

A Survey of Clear Air Turbulence and its Effect on “Seeing”

B. Roy Frieden

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Abstract

An introduction to random fluctuations in the refractive index of air and their effect on optical images, including the impact on MTF.

The aim of these notes is to study, in an introductory way, random fluctuations in the refractive index of air and their effect on optical images. Image quality is measured by a modulation transfer function (MTF) that characterizes the turbulent viewing, and also by a minimal resolution length. It will be seen that the MTF describing image formation for either “looking up” or “looking down” is describable by a $\frac{5}{3}$ -power exponential falloff with frequency. Also, the ultimate resolution for “looking up” follows a completely different dependence on altitude from “looking down”. In fact, for “looking up” the ultimate resolution increases indefinitely with altitude of source, whereas for “looking down” the ultimate resolution length approaches a constant, with altitude, for a source on the ground. The constant length is somewhere between 5 cm and 10 cm in extent.

This review will differ from others, cited below, in a few ways. First, it will necessarily be briefer. Cited references should help the interested student fill out desired sections. Second, we will take a “motivational approach” – e.g., why are “structure functions”, which seem at first such an alien concept, necessary? Third, we will use an intuitive, physical analysis of Lee and Harp (1969) as a basic approach to the problem, instead of the rather forbidding differential-equations approach of the Russian school that is usually taken up.

These notes will not consider the important phenomenon of radiative transfer into the atmosphere from the earth or oceans. However, the notes on “Optical Properties of the Oceans” will provide some coverage of this subject.

1 A Synopsis of the Approach

First, we introduce the purely mathematical concepts of correlation function and power spectra, relate the two, take up the special cases of shift invariance and isotropy, and relate three-dimensional quantities (which appear in the later

theory) to one-dimensional quantities (the physical measurables). Next, we specialize to the important case of *power laws* for all these quantities.

Then we introduce the physics of the situation and describe the source of turbulence – temporally random fluctuations in the real index of refraction function for the atmosphere. This obviously perturbs wavefronts that pass through it, and this in turn must adversely affect “seeing” through the MTF for spatial imagery.

We relate the long-term average MTF to an atmospheric quantity of significance, the “wave structure function” $D_w(r)$. This establishes the need for analyzing this function. In particular, how is D_w related to a measurable indicator of the refractive index randomness, such as its power spectrum $\phi_n(f)$?

This question demands that we physically analyze an optical wave (such as a spherical wave) as it propagates through a randomly varying medium of refraction. Lee and Harp’s model is used. It describes the medium as a contiguous sequence of planar scatterers, each plane containing a superposition of diffraction gratings that define the random refractive index variations. The result of this calculation is a basic relation between the predicted wave structure function D_w and the general index of refraction power spectrum $\phi_n(f)$. This link between D_w and ϕ_n is the key calculation of the entire study.

Next, it is found that $\phi_n(f)$ follows a simple power-law dependence $f^{-\frac{11}{3}}$, based on some physical, atmospheric considerations. Using the previously discussed relation between $D_w(r)$ and $\phi_n(f)$, we calculate $D_w(r)$ as obeying a $\frac{5}{3}$ -power law in r . Substitution into the MTF expression described previously then gives a $\frac{5}{3}$ -power exponential falloff for this point spread function. In turn, the spectrum (Fourier transform) of this function is evaluated to obtain the long-exposure point spread function. This is the “seeing disk” in astronomy.

An independent calculation yields the “short-exposure” MTF. This is shown to exceed the “long-exposure” MTF at all frequencies. Therefore, a sequence of short-term exposures contains more information than one long-term exposure of the same scene, a fact that has been exploited to enhance astronomical viewing.

The $\frac{11}{3}$ -power law for $\phi_n(f)$, which is called the “Kolmogoroff spectrum”, is next applied to establish the limiting resolution length $\delta\ell$ vs. L , where L is the distance to a point source for looking up and for looking down through the atmosphere. It is found that for looking up, $\delta\ell$ increases as $L^{\frac{7}{5}}$ for small L and linearly with L once L exceeds about 12 miles (20 km). In other words, seeing gets increasingly worse as the source recedes.

However, for looking down, $\delta\ell$ approaches a constant of about 10 cm once L increases beyond 12 miles. The upshot is that we can be “seen” but cannot “see” very well, vis-à-vis an observer in space. We live, effectively, in a glass house made of one-way mirrors!

The preceding results on $\delta\ell$ presume limitation on resolution due to turbulence alone. Actually, there is some interaction with lens diameter D as well. However, to achieve the above resolution effects, D need only be of finite size, and the necessary dependence of D upon L is found. For looking down, this is linear with L once L exceeds 12 miles.

Finally, the turbulent wavefront is analyzed for its dominant component

forms. Components due to constant deformation, tilt, spherical deformation, and hyperbolic deformation are analyzed for contribution to the overall turbulence. It is found that tilt is by far the dominant component. In fact, if the wave structure function D_w had a $\frac{6}{3}$ -power dependence instead of its actual $\frac{5}{3}$ dependence, tilt would be the *sole* constituent of a turbulent wavefront.

Next, we use the speckle interferometry methods of Labeyrie, Knox and Thompson, and Fienup for partially removing turbulence effects.

Sequence of Operations

1. Measure $\phi_n(f)$, the power spectrum of the index of refraction
2. Use $\phi_n(f)$ to find $D_w(f)$, the wave structure function, $D_w(f) \propto r^{\frac{5}{3}}$
3. $\{D_w(f)\} \rightarrow MTF$
4. $\mathcal{F}^{-1}\{MTF\} = psf$

2 Some Mathematical Relationships

We will be fundamentally interested in correlations both for the refractive index function and for the optical amplitude and phase functions after propagation through the random medium. For a general function f of this type, the correlation C_f between values of f evaluated at three-dimensional space points $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ is:

$$C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) \equiv \langle f^*(\vec{\mathbf{r}}_1) f(\vec{\mathbf{r}}_2) \rangle \quad (1)$$

The brackets used here and in subsequent equations will represent an *ensemble average* over time. The asterisk denotes a complex conjugate, although f will always be real in what follows.

The spectrum of $f(\vec{\mathbf{r}})$ is defined as $a(\vec{\mathbf{k}})$:

$$f(\vec{\mathbf{r}}) = \int d\vec{\mathbf{k}} a(\vec{\mathbf{k}}) e^{i\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}} \quad (2)$$

where $|\vec{\mathbf{k}}| = k$ is a wave number with units of radians per length. All integration limits are over the entire 3-D volume unless otherwise indicated. Substitution of eq.(2) into eq.(1) shows that:

$$C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = \int d\vec{\mathbf{k}} e^{i\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}_1} \int d\vec{\mathbf{k}}' e^{i\vec{\mathbf{k}}' \cdot \vec{\mathbf{r}}_2} \langle a^*(\vec{\mathbf{k}}) a(\vec{\mathbf{k}}') \rangle \quad (3)$$

This may be inverted to give:

$$\langle a^*(\vec{\mathbf{k}}) a(\vec{\mathbf{k}}') \rangle = \int d\vec{\mathbf{r}}_1 \int d\vec{\mathbf{r}}_2 C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}_1 - \vec{\mathbf{k}}' \cdot \vec{\mathbf{r}}_2)} \quad (4)$$

