Lecture 11: Fourier Transform Image Representation

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October 14, 2005

Abstract

The Fourier transform provides information about the global frequency-domain characteristics of an image. The Fourier description can be computed using discrete techniques, which are natural for digital images. Here we focus on the relationship between the spatial and frequency domains.

Fourier Transform and Image Representation

The Fourier transform is an algorithm that converts a signal or image to the frequency domain.

The inverse Fourier transform gets it back to the original domain.

What is the purpose of doing this?

1. The data representation in the frequency domain reveals information about the image.
2. Certain computations are more naturally expressed in frequency domain terms.
3. Eigenvectors of linear shift-invariant systems.
4. Certain processor hardware does its operations in the frequency domain (optics, filters).

Fourier Transform

We need to have a clear understanding of the Fourier transform algorithm and frequency domain data structure.

Two flavors of the Fourier transform are available, discrete and continuous.

Digital signal and image processing use the discrete Fourier transform, DFT.

Conversion between the continuous and discrete forms is a distinct operation that is related to Fourier analysis but a separate consideration.

Continuous Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du$$

- Both the $f(x)$ and $F(u)$ have infinite support.
- Both $f(x)$ and $F(u)$ are defined on a continuum of values.
- $f(x)$ and $F(u)$ must contain the same information.
- The definitions can be extended to multiple dimensions.

Discrete Fourier Transform

The DFT is defined by

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi ux/N}$$

for $u = 0, 1, 2, \ldots, N-1$. The inverse DFT can be computed by

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{i2\pi ux/N}$$

for $x = 0, 1, 2, \ldots, N-1$.

- The number N is the number of data values in either domain.
- Both $f(x)$ and $F(u)$, as defined by the equations above, are periodic with period N.
- $f(x)$ is defined for all x and $F(u)$ is defined for all u.
- The vectors $\mathbf{f} = \{f_0, f_1, \ldots, f_{N-1}\}$ or $\mathbf{F} = \{F_0, F_1, \ldots, F_{N-1}\}$ are sufficient to define the functions for all x and u.

Example

Let $f(x)$ be defined as

$$f(x) = \begin{cases} 1, & 0 \leq x \leq a \\ 0, & a < x \leq L \\ \text{not defined}, & \text{elsewhere} \end{cases}$$

where $0 < a < L$.

$f(x)$ can be expressed as a Fourier transform or a Fourier series, depending on how the function is extended beyond the primary interval $0 \leq x \leq L$.

The number of possibilities is infinite, since the extension is arbitrary.

Infinite Extension

Consider the specification $f(x) = 0$ in the extended interval. Then an equivalent definition is

$$f(x) = \begin{cases} 1, & 0 \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}$$

The Fourier transform is

$$F(u) = \int_0^a e^{-i2\pi ux} dx = \frac{\sin \pi ua}{\pi u} e^{-i\pi ua}$$

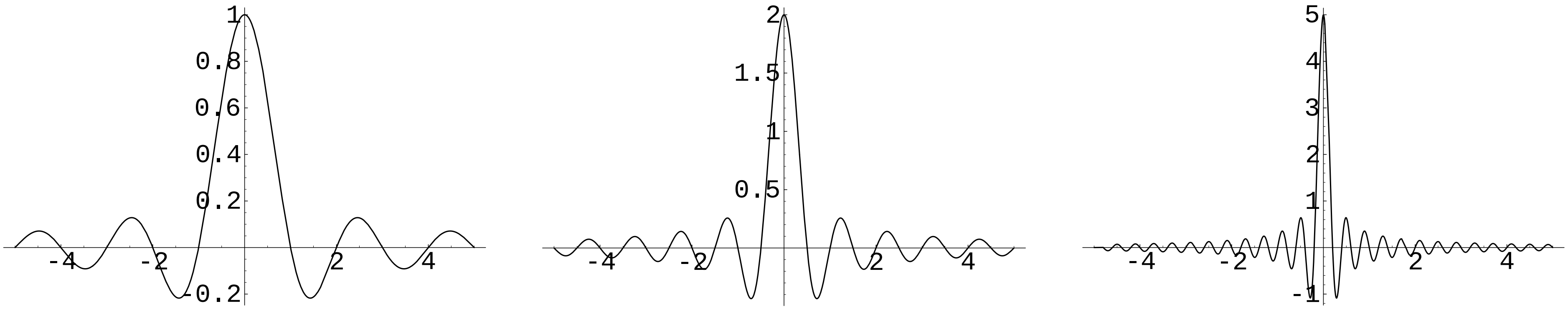
which is defined for all values of u.

Sketch the amplitude function

$$\frac{\sin \pi ua}{\pi u}$$

as a function of u for several values of a. Can you explain the behavior of the functions?

Example



$$\frac{\sin \pi u a}{\pi u}$$

is plotted above for $a = 1$, $a = 2$ and $a = 5$, respectively. Note the changes in width and height. As a increases the function approaches an impulse.

The interval $-W < u < W$ where $F(u)$ has a "significant" value is the bandwidth. Clearly, W is inversely proportional to a.

