Abstract
The Fourier transform provides information about the global frequency-domain characteristics of an image. The Fourier description can be computed using discrete techniques, which are natural for digital images. This lecture addresses the implementation of the 2D transform and provides examples of its use.
Image Transformation

Let $A$ be an $N \times M$ array that represents an image in the spatial domain. The frequency-domain representation is found by using the FFT.

$$AF = \text{FFT}(A)$$

The array $AF$ is complex and has dimensions $N \times M$. The frequency-domain origin of $AF$ is position $[0,0]$. Filtering is facilitated by shifting the origin to $[N/2, M/2]$ using the command

$$AFS = \text{Shift}(AF, N/2, M/2)$$
Effect of Shifting

The DFT is

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) e^{-i2\pi \left( \frac{ux}{N} + \frac{vy}{M} \right)} \]

\[ f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u, v) e^{i2\pi \left( \frac{ux}{N} + \frac{vy}{M} \right)} \]

Shifting \( F(u, v) \) is equivalent to replacing \((u, v) \rightarrow (u - u_0, v - v_0)\). The effect in the spatial domain is

\[ f_s(x, y) = \mathcal{F}^{-1} F(u - u_0, v - v_0) \]

\[ = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u - u_0, v - v_0) e^{i2\pi \left( \frac{ux}{N} + \frac{vy}{M} \right)} \]

\[ = \sum_{s=-u_0}^{N-1-u_0} \sum_{t=-v_0}^{M-1-v_0} F(s, t) e^{i2\pi \left( \frac{(s+u_0)x}{N} + \frac{(t+v_0)y}{M} \right)} \]
Effect of Shifting

\[ f_s(x, y) = \left( \sum_{s=-u_0}^{N-1-u_0} \sum_{t=-v_0}^{M-1-v_0} F(s, t) e^{i2\pi (\frac{sx}{N} + \frac{ty}{M})} \right) e^{i2\pi (\frac{u_0x}{N} + \frac{v_0y}{M})} \]

Everything in the summation is periodic with period \((N, M)\). Therefore, the starting index on the sum is not significant, and

\[ f_s(x, y) = \left( \sum_{s=0}^{N-1} \sum_{t=0}^{M-1} F(s, t) e^{i2\pi (\frac{sx}{N} + \frac{ty}{M})} \right) e^{i2\pi (\frac{u_0x}{N} + \frac{v_0y}{M})} \]

\[ = f(x, y) e^{i2\pi (\frac{u_0x}{N} + \frac{v_0y}{M})} \]

The image \( f_s(x, y) \) has the same amplitude information as a function of location, but it is modulated by a phase shift that depends on \((x, y)\). Since the phase shift is known, it can be removed.
Effect of Shifting

In particular,

\[ |f(x, y)| = |f_s(x, y)| \]

If \( f(x, y) \) is known to contain only non-negative real numbers as it does in images, then the phase modulation can be eliminated by using the magnitude

\[ f(x, y) = |f_s(x, y)| \]

**Shift by \((N/2, M/2)\)**

If \( N \) and \( M \) are even and \( u_0 = N/2 \) and \( v_0 = M/2 \) the phase is

\[ 2\pi \left( \frac{u_0 x}{N} + \frac{v_0 y}{M} \right) = \pi(x + y) \]

Therefore,

\[ f_s(x, y) = f(x, y)(-1)^{x+y} \]
Effect of Shifting

The hurricane image from the IDL distribution is shown below. The modulation effect produced by a \((N/2, M/2)\) shift is shown to the right.

Hurricane image (440×340)  Effect of shifting \(F(u, v)\) by (220,170)
**Effect of Shifting**

Corresponding small regions of the image arrays illustrate the effect.

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*From $f(x,y)$*  

*From $f_s(x,y)$*

Note the sign changes that follow the diagonal lines of the array.
Frequency Grid

We often want to process an image by using a filter that is described in the frequency domain.

\[ G(u, v) = F(u, v)H(u, v) \]

Many filter functions are constructed analytically. We need to be able to convert the analytical expression into an array of values of the proper dimensions.