3 Shift Invariance and Power Spectrum

Let us now assume a condition of shift invariance for C_f , i.e.,

$$C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = C_f(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) \quad (5)$$

Substitution into eq.(4) shows, after a change of variable to $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, that

$$\langle a^*(\vec{\mathbf{k}}) a(\vec{\mathbf{k}}') \rangle = \delta(\vec{\mathbf{k}} - \vec{\mathbf{k}}') \phi(\vec{\mathbf{k}}') \quad (6)$$

where

$$\phi(\vec{\mathbf{k}}) \equiv \int d\vec{\mathbf{r}} C_f(\vec{\mathbf{r}}) e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}})} \quad (7a)$$

This quantity is called the *power spectrum*, since from eq.(6) it is related to the spectral quantity $a(\vec{\mathbf{k}})$ through its “power” $\langle |a(\vec{\mathbf{k}})|^2 \rangle$. Because from eq.(1), C_f is even in $\vec{\mathbf{r}}$, eq.(7a) simplifies to

$$\phi(\vec{\mathbf{k}}) \equiv \int d\vec{\mathbf{r}} C_f(\vec{\mathbf{r}}) \cos(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}) \quad (7b)$$

Claim 1 Conjecture 2 Claim 3 $\mathcal{F}_3 \{C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)\} = \langle a^*(\vec{\mathbf{k}}_1) a(\vec{\mathbf{k}}_2) \rangle$

3.1 Isotropy

All correlation function C_f will also be assumed to depend on $\vec{\mathbf{r}}$ only through its magnitude r , i.e.,

$$C_f(\vec{\mathbf{r}}) = C_f(r) \quad (8)$$

How, then, does $\phi(\vec{\mathbf{k}})$ depend on C_f in this case? For example, is ϕ then itself isotropic in $\vec{\mathbf{k}}$?

Since $\vec{\mathbf{k}}$ is assumed constant in eq.(7), we will evaluate the integral there with axes having the z -direction along the direction of $\vec{\mathbf{k}}$,

Then $\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} = kr \cos(\theta)$. We also may use $d\vec{\mathbf{r}} = dr r d\theta r \sin(\phi) d\phi$. Then both the $d\phi$ and $d\theta$ integrations are easily evaluated, yielding:

$$\phi(\vec{\mathbf{k}}) \equiv \phi(k) = \frac{4\pi}{k} \int dr C_f(r) \cos(kr) \quad (9)$$

Hence $\phi(k)$ really *is* isotropic now.

3.2 Definition of Structure Function $D_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$

This function is defined as:

$$D_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) \equiv \langle [f(\vec{\mathbf{r}}_1) - f(\vec{\mathbf{r}}_2)]^2 \rangle. \quad (10)$$

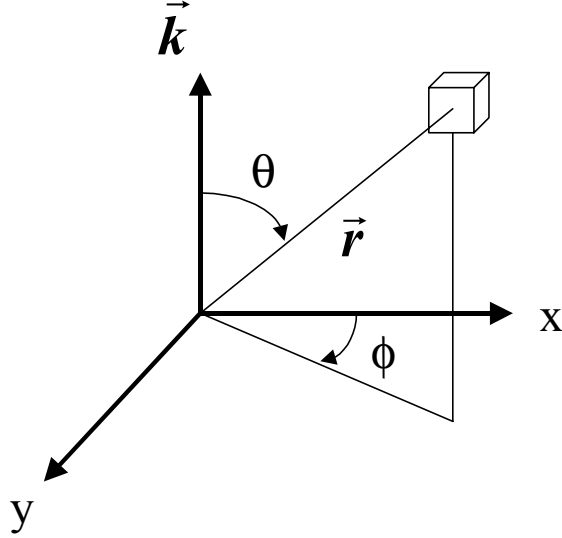


Figure 1: Coordinate system for $\vec{\mathbf{k}}$.

After squaring, evaluating the expectation term-wise, and using shift invariance, we have:

$$D_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) \equiv 2 [C_f(0) - C_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)]. \quad (11)$$

In fact, D_f is nearly always found from this relation rather than from eq.(10).

Claim 4 Remark 5 *Small value of the structure function $D_f \implies$ “low” turbulence*

3.3 Effect of Isotropy

Quantity D_f may be evaluated by inverting eq.(7a) and substituting into eq.(11) to give:

$$D_f(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) = 2 \int d\vec{\mathbf{k}} \phi(\vec{\mathbf{k}}) \left[1 - e^{-ik \cdot (\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)} \right] \quad (12)$$

Now let us assume isotropy in both $C_f(r)$ and $\phi(k)$, as above. We want to evaluate eq.(12) under this condition, and so take the analogous steps leading up to eq.(9). Construct the $\vec{\mathbf{k}}$ -axes with the z -direction along the fixed direction $(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)$. Now $d\vec{\mathbf{k}} = dk k d\theta k \sin(\theta) d\phi$. After integrations over $d\phi$ and $d\theta$, we have

$$D_f(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = D_f(r) = 8\pi \int_0^\infty dk k^2 \phi(k) \left[1 - \frac{\sin(kr)}{kr} \right] \quad (13a)$$

where

$$\vec{\mathbf{r}} = |\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2| \quad (13b)$$

3.4 Relation Between Three-Dimensional and One-Dimensional Power Spectra

By analogy with eq.(7b), which describes three-dimensional quantities, we may relate one-dimensional correlation $C_f(x)$ and power spectrum functions $V(k)$ through:

$$\int_0^{\infty} dx C_f(x) \cos(kx) = V(k), \text{ 1-D power spectrum} \quad (14)$$

$V(k)$ often is known experimentally from one-dimensional correlation measurements, whereas $\phi(k)$ appears in theoretical expressions. Therefore, to check theory against experiment, we first have to relate $V(k)$ to $\phi(k)$.

Remark 6 *Observe $C_f(x)$, find $V(k)$*

This may be done easily by integrating eq.(9) indefinitely to yield:

$$\begin{aligned} \frac{1}{4\pi} \int dk k \phi(k) &= - \int dr r C_f(r) \frac{\cos(kr)}{r} \\ &= - \int_0^{\infty} dr C_f(r) \cos(kr) = -V(k) \end{aligned} \quad (15)$$

by eq.(14). Therefore, differentiation yields:

$$2\pi k \phi(k) dk = -dV(k)$$

or

$$\boxed{\phi(k) = -\frac{1}{2\pi k} \frac{dV(k)}{dk}} \quad (16)$$

where $\phi(k)$ is the 3-D isotropic power spectrum and $V(k)$ is the measurable 1-D power spectrum, Note again that eq.(16) assumes shift invariance and isotropy.

Remark 7 *This allows the 3-D power spectrum $\phi(k)$ to be calculated from the measurable 1-D power spectrum $V(k)$*

3.5 One-Dimensional Structure Function

By analogy with eq.(12),

$$D_f(x) = 2 \int_{-\infty}^{+\infty} dk V(k) [1 - \cos(kx)], \quad (17a)$$

where $x = x_1 - x_2$.