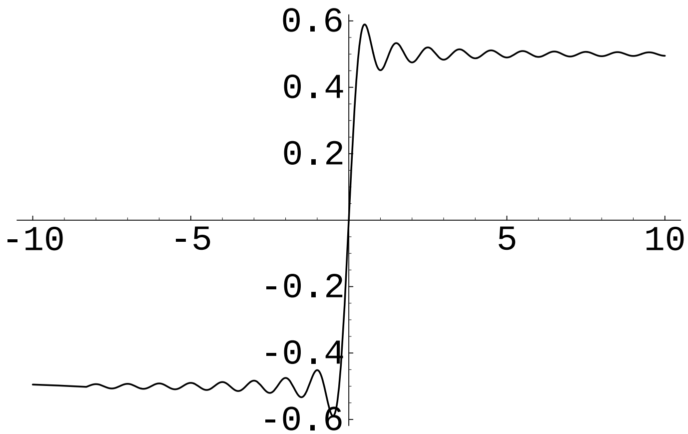
Inverse Transform

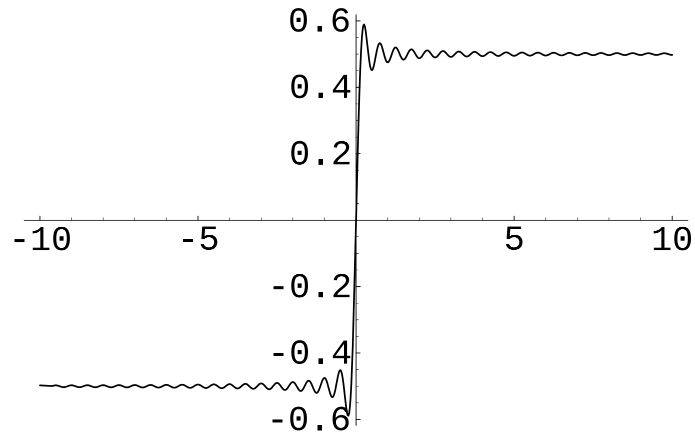
The function $f(x)$ can be recovered by the inverse transform

$$
\begin{aligned}
f(x) &= \lim_{P\to\infty} \int_{-P}^{P} F(u) e^{i2\pi ux} du \\
&= \lim_{P\to\infty} \int_{-P}^{P} \frac{\sin \pi ua}{\pi u} e^{i2\pi u(x-a/2)} du \\
&= \lim_{P\to\infty} \left[\frac{1}{i2\pi} \int_{-P}^{P} \frac{1}{u} e^{i2\pi ux} du - \frac{1}{i2\pi} \int_{-P}^{P} \frac{1}{u} e^{i2\pi u(x-a)} du \right] \\
&= \lim_{P\to\infty} \frac{1}{\pi} \left[\mathsf{Si}(2\pi Px) - \mathsf{Si}(2\pi P(x-a)) \right]
\end{aligned}
$$

Sine Integral

The function $\frac{1}{\pi}\text{Si}(2\pi Px)$ is the sine integral, plotted below for $P = 1$ and $P = 2$. As $P \to \infty$ it approaches a step function.

