All these packages allow computations involving most aspects of matrix algebra and complex analysis to be evaluated quickly and (more or less) painlessly. They also have graphing routines which may assist in visualizing concepts. In my opinion, most of the packages have a fairly steep learning curve -- you can't do much that is useful very quickly. The programs also have their respective advantages and disadvantages, e.g., my opinion is that the interfaces to **Mathematica™** and **MathCAD™** are not very intuitive, which means that new users have to travel the learning curve. Conversely, experienced users are rewarded by quicker answers.

For some specific applications in 1-D linear systems, my program for PCs ("**SIGNALS**") may be useful. It was written with the intent of being easy to use, though you must decide for yourself if it succeeds. It is available gratis if you supply the diskette, and is installed on the PCs in the CIS computing complex (click on the **SIGNALS** icon or type "signals" at the command prompt) and may be downloaded from the Blackboard site or from the CIS website at:

http://www.cis.rit.edu/resources/software/index.html

The user manual also is available online at:

This course presents mathematical descriptions for functions and systems and demonstrates their application to solving imaging problems. Alternative descriptions of images and imaging systems will be derived that are conceptually based on the \textit{projection} of one vector onto another, which leads naturally to the concepts of \textit{orthogonal} vectors and \textit{orthogonal functions}.

\textbf{Course Outline:}

\textbf{I. Introduction and motivation}
\begin{itemize}
  \item A. The Imaging “Chain”
  \item B. Mathematical expression for an imaging system: \( \mathcal{O} \{ f(x, y, \ldots) \} = g(x, y, \ldots) \)
  \item C. The three imaging “tasks” (direct problem, inverse problem, system analysis/synthesis)
  \item D. Examples of imaging systems and mathematical models
    \begin{itemize}
      \item 1. “imaging” of optical rays
        \begin{itemize}
          \item a. imaging without a lens
          \item b. “imaging” of optical rays by selection with pinhole
          \item c. multiple pinholes
        \end{itemize}
      \item 2. “redirection” of rays by mirror or lens
      \item 3. Examples of “imaging tasks” in medicine
        \begin{itemize}
          \item a. Gamma-ray imaging
          \item b. Radiography
          \item c. Computed Tomographic Radiography (CT)
        \end{itemize}
      \item 4. Image “quality”
    \end{itemize}
  \item E. Necessity to constrain possible action of system for mathematically tractible description
\end{itemize}

\textbf{II. Functions}
\begin{itemize}
  \item A. Continuous and Discrete Domains
  \item B. Continuous and Discrete Ranges
  \item C. Discrete Domain and Range, “Digital” Functions
  \item D. Periodic, Aperiodic, and Harmonic Functions
  \item E. Symmetry Properties of Functions
\end{itemize}

\textbf{III. Vector and Matrix Concepts}
\begin{itemize}
  \item A. Scalars and vectors with real-valued components
    \begin{itemize}
      \item 1. vector addition
      \item 2. scalar multiplication
      \item 3. triangle inequality
      \item 4. scalar (dot) product
        \begin{itemize}
          \item a. length (norm)
          \item b. projection of one vector onto another
          \item c. Cauchy-Schwarz inequality
        \end{itemize}
      \item 5. Matrices as multiple scalar products
        \begin{itemize}
          \item a. matrix-vector product
          \item b. matrix-matrix product
          \item c. square matrices
          \item d. diagonal matrices
          \item e. identity matrix
        \end{itemize}
      \item 6. vector spaces
      \item 7. basis vectors
        \begin{itemize}
          \item a. Constructing different sets of basis vectors
          \item b. Gram-Schmidt orthogonalization
        \end{itemize}
\end{itemize}
c. rotation of vectors

IV. Complex numbers
A. as real-valued vectors
B. representations
   1. real/imaginary parts
   2. magnitude/“phase”
C. Graphical representation on phasor/Argand diagram
D. complex arithmetic
E. Euler relation, deMoivre's theorem
F. Complex functions of a real variable, e.g., \( f(x) \)
G. Complex functions of a complex variable, e.g., \( w(z) \)
   1. examples, e.g., \( z^+, z^*, z^{-1} \)
   2. Introduction to path integrals

V. Vectors with complex-valued components
A. inner product
   1. norm (length)
   2. projections
B. Operators on vectors, matrices
   1. matrix-vector multiplication as multiple scalar products or as simultaneous linear
equationsrepresentations of arbitrary vectors, projection onto different basis sets
   2. imaging problems in matrix-vector form
      a. matrix inverses
      b. pseudoinverses
      c. shift invariance, circulant matrices

VI. Eigenvectors and Eigenvalues
A. diagonal forms of matrix operator
B. diagonalization operators
C. diagonalization of circulant matrix
D. discrete Fourier transform (DFT)

VII. Matrix-Vector Formulations of the Imaging “Tasks”
A. Inverse Task
   1. matrix inverse
   2. Solution of inverse problem by diagonalization
B. Matrix-vector formulation of system analysis
C. Alternative representation of shift invariant system, rotation of basis vectors

VIII. Functions of continuous variables \( f(x) \)
A. Classification
   1. domain and range (real/complex, continuous/discrete)
   2. form (linear/nonlinear, periodic, harmonic)
   3. symmetry (even/odd)
B. Representations obtained by projecting onto different sets of basis functions
   1. inner product, relation to scalar product
   2. orthogonal/orthonormal sets of functions
   3. power series representations, Taylor series
C. Representations of functions
   1. real/imaginary parts
   2. magnitude (modulus)/phase
   3. Argand-phasor diagram (Lissajou figures)

IX. Continuous-Domain Analogues of VectorOperations
A. Inner Product of Continuous Functions
B. Projections of Continuous Functions
C. Special Functions
   a. 1-D real-valued functions (constant, RECT, TRI, SGN, COS, CHIRP, GAUS)
   b. 1-D Dirac delta function (impulse) and related functions
   c. 1-D complex-valued sinusoid

X. Mathematical representation of systems, operators
   A. “Linearity”
   B. “Shift (space-, time-) invariance”
   C. Linear and shift-invariant (LSI) systems, action of system is convolution (filtering)
   D. Representations of systems
      1. linear and discrete (matrix-vector multiplication)
      2. linear and continuous (superposition integral)
      3. LSI and discrete (circulant matrix, diagonalizing transformation)
      4. LSI and continuous (convolution integral)
      5. impulse response/point-spread function as descriptor of system
   E. Crosscorrelation and autocorrelation

XI. Implementation of the DFT

XII. Fourier transforms of 1-D continuous functions
   A. Direct integration
   B. Fourier transforms of special functions
   C. Theorems of the Fourier transform