Remark 8 *Tatarski, eq. 1.15, Tatarski's $W(\omega) = \text{Frieden's } |V(k)|$, $W(\omega) \geq 0 \implies W(\omega) = |V(k)|$*

But of course, the one-dimensional and three-dimensional structure functions are identical, giving:

$$D_f(r) = D_f(x) \quad (17b)$$

3.6 The All-Important Power Law Case

Remark 9 *The 1-D structure function is a power law.*

It will be found that based on physical grounds:

$$D_f(r) = \alpha r^p, \quad \alpha \text{ is a constant} \quad (18)$$

where f is the random refractivity function of the atmosphere. Also, $p = \frac{2}{3}$ in this case. What are the ensuing $V(k)$ and $\phi(k)$ functions?

To solve this problem, it is easier to *assume* that $V(k)$ is of the form:

$$V(k) = A k^{-(p+1)}, \quad 0 \leq p \leq 2 \quad (19)$$

and then find $D_f(r)$ to check if this is of the form in eq.(18). Substitution of the form in eq.(19) into eq.(17a) yields:

$$D_f(r) = \frac{2A\pi}{\Gamma(p+1) \sin\left(\frac{\pi p}{2}\right)} \quad (20)$$

which checks.

Notation 10 *Gradshteyn and Ryshik, p. 421, 3.7619*

Then, use eq.(19) in eq.(16) to obtain:

$$\phi(k) = \beta k^{-(p+3)}, \quad \beta \text{ is a constant} \quad (21)$$

as the power spectrum.

Remark 11 $\phi(k) = -\frac{1}{2\pi k} \frac{dV}{dk}$

In the important case of $p = \frac{2}{3}$, eq.(21) yields the ‘‘Kolmogoroff spectrum’’ or ‘‘ $\frac{11}{3}$ -power law’’,

$$\phi(k) = \beta k^{-\frac{11}{3}} \quad (22)$$

We will have much use for this below.

Remark 12 *Roddier, Prog. Optics XIX, 1981, p.286, eq.(2.4)*

4 The Physics of Turbulence

At optical wavelengths, the refractive index n at any point $\vec{\mathbf{r}}$ at time t obeys:

$$\begin{aligned} n &= n_0 + n_1 \\ &= n_0 + (0.776 \times 10^{-4}) \frac{P}{T} \end{aligned} \quad (23)$$

where P is the local pressure in millibars and T is the temperature in Kelvins. P does not usually change rapidly with time, but T does, due to fluctuations

caused by the local wind velocity. It is this known physical link from wind velocity to temperature to refractive index that ultimately gives us an optical theory of turbulence.

By differentiating eq.(23) due to random temperature changes ΔT , we get a random change in refractive index:

$$\Delta n = - (0.776 \times 10^{-4}) \left(\frac{P}{T^2} \right) \Delta T \quad (24)$$

This is what causes optical wavefronts to distort randomly in time.

The fluctuating part of n , which is n_1 , is called the “refractivity”. It has a correlation function defined as in eq.(1):

$$C_n (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) \equiv \langle n_1 (\vec{\mathbf{r}}_1, t) f (\vec{\mathbf{r}}_2, t) \rangle, \quad (25)$$

and a structure function $D_n (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ as in eq.(11). In anticipation of future needs, we next give a heuristic derivation (no other exists!) of the form of the latter function.

4.1 The Structure Function of Refractivity

Turbulence is supposed to occur as follows. Initially, there is a smooth, uniform, laminar flow of air. Then, usually due to rising heat, a velocity fluctuation v occurs while a portion of the flow passes through a distance L . Then, the time t taken for the generation of this fluctuation is $t = \frac{L}{v}$. Then the energy/time-mass which goes over from “laminar flow” to “turbulence” is:

$$\frac{\left(\frac{\frac{1}{2}mv^2}{t} \right)}{m} = \frac{\frac{1}{2}v^2}{\left(\frac{L}{v} \right)} = \frac{1}{2} \frac{v^3}{L} \quad (26)$$

Furthermore, this energy conversion is supposed to be going on at a constant rate, or nearly so, and hence:

$$\frac{1}{2} \frac{v^3}{L} = \varepsilon, \text{ a constant}$$

so that:

$$v^2 = AL^{\frac{2}{3}}, A \text{ is a constant} \quad (27)$$

Now the structure function for wind velocity fluctuation over distance L is defined as:

$$D_v (L) = C_f (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) \equiv \left\langle [v (\vec{\mathbf{r}}_1 + L) - v (\vec{\mathbf{r}}_1)]^2 \right\rangle. \quad (28)$$

Imagine the two points $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_1 + L$ to be fixed, and the air mass of velocity fluctuation v repeatedly progressing from $\vec{\mathbf{r}}_1$ to $\vec{\mathbf{r}}_1 + L$. All this means is that the turbulence over this region is stable for a finite amount of time. This enables the averaging indicated in eq.(28) to be carried through as well. At the start

of each repetition of the wave progression, $v(\vec{r}_1) = v$, while $v(r_1 + L) = 0$ (the air mass has not arrived there yet). At the end of each repetition the two velocities are switched. In between, if the air mass has finite extent, the same values v and θ are also present. Hence, it is reasonable to estimate that

$$D_v(L) = v^2 \quad (29)$$

or

$$D_v(L) = V L^{\frac{2}{3}} \quad (30)$$

by eq.(27).

A further premise of turbulence theory is that wind fluctuations directly cause temperature fluctuations; but by eq.(24), temperature fluctuations cause unique refractivity fluctuations. Hence, it is reasonable to assume that the statistics of refractivity follow those of wind fluctuations. In this case, from eq.(30):

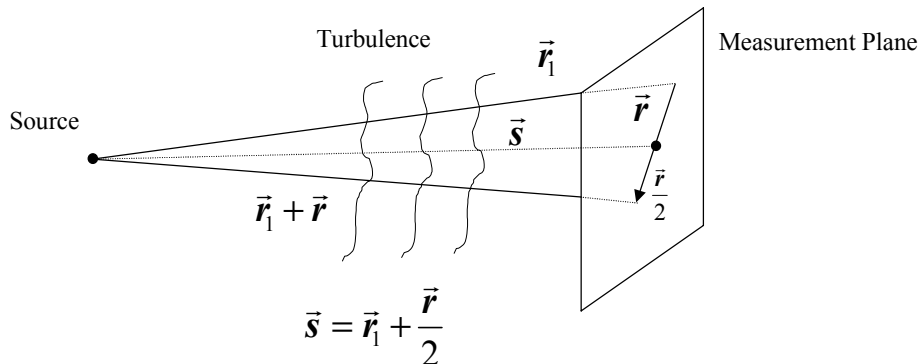
$$\boxed{D_n(r) = A r^{\frac{2}{3}}}, \quad A \text{ is a constant} \quad (31)$$

for the structure function of refractivity. This is the basis for all theory that follows.

Remark 13 *Relates to Lena, **Observ. Astrophys**, p. 42*

4.2 Refractive Index Structure Constant

The constant A measures the “strength” of the turbulence. In general, this strength will vary along the path of propagation of the light beam to be considered.