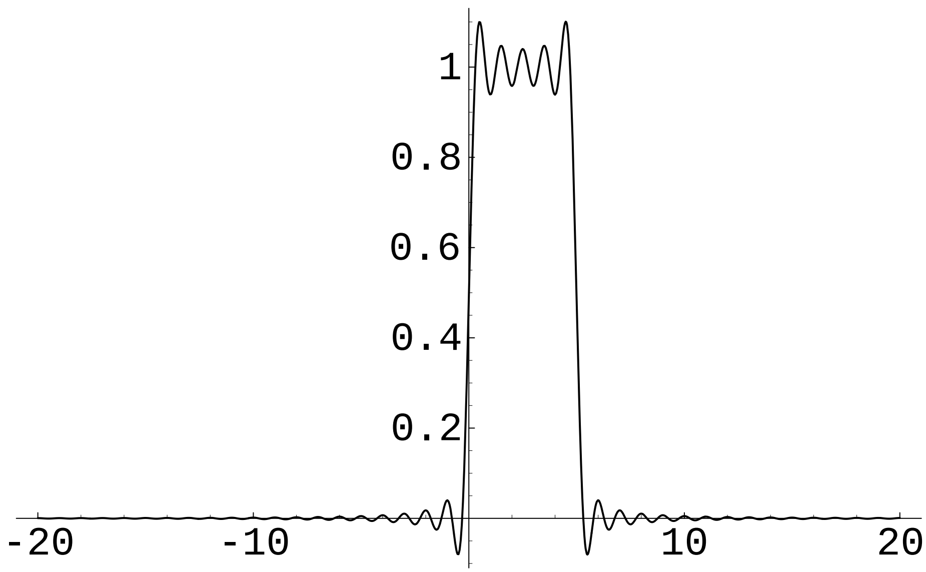




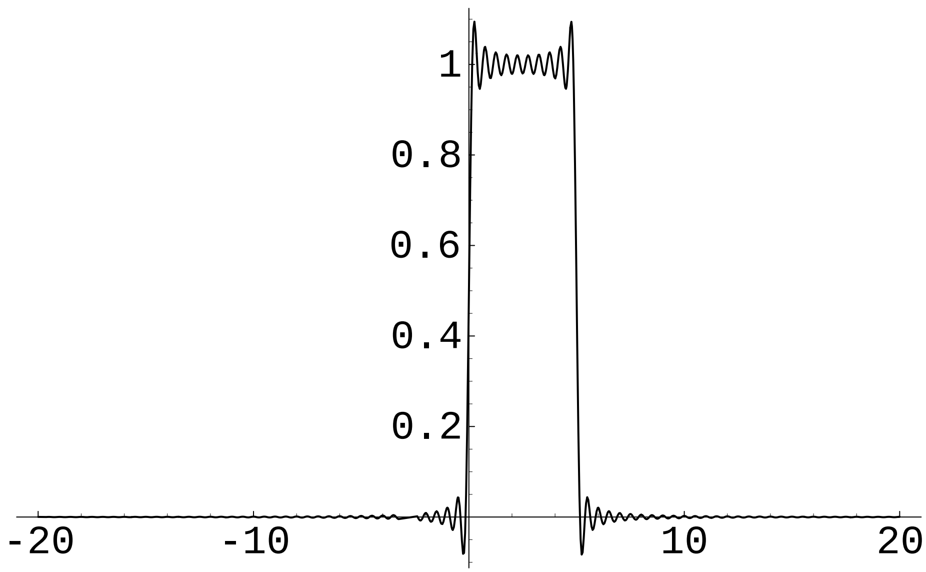
Inverse Transform

The function $f(x)$ is the difference between the two "step" functions, with one shifted by a.

The approximation to $f(x)$ is plotted below for $P = 1, 2$ and $a = 5$.



$P = 1,\ a = 5$



$P = 2,\ a = 5$

Example–with Periodic Extension

$f(x)$ was defined as

$$f(x) = \begin{cases} 1, & 0 \leq x \leq a \\ 0, & a < x \leq L \\ \text{not defined}, & \text{elsewhere} \end{cases}$$

where $0 < a < L$.

Let us require that $f(x)$ repeat periodically in the undefined region. We can then write it in the form of a Fourier series.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/L}$$

The constants are computed by

$$c_n = \frac{1}{L} \int_0^L f(x) e^{-i2\pi nx/L} dx$$

Example–with Periodic Extension

The expression is now a hybrid. The information about $f(x)$ for all x is contained in an infinite list of numbers $\{c_n\}$.

The Fourier coefficients are related to the Fourier transform by

$$c_n = \frac{1}{L} F(\frac{n}{L})$$

We have already found

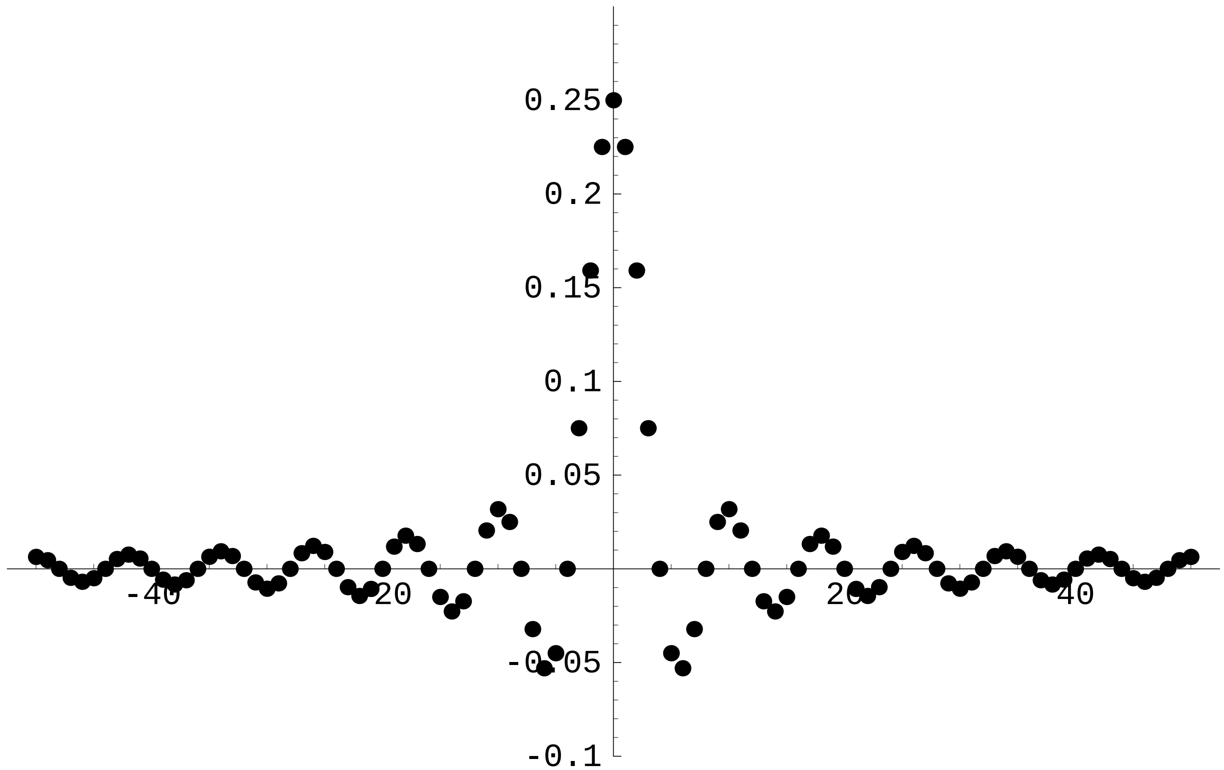
$$F(u) = \frac{\sin \pi u a}{\pi u} e^{-i\pi u a}$$