The conversion can be done by representing \( U \) and \( V \) as coordinate arrays and then evaluating \( H(U, V) \).

\[
\begin{array}{cccccccc}
-3 & -2 & -1 & 0 & 1 & 2 & -3 & -3 \\
-3 & -2 & -1 & 0 & 1 & 2 & -2 & -2 \\
-3 & -2 & -1 & 0 & 1 & 2 & -1 & -1 \\
-3 & -2 & -1 & 0 & 1 & 2 & 0 & 0 \\
-3 & -2 & -1 & 0 & 1 & 2 & 1 & 1 \\
-3 & -2 & -1 & 0 & 1 & 2 & 2 & 2 \\
\end{array}
\]

Array \( U \)

\[
\begin{array}{cccccccc}
-3 & -2 & -1 & 0 & 1 & 2 & -3 & -3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

Array \( V \)

The array origin must be constructed to match the origin of \( F(u, v) \).
Frequency Grid

IDL array multiplication operators can be used to construct the desired grid arrays. Let $uu$ and $vv$ be row vectors of lengths $N$ and $M$, respectively, that contain the $(u, v)$ coordinates. Then $U$ and $V$ can be constructed by

$$U = uu \# \text{Replicate}(1, M)$$
$$V = vv \# \# \text{Replicate}(1, N)$$

For example, to construct arrays that have the origin at $\left(\frac{N}{2}, \frac{M}{2}\right)$

$$U = (\text{Findgen}(N) - N/2) \# \text{Replicate}(1, M)$$
$$V = (\text{Findgen}(M) - M/2) \# \# \text{Replicate}(1, N)$$
Lowpass Filter

We can construct a symmetric lowpass filter like the one shown by use of the analytic expression

\[ H(U, V) = \frac{1}{1 + \left( \frac{U^2 + V^2}{D_0^2} \right)^p} \]

In terms of IDL

\[
\begin{align*}
D2 &= U^2 + V^2 \\
H &= 1/(1 + (D2/D0^2)^p)
\end{align*}
\]

For this example \( N = M = 101 \), \( p = 1 \) and \( D_0 = 10 \).
Non-Isotropic Filters

When distance is measured using the standard Euclidean scale then the filter turns out isotropic—meaning that the response depends only on the distance from the frequency origin.

An elliptical measure can be used to make a non-isotropic response. The equation of a simple ellipse is

\[ \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \]

The ellipse intersects the \( u \)-axis at \( \pm a \) and the \( v \)-axis at \( \pm b \).

Distance from the origin is measured in units of \( a \) and \( b \). A distance array is now

\[ D2 = \left(\frac{U}{a}\right)^2 + \left(\frac{V}{b}\right)^2 \]
The Butterworth LPF response is computed with the elliptical distance array

\[ H = \frac{1}{1 + (D^2)^p} \]

The footprint of the filter is controlled by the \((a, b)\) values.
Coordinate Rotation

Assume that you have an initial coordinate grid \((u, v)\). To rotate by an angle \(\alpha\) you need new coordinate values.

\[
\begin{align*}
    s &= u \cos \alpha + v \sin \alpha \\
    t &= u \sin \alpha - v \cos \alpha
\end{align*}
\]

Then an array can be constructed using the \((s, t)\) coordinates, but referenced using the \((u, v)\) coordinates. This produces a rotation by \(\alpha\).

A function \(H(s, t)\) will be a rotated version of \(H(u, v)\).
Coordinate Rotation

Construct an ellipse that is rotated with respect to the coordinate axes.

\[ D_2 = \left(\frac{s}{a}\right)^2 + \left(\frac{t}{b}\right)^2 \]

\[ H = \frac{1}{1 + D_2^p} \]

Contour, H, s, t
Contour, H, u, v
Rotated Elliptical Butterworth Filter

Axis rotation can be used to produce the array for a rotated elliptical Butterworth filter.

Equal distance contours  Butterworth LPF
Filter Rotation

Coordinate rotation can be used with any filter response function.

1. Construct arrays for $u$ and $v$ (or $x$ and $y$ if in the spatial domain).
2. Rotate the axes by $\alpha$ using

\[
\begin{align*}
    s &= u \cos \alpha + v \sin \alpha \\
    t &= u \sin \alpha - v \cos \alpha
\end{align*}
\]

3. Construct $H(s, t)$. 
Rectangular Filter

\[ N = 401 \text{ and } M = 401 \]
\[ u = (\text{Findgen}(N) - N/2) \# \text{Replicate}(1, M) \]
\[ v = (\text{Findgen}(M) - M/2) \# \# \text{Replicate}(1, N) \]
\[ \alpha = \pi / 6 \]
\[ s = u \cos(\alpha) + v \sin(\alpha) \]
\[ t = u \sin(\alpha) - v \cos(\alpha) \]
\[ H_1 = (\text{Abs}(u) \leq 70) \text{ AND } (\text{Abs}(v) \leq 50) \]
\[ H_2 = (\text{Abs}(s) \leq 70) \text{ AND } (\text{Abs}(t) \leq 50) \]
\[ \text{Window, /Free, xsize} = N, \text{ysize} = 2M + 1 \]
\[ \text{TVSCL, } H_1, 0, M + 1 \]
\[ \text{TVSCL, } H_2 \]
Example 1: Additive Harmonic Distortion