$|\bar{s}|$ is the average distance from source to measurement plane

Figure 2: Definition of Terms

If the two points defining D_n are $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_1 + \vec{\mathbf{r}}$ and $|\vec{\mathbf{s}}|$ is the average distance from the light beam source to these points, then:

$$D_n(\vec{\mathbf{r}}; \vec{\mathbf{s}}) = C_N^2(s) r^{\frac{2}{3}} \quad (32)$$

The quantity $C_N^2(s)$ is called the “refractive index-structure constant”. It is a slowly varying function of s . When s represents “altitude”, C_N^2 has been calculated to have the following shapes:

Remark 14 *Structure function = Refractive-index structure constant $\times r^{\frac{2}{3}}$*

Remark 15 *Lena, p. 44*

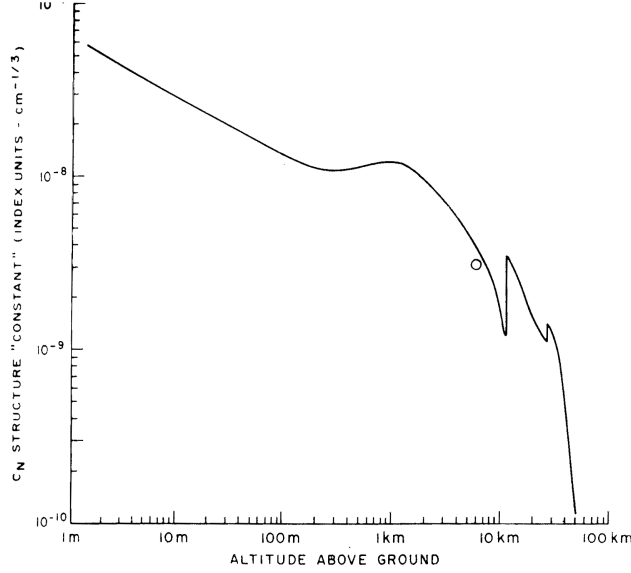


Figure 3: Model of refractive-index structure parameter proposed by Hufnagel (units of $cm^{-\frac{1}{3}}$). The discontinuity appears at the tropopause. Note that C_N^2 increases with altitude when looking down.

By eq.(18) to (22), the Kolmogoroff spectrum then does represent ϕ for refractivity, and in terms of $C_N^2(s)$

$$\phi_n(\vec{\mathbf{k}}; \vec{\mathbf{s}}) = 8.16 C_N^2(\vec{\mathbf{s}}) k^{-\frac{11}{3}} \quad (33)$$

This result will be used extensively below.

Remark 16 *Lena, p. 44*

Remark 17 *Tatarski, $\phi_n(\vec{\mathbf{k}}; \vec{\mathbf{s}}) = 8.16 C_N^2(\vec{\mathbf{s}}) k^{-\frac{11}{3}} e^{-\frac{k^2}{k_m^2}}$, $k_m = \frac{5.92}{t_0}$*

4.3 Average MTFs for Seeing Through Turbulence

The effect of turbulence is, of course, to reduce visibility. This occurs as a loss of resolution and as a loss of contrast. Here we consider the latter, as measured by two modulations transfer functions – one for long-term viewing through turbulence, and one for short-term viewing. These MTFs will be seen to be a direct function of the so-called “wave structure function”, $D_w(r)$, a quantity which is the sum of a D for long-amplitude and a D for phase in the optical pupil. This dependence is the basic reason for our interest in such a structure function.

4.3.1 Derivation: Long-Exposure Case

In general, the MTF $T(f)$ is related to the pupil amplitude $U(v)$ of an imaging system via:

$$T(f) = A \int_{pupil} dx U^*(x - \lambda Rf) U(x), \quad A \text{ is a constant} \quad (34)$$

where λ is the wavelength of light and R is the image conjugate distance. Also, x and f are two-dimensional vector quantities. Let us assume a lens system that has an intrinsic pupil function $U_o(x)$, a fixed part due to aberrations and/or coating. The corresponding MTF via eq.(34) will be $T_o(f)$.

However, because of turbulence in the intervening medium between object and image planes, the new pupil amplitude U has *both* a fixed part $U_o(x)$ and a randomly varying part $A(x) \exp(i\phi(x))$ due to turbulence. Since amplitude $A(x) \geq 0$, we can define a quantity $\ell(x) = \ln[A(x)]$ called the “log-amplitude”. Then, eq.(34) becomes:

$$T(f) = A \int_{pupil} dx U_o(x) U_o(x - \lambda Rf) e^{[\ell(x) + \ell(x - \lambda Rf)]} e^{i[\phi(x) - \phi(x - \lambda Rf)]} \quad (35)$$

Since $\ell(x)$ and $\phi(x)$ are random variables, so is $T(f)$. We next consider average values of $T(f)$ over time.

The long-term average value of $T(f)$ is defined as an average over many sequential exposures of duration $\gtrsim 0.01$ seconds, each of which is long enough to permit $\ell(x)$ and $\phi(x)$ to fluctuate through their characteristic values. This average of eq.(35) yields:

$$\langle T(f) \rangle = A \int_{pupil} dx U_o(x) U_o(x - \lambda Rf) \left\langle e^{[\ell(x) + \ell(x - \lambda Rf)]} e^{i[\phi(x) - \phi(x - \lambda Rf)]} \right\rangle \quad (36)$$

In order to mathematically evaluate this expectation, we have to define the statistics of $\ell(x)$ and $\phi(x)$. It can be shown that they have the following properties:

1. $\ell(x)$ obeys normal statistics at each x , where the means $\bar{\ell} = -C_\ell(0)$. (Respective proofs based on the central-limit theorem and conservation of energy);

2. $\phi(x)$ obeys Gaussian statistics at each x ;
3. Quantity $\ell(x) + \ell(x - \lambda Rf)$ is independent of $\phi(x) - \phi(x - \lambda Rf)$ (proof based on isotropy and shift invariance of statistics).

The last property permits the separate averaging of the two exponential factors within angle brackets in eq.(36). Properties (1) and (2) permit us to use the identity:

$$\langle e^{ax} \rangle = e^{\left(\frac{a^2 \sigma^2}{2} + a\bar{x}\right)} \quad (37)$$

for a normal random variable x and a constant a .

Remark 18 *Lena, p. 253*

Finally, use of definition (1) and eq.(11) for functions $\ell(x)$ and $\phi(x)$ gives the final result:

$$\boxed{\langle T(f) \rangle_{LE} = T_o(f) e^{-\frac{1}{2} D_w(\lambda Rf)}} \quad (38a)$$

where

$$D_w(r) \equiv D_\ell(r) + D_\phi(r). \quad (38b)$$

D_w is called the “wave structure function” for light amplitude in the pupil.

Discussion: This result shows that the long-term MTF for turbulence factors into a part due to the fixed lens properties, and a part due to turbulence. The exponential factor in eq.(38a) is often called the “MTF for long-term turbulence.”

We also see that the structure function $D_w(r)$ for light amplitude in the pupil must be calculated to evaluate $\langle T(f) \rangle_{LE}$. We shall do this in a later section and substitute back into eq.(38a) to complete this calculation.

4.3.2 Short-Exposure Case:

The short-exposure situation is defined by exposures of time with durations $\lesssim 0.001$ seconds. It will be seen later that the main wavefront deformation of turbulence is a random tilt in time. If such tilts could be removed effectively, say by taking a sequence of short-exposure pictures and suitably re-registering each to compensate for the individual tilts, then the MTF would be much better.