Hence,

$$c_n = \frac{\sin n\pi a/L}{n\pi} e^{-in\pi a/L}$$

Example–Periodic Extension (cont)

The coefficient amplitude is shown below for the case $a = 5$ $L = 20$



We can approximate $f(x)$ by using a finite number of the coefficients.

Symmetry Makes Things Nice

It is always the case that when $f(x)$ is real then $c_{-n} = c_n^*$. Applying this rule to this example gives

$$f(x) = \frac{a}{L} + 2\sum_{n=1}^{\infty} \frac{\sin n\pi a/L}{n\pi} \cos\left(\frac{2\pi n(x - a/2)}{L}\right)$$

This is a form of the general expression

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(2\pi n x/L + \theta_n\right)$$

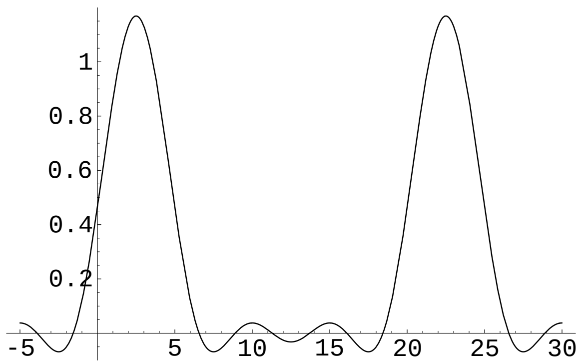
It is common to approximate $f(x)$ with

$$\hat{f}(x) = \sum_{n=0}^{N_0-1} A_n \cos\left(2\pi n x/L + \theta_n\right)$$

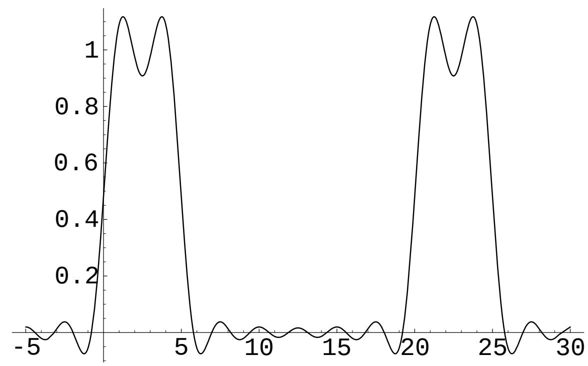
for some suitably large N_0.

Finite Fourier Sums

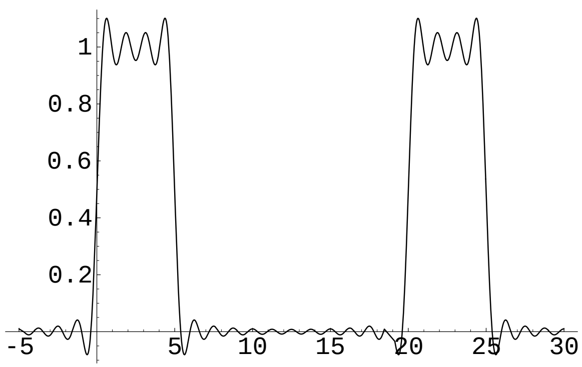
Shown below are the results of adding $N = 4, 8, 16, 32$ Fourier components for $f(x)$.



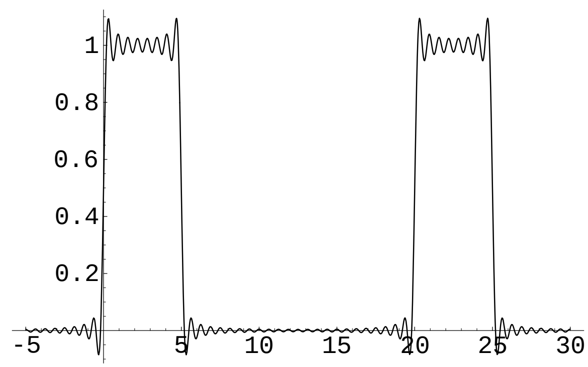
$N_0 = 4$



$N_0 = 8$



$N_0 = 16$



$N_0 = 32$

Band Limited Approximation

Choose a number N_0 such that $f(x) - \hat{f}(x)$ is sufficiently small. This is application dependent. Then the function is described by the $2N_0 - 1$ coefficients $\mathbf{c} = \{c_{-N_0+1}, \ldots, c_{-1}, c_0, c_1, \ldots, c_{N_0-1}\}$.

$$\hat{f}(x) = \sum_{n=-N_0+1}^{N_0-1} c_n e^{i2\pi nx/L}$$

This is equivalent to the sum

$$\hat{f}(x) = \sum_{n=0}^{N_0-1} A_n \cos\left(2\pi nx/L + \theta_n\right)$$

The coefficients can be computed using the orthogonality of the exponential functions over the interval $[0, L]$. They can also be computed directly from a finite set of samples of $\hat{f}(x)$.