Shown below is an original picture and the same picture with additive harmonic distortion. The goal is the recovery of the original image.
Plan

1. Analyze the distorted image to determine the distortion frequencies.

2. Construct a filter that removes the distortion components.

3. Filter the distorted image by multiplication in the frequency domain.

4. Transform the filtered image to the spatial domain.

5. Display the result.
Step 1

We will read the image array, find its dimensions, do the transform, and find the distortion components.

```plaintext
B = Read Image('barb.bmp')
SB = Size(B,/dimensions) & N = SB[0] & M = SB[1]
u = (Findgen(N)-N/2)#Replicate(1,M)
v = (Findgen(M)-M/2)##Replicate(1,N)
BF = Shift(FFT(B),N/2,M/2)
BF0 = BF & BF0[N/2,M/2] = 0
; Suppress (0,0) component for display
Window,0,xsize=400,ysize=400
Contour,ABS(BF0),u,v
Window,1,xsize=400,ysize=400
Surface,ABS(BF0),u,v
```
Step 1: Results

Contour and surface plots are shown below. Evidently the distortion is a harmonic at frequency $(u_0, v_0) = (40, -20)$. (Only the central part of the array was displayed, and gridlines were added to the contour plot.)
Step 2: Construct a Filter

We want to suppress the frequency components in a small zone around $(u_0, v_0)$ and $(-u_0, -v_0)$. Use circular regions with radius $c$.

\[
\begin{align*}
    u_0 &= 40 \quad \text{and} \quad v_0 &= -20 \quad \text{and} \quad c = 3 \\
    H &= 1 - \left( \left( (u - u_0)^2 + (v - v_0)^2 \right) \text{LT} \ c^2 \right) \\
        &\quad + \left( \left( (u + u_0)^2 + (v + v_0)^2 \right) \text{LT} \ c^2 \right) \\
\end{align*}
\]

Steps 3-5: Remove the Distortion

\[
BFH = BF * H; \text{ Filter the image}
\]

\[
BFHI = \text{Abs(FFT(BFH, /Inverse))}; \text{Back Transform Window,1,xsize=N,ysize=M}
\]

TV,BFHI

Original

Restored
Example 2: Non-harmonic Banding

The image below has been corrupted non-harmonic banding.
Image Analysis

Contour and surface plots are shown below. Evidently the distortion is a frequency sweep (chirp) with a $1 : 2$ slope.
Step 2: Construct a Filter

We want to suppress the frequency components along the line with a $1:2$ slope but leave some of the central area or we will erase picture content.

\[
\begin{align*}
\phi &= \text{Atan}(1,2) \\
s &= u \cos(\phi) - v \sin(\phi) \\
t &= u \sin(\phi) + v \cos(\phi) \\
H &= 1 - (\text{Abs}(s) \leq 20 \text{ and } \text{Abs}(s) > 5 \text{ AND } \text{Abs}(t) \leq 1) \\
\end{align*}
\]

The filter response is shown on the next slide.
Filter Design

Filter Response

Filter Response
Results

\[
BFH = BF*H; \text{ Filter the image}
\]
\[
BFHI = \text{Abs(FFT(BFH, /Inverse))} ; \text{Back Transform}
\]

The image is partially restored but the result is not perfect. A more detailed filter design may be more successful.
Example 3 – Sharpening Filter

A high-pass filter can be used to emphasize high frequency information. If one has an analytical expression for a LP filter, then one can form

\[ H_{hp}(u, v) = 1 - H_{lp}(u, v) \]

For a Butterworth filter of order \( p \) and cutoff \( D_0 \) this becomes

\[ H_{hp}(U, V) = \frac{\left( \frac{U^2 + V^2}{D_0^2} \right)^p}{1 + \left( \frac{U^2 + V^2}{D_0^2} \right)^p} \]
Example 3

The test chart image is of size $500 \times 500$.

A filter array with parameters $D_0 = 15$ and $p = 2$ is constructed by:

\begin{align*}
N &= 500 \quad \text{and} \quad M = 500 \\
D_0 &= 15 \quad \text{and} \quad p = 2 \\
U &= (\text{Findgen}(N)-N/2)\#\text{Replicate}(1,M) \\
V &= (\text{Findgen}(M)-M/2)\#\text{Replicate}(1,N) \\
D &= ((U^2+V^2)/D_0^2)^p \\
H &= D/(1+D)
\end{align*}
Example 3

\[ D_0 = 15 \]

\[ D_0 = 80 \]