This may be shown by introducing a compensating tilt term of $i\vec{\mathbf{a}} \cdot \vec{\mathbf{x}}$ into the exponent of U before taking the average indicated in eq.(36). Tilt direction-cosines vector $\vec{\mathbf{a}}$ is defined so that it minimizes the mean-square net phase across the pupil after subtraction from the total phase $\phi(x)$. Hence, $\vec{\mathbf{a}}$ is defined by

$$\frac{\partial}{\partial a_i} \int_{pupil} d\vec{\mathbf{x}} [\phi(\vec{\mathbf{x}}) - \vec{\mathbf{a}} \cdot \vec{\mathbf{x}}]^2 = 0, \quad i = 1, 2 \quad (39)$$

The solution is easily found to be

$$a_1 = \left(\frac{2}{\sqrt{\pi}R^2} \right)^2 \int_{pupil} d\vec{x} x \phi(\vec{x}) \quad (40a)$$

$$a_2 = \left(\frac{2}{\sqrt{\pi}R^2} \right)^2 \int_{pupil} d\vec{x} y \phi(\vec{x}) \quad (40b)$$

But this is not enough, because eq.(40) is true at each instant of time, whereas the expectation value in eq.(36) will ultimately demand that we know:

$$\left\langle \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{f}} \right)^2 \right\rangle \quad (41)$$

That is, an evaluation of eq.(36) for this short-exposure case yields the result:

$$\langle T(f) \rangle_{SE} = \langle T(f) \rangle_{LE} \exp \left(+\frac{1}{4} (\lambda R)^2 \left\langle \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{f}} \right)^2 \right\rangle \right) \quad (42)$$

Note that because of the plus sign in the exponent, we have proved that

$$\langle T(f) \rangle_{SE} > \langle T(f) \rangle_{LE} \quad (43)$$

at all frequencies f . We return to this point later.

Regarding the still-unknown exponent in eq.(42), Fried (1966a) showed that:

$$(\lambda R)^2 \left\langle \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{f}} \right)^2 \right\rangle = 0.513 D_w(\lambda R f) \left[1 - 0.5 \left(\frac{\lambda R f}{D} \right)^{\frac{1}{3}} \right] \quad (44)$$

where D is the pupil diameter. Hence, once again, $D_w(r)$ must be known. At this point, we shall compute $D_w(r)$, using a simple laminations model of the atmosphere due to Lee and Harp (1969). Following this, we shall substitute the known D_w expression back into the preceding expressions for $\langle T(f) \rangle_{LE}$ and $\langle T(f) \rangle_{SE}$ to complete their calculation.

4.4 Calculation of the Optical Structure Function by a Simple Model

Here we are considering the problem of finding parameters of the optical wave, i.e., its structure functions for phase and amplitude in terms of the statistics of the turbulent medium through which it passes. This is the vital link in the entire theory of optical turbulence. The traditional method of calculation (Tatarski) requires solution of the wave equation in a statistically varying medium, no mean trick. Instead, we outline here the elegantly simple approach of Lee and Harp to the same problem. It gives the correct results without requiring a great deal of mathematical manipulation, relying instead on some reasonable physical modeling. It is based on direct use of Huyghen's wave construction in optics.

The situation is shown below. A point source is located at O, with a receiving plane on the far right. Correlations in optical amplitude and phase are to be measured there, (at the image). The intervening space between the source and receiving plane is filled with a turbulent medium. The medium is modeled as consisting of contiguous thin slabs perpendicular to the propagation path. One of these is shown. The refractivity field $n_1(r, t)$ within each slab is expressed as a sum of its Fourier components, each having a specified wave number and orientation (in polar coordinates). The effect produced upon the optical wave by each of these Fourier components is then easily determined, since it acts exactly like a diffraction grating.

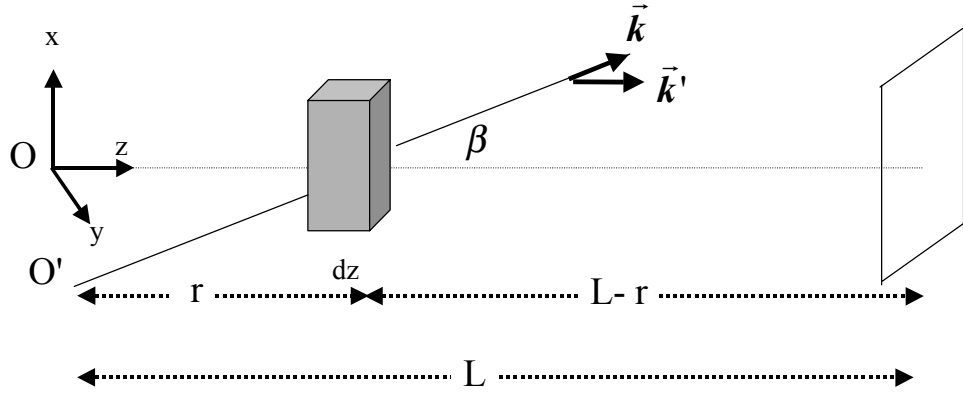


Figure 4: Schematic of light propagation through the turbulent atmosphere. The “grating” of thickness dz diffracts the light from O so that it appears to be emitted from O’.

In fact, the effect is to make the single source O effectively appear to be *three* distinct point sources, each radiating its own spherical wave. Point O’ is one of two virtual sources located symmetrically on each side of O. These spherical waves are easily known at the receiving plane, by mathematical “propagation”, so that the two-point correlation in amplitude and in phase can be formed there by integration. The results are a log-amplitude correlation function:

$$C_\ell(r) = 4\pi^2 k^2 \int_0^k du \int_0^L ds u \phi_n(u) J_0\left(\frac{rus}{L}\right) \sin^2\left(\frac{u^2 s(L-s)}{2kL}\right) \quad (45)$$

and a phase correlation function:

$$C_\phi(r) = 4\pi^2 k^2 \int_0^k du \int_0^L ds u \phi_n(u) J_0\left(\frac{rus}{L}\right) \cos^2\left(\frac{u^2 s(L-s)}{2kL}\right) \quad (46)$$

These relate optical quantities C_ℓ, C_ϕ to the refractivity power spectrum $\phi_n(u)$.

Next, relation (11) may be used to form structure functions, and these are added to form $D_w(r)$, the total or “wave” structure function,

$$\boxed{D_w(r) = 8\pi^2 k^2 \int_0^k du \int_0^L ds u \phi_n(u) \left[1 - J_0\left(\frac{rus}{L}\right)\right]} \quad (47)$$

In this calculation, the following physical assumptions are made:

1. Each wave reaching the Receiving Plane has been scattered only once.
2. The scattering angle β is small ($\beta \lesssim 0$).
3. In eq.(45) and eq.(46), variable u represents wavenumbers for the spectrum of the refractivity function $n_1(r)$. These are assumed to be small compared to the light wavenumber k .

4.4.1 Case of Kolmogoroff Turbulence:

The very important case of Kolmogoroff turbulence, where the $\frac{11}{3}$ power law of eq.(33) describes $\phi_n(u)$, may be applied to eq.(47). In this case, Fried (1966b) showed simply that:

$$D_w(r) = 2.91 k^2 r^{\frac{5}{3}} \int_0^L ds C_N^2(s) \left(\frac{s}{L}\right)^{\frac{5}{3}} \quad (48)$$

for a general structure constant $C_N^2(s)$.