Error vs N_0

The mean-squared error in the approximation is

$$\mathcal{E}(N_0) = \frac{1}{L}\int_0^L \left(f(x) - \hat{f}(x)\right)^2 dx$$

After substitution for $\hat{f}(x)$ and simplification,

$$\mathcal{E}(N_0) = \frac{1}{L}\int_0^L f^2(x)dx - \left(c_0^2 + 2\sum_{k=1}^{N_0-1} |c_k|^2\right)$$

The first term is the "average power" of $f(x)$ over the interval L. The second term is monotonically increasing.

We can choose N_0 to make

$$\frac{\mathcal{E}(N_0)}{\frac{1}{L}\int_0^L f^2(x)dx} < \eta \quad \text{for some small } \eta > 0$$

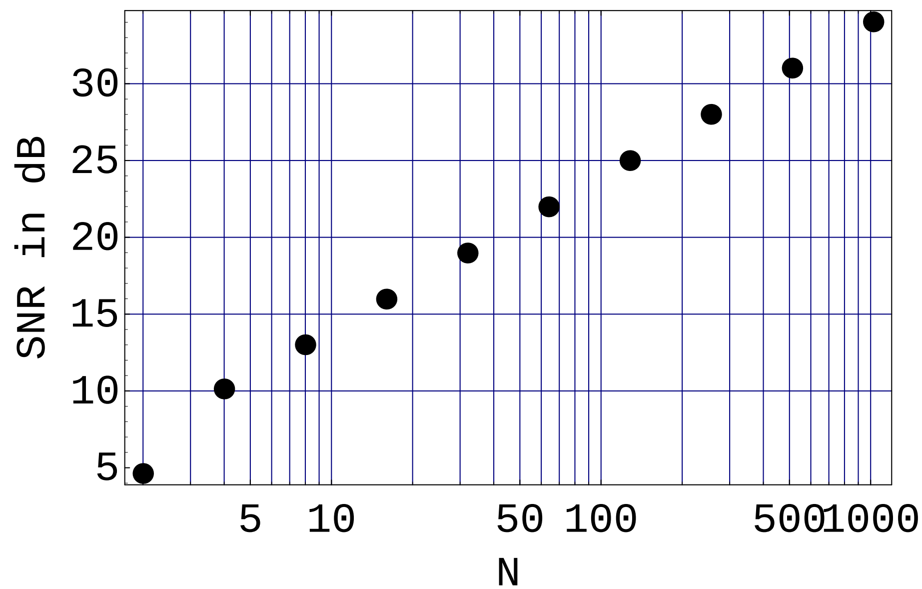
Error vs N_0

The quality of the approximation can be expressed as a "signal-to-noise" ratio

$$\text{SNR} = 10 \log_{10} \left(\frac{\mathcal{P}}{\mathcal{E}(N_0)} \right) \text{ dB}$$

where

$$\mathcal{P} = \frac{1}{L} \int_0^L f^2(x) dx$$



Bridge to the DFT Relationship

Let $\Delta_x = L/(2N_0 - 1)$ and consider the finite set of samples

$$\hat{f}(m\Delta_x) = \sum_{n=-N_0+1}^{N_0-1} c_n e^{i2\pi mn/(2N_0-1)}$$

This can be solved for the c_k by using the orthogonality of exponential sums. For any integers N, n, k,

$$\sum_{m=0}^{N-1} e^{i2\pi m(n-k)/N} = \begin{cases} N & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Choose $N = 2N_0 - 1$ and evaluate the sum

$$\begin{aligned} \sum_{m=0}^{2N_0-2} \hat{f}(m\Delta_x) e^{-i2\pi mk/(2N_0-1)} &= \sum_{-N_0+1}^{N_0-1} c_n \sum_{m=0}^{2N_0-2} e^{i2\pi m(n-k)/(2N_0-1)} \\ &= (2N_0 - 1)c_k \end{aligned}$$

DFT Relationship

It is convenient to use N in place of $2N_0 - 1$ in the relationships. Also, use the notation $\hat{f}_k = \hat{f}(k\Delta_x)$. Then

$$\hat{c}_k = \frac{1}{N} \sum_{m=0}^{N-1} \hat{f}_m e^{-i2\pi km/N}$$

By a symmetric analysis we find

$$\hat{f}_m = \sum_{k=0}^{N-1} \hat{c}_k e^{i2\pi km/M}$$

This is the DFT relationship. It is between the vectors $\hat{\mathbf{f}} = \{\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{N-1}\}$ and $\hat{\mathbf{c}} = \{\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{N-1}\}$

We need to know how $\hat{\mathbf{f}}$ and $\hat{\mathbf{c}}$ relate to $f(x)$ and $F(u)$.