4.4.2 Case of Vertical Propagation – the “ $\frac{5}{3}$ Power Law”

In the particular case of vertical propagation upward (looking down), a model

$$C_N^2(s) = 4.2 \times 10^{-14} s^{-\frac{1}{3}} e^{-\frac{s}{h_0}}, \quad h_0 = 3200 \text{ m} \quad (49)$$

may be used (Hufnagel and Stanley, 1964). Substituting this into eq.(48) yields

$$\boxed{D_w(r) = 2.19 \times 10^{-5} k^2 \left(\frac{r}{L}\right)^{\frac{5}{3}} I\left(\frac{L}{h_0}\right)} \quad (50)$$

where $I(x)$ is the normalized incomplete gamma function of order $\frac{7}{3}$. It has the important property of monotonically increasing from 0 to 1 as x varies upward from 0, with saturation once $x \gtrsim 5.0$ (Fried, 1966a). Hence:

$$\boxed{D_w(r) = 2.19 \times 10^{-5} k^2 \left(\frac{r}{L}\right)^{\frac{5}{3}}} \quad (51)$$

for $L \gtrsim 5h_0$, and D_w obeys a simple “ $\frac{5}{3}$ power law” dependence on r .

In the complementary case of looking upward, geometry indicates that $\frac{s}{L}$ in eq.(48) should simply be replaced by $1 - \frac{s}{L}$. Sutton (1969) has carried through the remainder of this calculation.

4.5 Evaluation of Long-Exposure and Short-Exposure MTFs for Downward Viewing

We have now laid the groundwork for completing the MTF study ending at eq.(38) and eq.(42). Since $D_w(r)$ is now known, it may be substituted into these equations to produce final evaluation of the long- and short-term exposure MTFs.

It is first useful to express the known $D_w(r)$ at eq.(50) as

$$D_w(r) \equiv 6.88 \left(\frac{r}{r_0} \right)^{\frac{5}{3}} \quad (52a)$$

where the new parameter r_0 obeys

$$r_0 = 1.99 \times 10^3 k^{-\frac{6}{5}} L \left[I \left(\frac{L}{h_0} \right) \right]^{-\frac{5}{3}} \quad (52b)$$

The physical significance of this length r_0 will soon become apparent.

Substitution of eq.(52) into eq.(38) and eq.(42), supplemented by eq.(44), produces analytical expressions for the required quantities:

$$\boxed{\langle T(f) \rangle_{LE} = T_0(f) \exp \left(-3.44 \left(\frac{\lambda R f}{r_0} \right)^{\frac{5}{3}} \right)} \quad (53a)$$

and

$$\boxed{\langle T(f) \rangle_{SE} = T_0(f) \exp \left(-3.44 \left(\frac{\lambda R f}{r_0} \right)^{\frac{5}{3}} \right) \left[1 - \alpha \left(\frac{\lambda R f}{D} \right)^{\frac{1}{3}} \right]} \quad (53b)$$

where quantity $\alpha = 1$ for near-field viewing of the object, and $\alpha = \frac{1}{2}$ for far-field viewing.

4.6 Discussion:

Equation (53a) very neatly shows the advantage of short-term exposures over long ones. The cutoff frequency in $T_o(f)$ is known from diffraction theory to be $f_c = \frac{D}{\lambda R}$. Eq.(53b) shows that, for f somewhere near the cutoff frequency, for near-field viewing ($\alpha = 1$), then

$$\boxed{\langle T(f) \rangle_{SE} = T_0(f)}. \quad (54)$$

That is, short-exposure imagery allows nearly diffraction-limited performance at the high-frequency end of the spectrum! This is a surprising and important result.

5 Resolution:

A useful measure of resolution, related to a known transfer function $T(f)$, is the “effective bandwidth” quantity:

$$\mathcal{R} \equiv \int_{-\infty}^{+\infty} T(\vec{f}) d\vec{f} \quad (55)$$

That this parameter truly measures resolution may be shown as follows.

By the form of eq.(55), \mathcal{R} represents the square of a net cutoff frequency. Now, a cutoff frequency f_c in *line pairs per mm* represents $(2f_c)^{-1}$ units of length per line. Let δs represent this “resolution length”. Hence, this length is:

$$\delta s = \frac{1}{2\sqrt{\mathcal{R}}} \quad (56)$$

We may now apply the known average MTFs given at eq.(53) to the resolution formula in eq.(55) for evaluation. It is useful to first carry through for the case $D \rightarrow \infty$, i.e., perfect optics. Here, the image degradation is *solely* due to turbulence, and we get a maximum of the resolution \mathcal{R} . Here, $T_o(f) = 1$. For the long-exposure case, we get in this way:

$$\mathcal{R}_{\max} = \frac{\pi}{4} \left(\frac{r_0}{\lambda R} \right)^2 \quad (57)$$

5.1 Turbulence Parameter r_0 as an Equivalent Lens Diameter D

This relation permits us to interpret r_0 optically. Use of eq.(56) gives:

$$\delta s_{\min} = \frac{1}{\sqrt{\pi}} \left(\frac{\lambda R}{r_0} \right) \quad (58)$$

This is the resolution in the image plane. It is useful to translate this to ultimate ground resolution $\delta \ell_{\min}$, which entails multiplication by $\frac{L}{R}$, where L is the altitude of the observer. Then

$$\delta \ell_{\min} = \frac{1}{\sqrt{\pi}} \left(\frac{\lambda L}{r_0} \right). \quad (59)$$

Compare this to the ultimate ground resolution length $\delta \ell_{diff}$ due to diffraction alone (no turbulence):

$$\delta \ell_{diff} = 1.22 \left(\frac{\lambda L}{D} \right) \quad (60)$$

By comparing to eq.(60a) and eq.(60b), we see an approximate relation between r_0 and D :

$$r_0 \simeq \frac{D}{2} \quad (61)$$

i.e., the *turbulence* parameter r_0 physically represents the radius of a *purely* diffraction-limited lens system which has the *same* ground resolution as for the given turbulent situation at altitude L .

5.2 Downward Resolution for Various Lens Diameters

Eq.(53a) and (53b) were substituted in eq.(55) and integrated. These values of \mathcal{R} were then normalized to \mathcal{R}_{\max} given by eq.(57). The ratios $\frac{\mathcal{R}}{\mathcal{R}_{\max}}$ represent the relative resolution performance due to finite lens diameter D relative to performance with infinite lens diameter. The dependence on normalized lens diameter is shown in the figure.

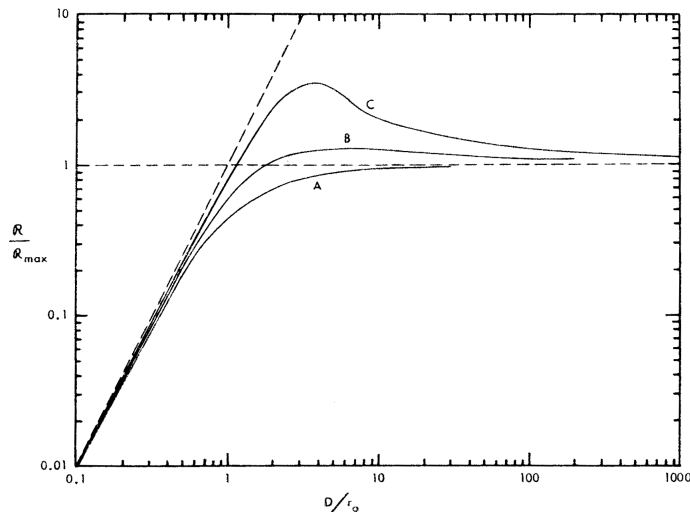


Figure 5: Dependence of normalized resolution $\frac{\mathcal{R}}{\mathcal{R}_{\max}}$ on normalized lens diameter $\frac{D}{r_0}$. (A) long-exposure result $\frac{\mathcal{R}_{\infty}}{\mathcal{R}_{\max}}$. (B) short-exposure far-field result. (C) short-exposure near-field result. (Fried 1966a Figure 1)

Some effects are worth noting here. First, resolution is not improved by increasing the physical lens diameter indefinitely. Next, the short exposure MTF (curve C) is everywhere above the long-exposure MTF (curve A), and significantly so (note the log scale) once the physical D exceeds about $2r_0$. Third, for short exposures, there is a distinct optimum diameter D of about $3.5r_0$. Since r_0 is related to altitude L via eq.(52b), this gives a unique best lens diameter for a given altitude.

Regarding the long-exposure curve (A), we see that for a value of $D = 2r_0$, the obtained resolution is about 70% of the maximum possible (with $D = \infty$). This tells us how D should vary with altitude L for long exposures.

The *limiting* resolution performance was also calculated. Eq.(60a), for minimum ground resolution length, was evaluated with $\lambda = 500$ nm at varying heights L . This produced the following result:

Here we see that, as altitude increases indefinitely, ground resolution length gets no worse than a fixed number! (Remember D must increase with altitude

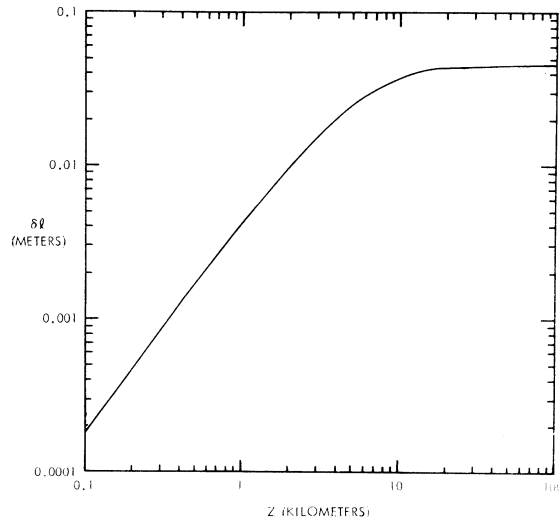


Figure 6: Dependence of minimum resolvable length on the ground vs. altitude of observation z . (Fried, 1966b, Figure 2)

as about $2r_0$ to achieve this effect.) This limiting ground resolution length is about 4.8 cm *at any height*.

The dependence of r_0 upon altitude L (again, via eq.(52b)) was also calculated. By $D \simeq 2r_0$, this also gives the lens diameter vs. altitude necessary to produce the optimum resolution given in the preceding figure.

Surprisingly, this shows a *drop* in required D with increasing altitude until an altitude of about 3 km is attained, after which D rises nearly in proportion to altitude. Notice that at 100 km altitude, the required lens diameter would be about 1.2 m. This is still a practical (albeit expensive) system to attain.

5.3 Limiting Resolution for Upward Viewing (Sutton, 1969)

A similar sequence of calculations may be carried through for upward viewing. The limiting resolution, based on lens diameters $D \gtrsim 2r_0$, is found to depend on altitude in the following manner:

Here, the resolution monotonically *worsens* with altitude, the exact opposite of the trend when looking downward. The required lens radius r_0 for attaining these limiting resolution values varies with altitude of the source in the following manner:

This is asymptotic to a diameter of about 4 cm as altitude increases, and is the origin of the well-known maxim: “*on the basis of resolution, when star-gazing there is no need to use a telescope lens larger than about 4 cm in diameter*”. (Of course, this ignores the matter of brightness, which does increase with diameter).

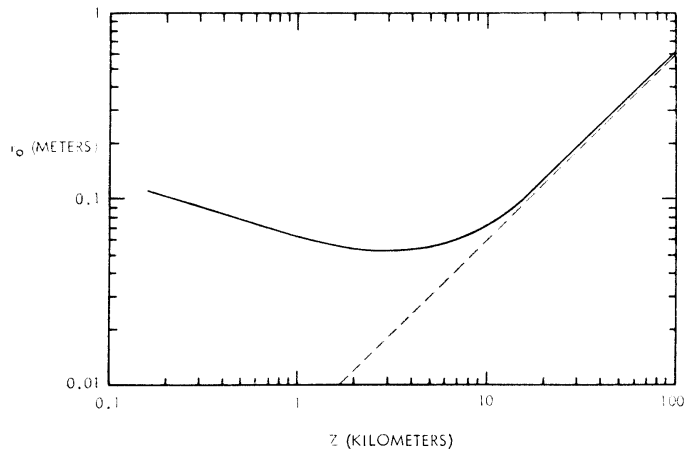


Figure 7: Dependence of r_0 (for propagation of a spherical wave from the ground) upon altitude z at which measurements are made. To achieve the resolution $\delta\ell$ indicated in Figure ??, a lens diameter two or more times larger than r_0 should be used.

The long-exposure transfer function $\langle T(f) \rangle_{LE}$ and its Fourier transform, the point spread function or “seeing disc”, were evaluated for altitude $L \rightarrow \infty$ and maximum resolution case $\frac{D}{r_0} \rightarrow \infty$. Results are as follows:

5.4 Components of the Turbulent Wavefront (Fried, 1965)

The turbulent wavefront is often modeled heuristically as being chiefly composed of randomly tilted sections. That is, at any one time, the image of a point is composed of, at most, a few Airy discs; and that as time progresses these discs preserve their basic Airy form and merely move randomly about in the image plane. How correct is this model?

The situation may be analyzed by representing the wavefront phase $\phi(\vec{x})$ by a set $\{F_n(\vec{x})\}$ of distortion functions,

$$\phi_6(\vec{x}) = \sum_{n=1}^6 a_n F_n(x) \quad (62)$$

These distortions are of the following type:

1. F_1 measures average phase and is constant;
2. $F_2 = c_2x$ and $F_3 = c_2y$ measure average tilt;
3. $F_4 = c_4 \left(x^2 + y^2 - \frac{D^2}{8} \right)$ measures spherical deformation (aberration); and

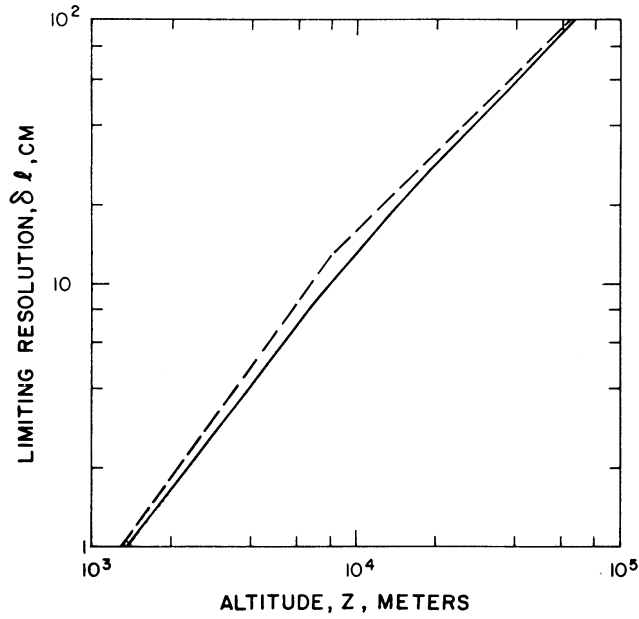


Figure 8: Limiting resolution as a function of object altitude for ground-level aperture $D \gg r_0$. The “lower” dashed line is proportional to $k^{\frac{7}{5}}$ and the upper to k^1 . (Sutton, 1969, Figure 2).