DFT Relationship

The sequences $\{\hat{c}_k,\ 0 \leq k \leq N-1\}$ and $\{\hat{f}_m,\ 0 \leq m \leq N-1\}$ can be extended beyond their range of definition by noting that both are periodic with period N. For any integer p,

$$\hat{c}_{k+pN} = \frac{1}{N} \sum_{m=0}^{N-1} \hat{f}_m e^{-i2\pi(k+pN)m/N} = \hat{c}_k$$

since

$$e^{-i2\pi pm} = 1$$

Similarly, $\hat{f}_{k+pN} = \hat{f}_k$

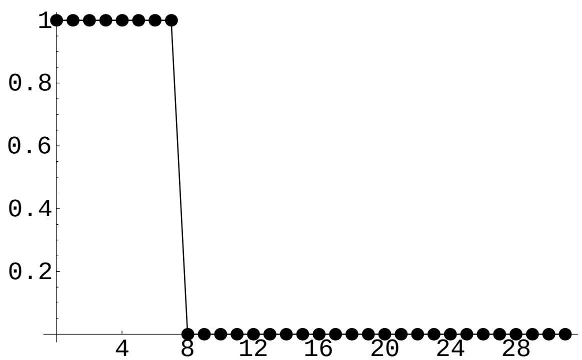
Any integer j can be written in the form $j = k + pN$ for some integer p and some $k,\ 0 \leq k \leq N-1$. Thus, $\hat{c}_j = \hat{c}_{(j \mod N)}$ and $\hat{f}_j = \hat{f}_{(j \mod N)}$ are defined for any integer j.

DFT Relationship

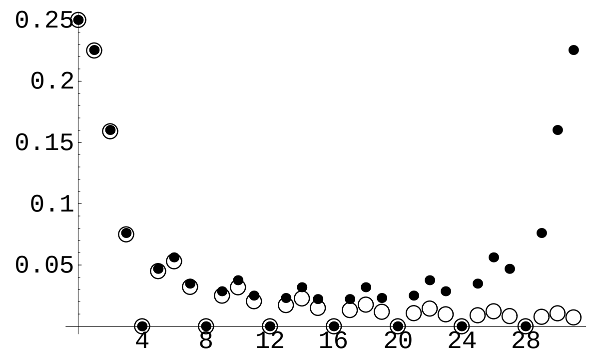
Recall that the Fourier series coefficients were related to the Fourier transform by

$$c_k = \frac{1}{L} F(\frac{k}{L})$$

The sampled functions $\hat{f}_m$ and $\hat{c}_k$ are shown for the case $N = 32$.



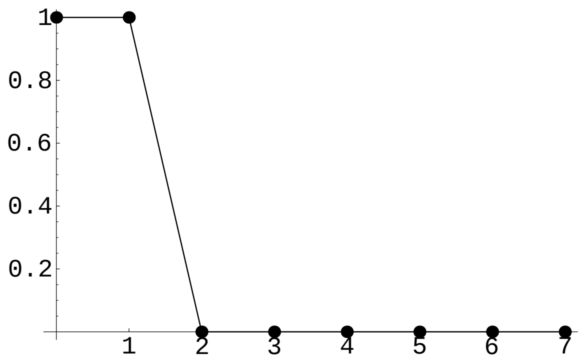
Sampled function with $N = 32$.



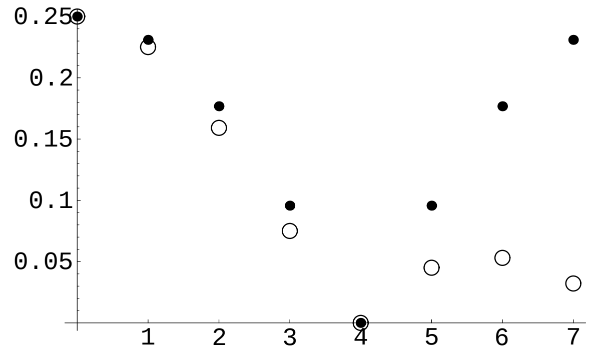
$\hat{c}_k$ (dots) and c_k (circle)

$\hat{c}_k$ and c_k agree closely for small values of k and begin to diverge near $k = N/2$.

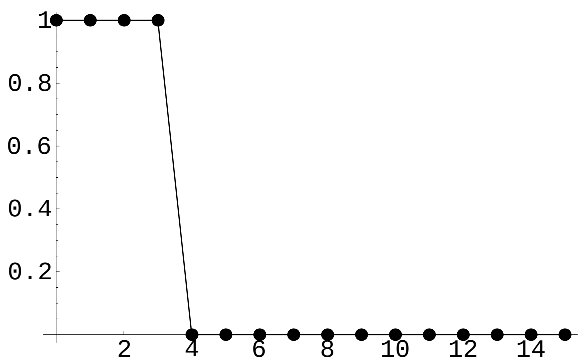
DFT Relationship



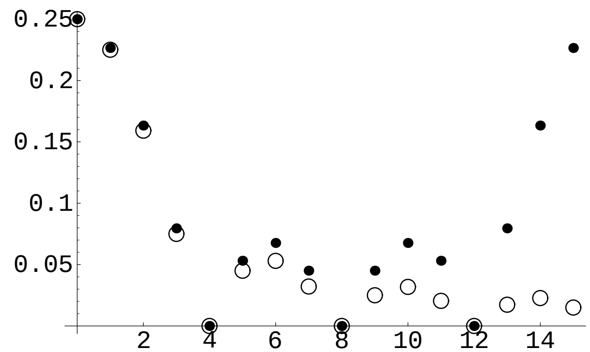
$N = 8$



$N = 8$



$N = 16$



$N = 16$

W, L, N Relationship

The number N should be chosen so that $\hat{c}_k$ and c_k are in good agreement over $0 \leq k \leq N/2$.

The index $k = N/2$ should correspond to the bandwidth W of $F(u)$.

c_k and $F(u)$ are related by

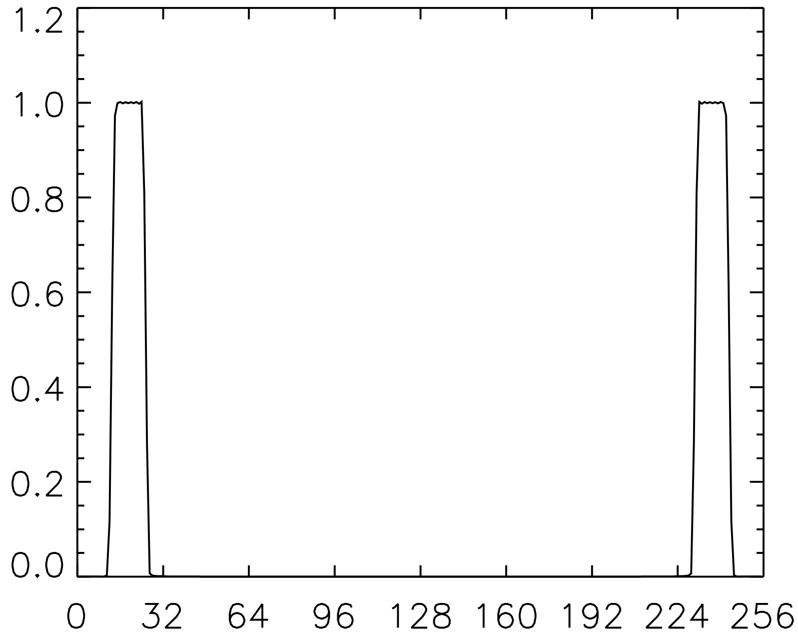
$$c_k = \frac{1}{L}F(\frac{k}{L})$$

Hence, we require

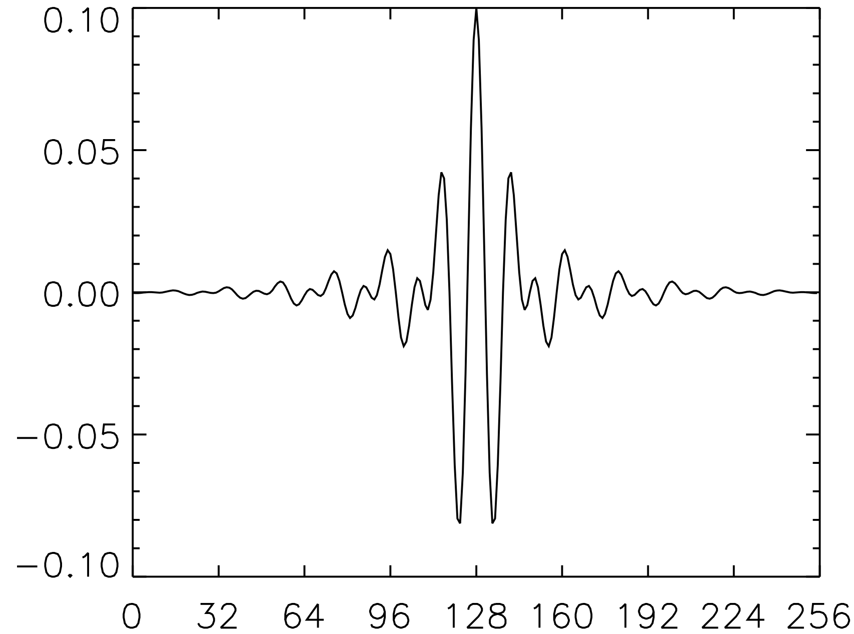
$$W \approx \frac{N}{2L}$$

The number of coefficients that are required are $N \approx 2WL$.

Bandpass Filter



Bandpass Filter



Impulse Response

Filtering

A linear shift-invariant (LSI) filter operates independently on each Fourier component of a signal. An input sequence $e^{i2\pi nu/N}$ produces an output sequence $H_u e^{i2\pi nu/N}$

Let a function $f(x)$ be represented by the DFT pair $\{\mathbf{f}, \mathbf{c}\}$ (we'll drop the "hat" notation now).

$$f_m = \sum_{k=0}^{N-1} c_k e^{i2\pi km/N}$$

The response of the filter to this input is

$$g_m = \sum_{k=0}^{N-1} H_k c_k e^{i2\pi km/N}$$

By inspection, the Fourier coefficients for the output sequence are

$$d_k = H_k c_k$$

Convolution

The filter output can be computed by convolution. Let h_m be the impulse response.