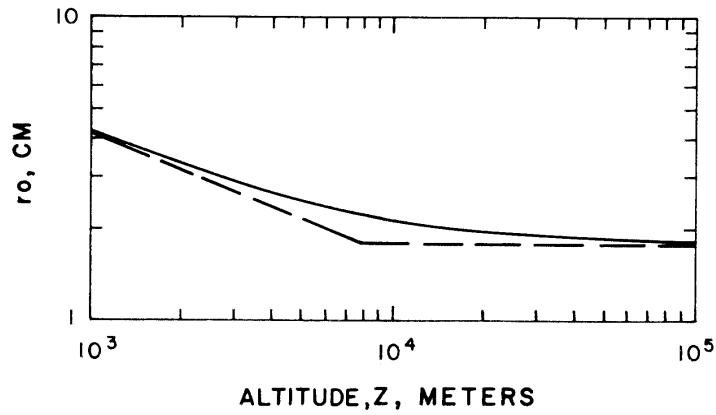


Figure 9: Ground-level aperture-scale size as a function of object altitude (Sutton, 1969, Figure 1).

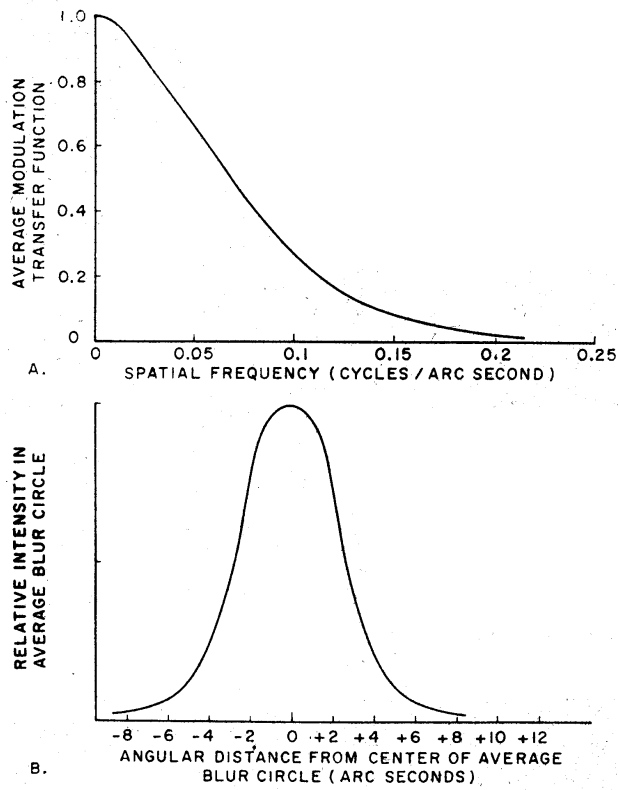


Figure 10: Average transfer function (A) and average intensity distribution (B) for zenith viewing through earth atmosphere. (Hufnagel and Stanley, 1967, Figure 7).

4. $F_5 = c_6(xy)$ measures hyperbolic deformations.

The question is, which of these functions statistically dominates the behavior of $\phi(\vec{x})$? That is, suppose an error function:

$$\Delta = \int_{pupil} d\vec{x} \langle [\phi(\vec{x}) - \phi_6(\vec{x})]^2 \rangle \quad (63)$$

is formed, and observing eq.(62), this error is minimized through choice of the coefficients $\{a_n\}$. Which of these coefficients if *deleted* from the series in eq.(62) would most increase the error $\langle \Delta \rangle$?

The analysis is pretty straightforward, since the coefficients $\{c_n\}$ are selected so that the $\{F_n(\vec{x})\}$ constitute an orthogonal set of functions over a circular pupil, i.e.,

$$\int_{pupil} d\vec{x} F_m(x) F_n(x) = \delta_{mn} \quad (64)$$

Then the solution for unknown coefficients $\{a_n\}$ for minimizing $\langle \Delta \rangle$ in eq.(63) is:

$$a_n = \int_{pupil} d\vec{x} \phi(x) F_n(x) \quad (65)$$

The evaluation of *average* error $\langle \Delta \rangle$ for different combinations of the $\{a_n\}$ present in the series in eq.(62) requires statistical knowledge of the wavefront through the structure function $D_w(r)$. The “ $\frac{5}{3}$ power law” in eq.(52) was used.

The combinations of $\{a_n\}$ that were tested are:

$$\begin{aligned} a_c^2 &= a_1^2 \\ a_L^2 &= a_2^2 + a_3^2 \\ a_s^2 &= a_4^2 \\ a_Q^2 &= a_4^2 + a_5^2 + a_6^2 \end{aligned} \quad (66)$$

These measure, in turn, constant, linear, spherical, and quadratic contributions to the wavefront, as suggested by the subscripts.

Results obtained for the average contributions to the wavefront are:

$$\begin{aligned} \langle a_L^2 \rangle &= 0.883 \left(\frac{D}{r_0} \right)^{\frac{5}{3}} \\ \langle a_s^2 \rangle &= 0.021 \left(\frac{D}{r_0} \right)^{\frac{5}{3}} \\ \langle a_Q^2 \rangle &= 0.067 \left(\frac{D}{r_0} \right)^{\frac{5}{3}} \end{aligned} \quad (67)$$

From the sizes of the numerical coefficients, we see that the linear component, defining random tilts, is by far the dominant deformation.

These results, of course, follow from the particular form of $D_w(r)$ assumed in eq.(52). This is a $\frac{5}{3}$ law, and one is led to ask what the effect would be if

$D_w(r)$ followed a square law ($\frac{6}{3}$), i.e., a simple quadratic dependence. Fried shows that, remarkably, the results in eq.(67) would change to:

$$\begin{aligned}\langle a_L^2 \rangle &= 1.0 \left(\frac{D}{r_0} \right)^{\frac{5}{3}} \\ \langle a_s^2 \rangle &= 0 \\ \langle a_Q^2 \rangle &= 0\end{aligned}\tag{68}$$

That is, nothing but tilt in the random wavefronts! We may conclude, then, that had Nature been a bit kinder and increased the exponent by only $\frac{1}{3}$, our entire analysis of atmospheric turbulence would be much simpler, as would our physical handling of the effect.

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