$$H_k = \frac{1}{N} \sum_{n=0}^{N-1} h_n e^{-i2\pi kn/N}$$

$$g_m = \frac{1}{N} \sum_{n=0}^{N-1} h_n \sum_{k=0}^{N-1} c_k e^{i2\pi k(m-n)/N} = \frac{1}{N} \sum_{n=0}^{N-1} h_n f_{m-n}$$

The expression can be computed by convolution:

$$g_m = \frac{1}{N} \text{CONVOL}(f, h)$$

Power Sum

The power can be computed in either domain.

$$\sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_n f_m^* \sum_{k=0}^{N-1} e^{i2\pi k(m-n)/N}$$

$$\sum_{k=0}^{N-1} e^{i2\pi k(m-n)/N} = \begin{cases} 0, & m \neq n \\ N, & m = n \end{cases}$$

$$\sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |f_n|^2$$

Similarly, on the output side,

$$\frac{1}{N} \sum_{n=0}^{N-1} |g_n|^2 = \sum_{k=0}^{N-1} |d_k|^2 = \sum_{k=0}^{N-1} |c_k H_k|^2$$

Filter Example

The IDL program DIGITAL_FILTER can be used to determine the coefficients for a FIR digital filter.

```
h = DIGITAL_FILTER( Flow, Fhigh, A, Nterms )
```

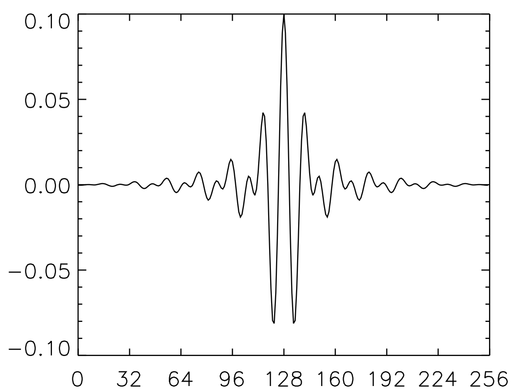
No Filtering:	$F_{low} = 0,\ F_{high} = 1$
Low Pass:	$F_{low} = 0,\ 0 < F_{high} < 1$
High Pass:	$0 < F_{low} < 1, F_{high} = 1$
Band Pass:	$0 < F_{low} < F_{high} < 1$
Band Stop:	$0 < F_{high} < F_{low} < 1$

The frequency $f = 1$ corresponds to the Nyquist sampling frequency.

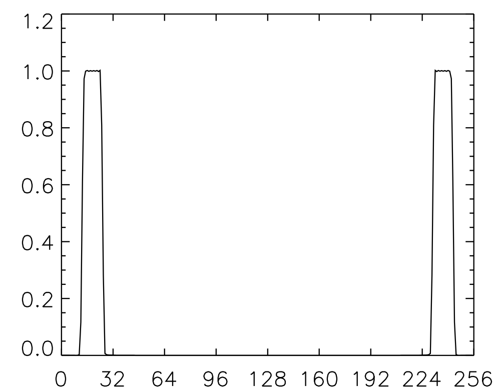
h is a vector of 2*Nterms+1 filter coefficients.

Bandpass Filter

```
h=Digital_Filter(0.1,0.2,50,128)
h=h[0:255]; Drop h[256]=h[0]
plot,h,xtickv=indgen(9)*32,xticks=8
hf=FFT(h)
plot,256*Abs(hf),xtickv=indgen(9)*32,xticks=8
```



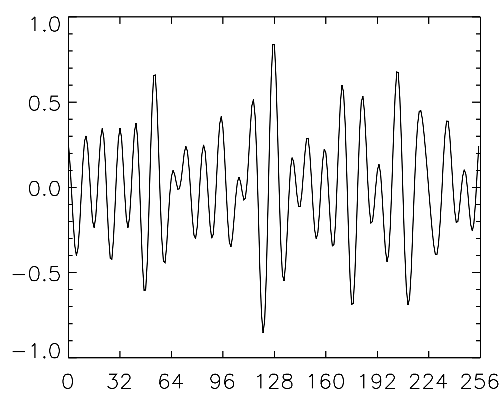
Impulse Response



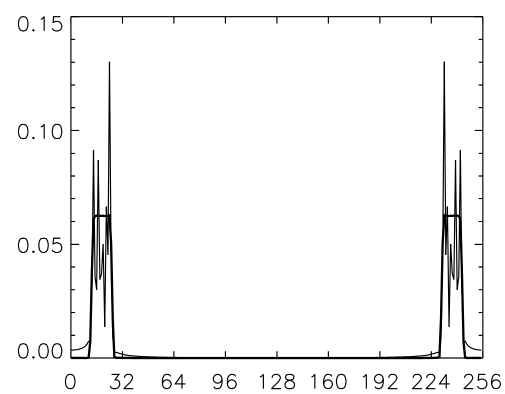
Frequency Response

Response to Random Input

```
x=Randomn(seed,1000)
y=convol(x,h)
hy=FFT(y[200:455]); Get FFT of a section of length 256.
plot,y[200:455],xtickv=indgen(9)*32,xticks=8
plot,Abs(hy),xtickv=indgen(9)*32,xticks=8
oplot,Abs(hf)*16
```



Section of y_n



Spectrum with Filter