SIMG-782 Digital Image Processing
Homework 5

Due November 8, 2005

Ex. 1 — Define the function

$$f(x) = e^{-\pi x^2}$$

It can be shown that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

(a) Show that

$$E_f = \int_{-\infty}^{\infty} f^2(x)dx = \frac{1}{\sqrt{2}}$$

(This is called the *energy* in $f(x)$).

(b) Show that $f(x)$ has the Fourier transform

$$F(u) = e^{-\pi u^2}$$

(c) Consider the periodic function

$$f_p(x) = \sum_{k=-\infty}^{\infty} f(x - kL)$$

Find an expression for the Fourier coefficients in

$$f_p(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/L}$$

(d) Explore what happens when $f_p(x)$ is approximated by

$$f_p(x) = \sum_{n=-N_0+1}^{N_0-1} c_n e^{i2\pi nx/L}$$

with different choices for L and N_0.

(e) What does the sum

$$c_0^2 + 2\sum_{n=1}^{N_0-1} |c_n|^2$$

represent? What value do you expect it to approach for large L and large N_0?

Answer (ex. 1) — (Repetitions of a Gaussian pulse)

(a)

$$\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} e^{-2\pi x^2} dx \quad \text{Let } t = \sqrt{2}x$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\pi t^2} dt = \frac{1}{\sqrt{2}}$$

(b)

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} e^{-\pi x^2 - i2\pi ux} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(x^2 - i2ux)} dx \quad \text{Complete the square in the exponent} \\
&= e^{-\pi u^2} \int_{-\infty}^{\infty} e^{-\pi(x-iu)^2} dx \\
&= e^{-\pi u^2}
\end{aligned}
$$

since

$$
\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1
$$

(c) The Fourier coefficient is found by integrating over one period of the *periodic* function. Although the Gaussian pulse has infinite extent, it still "folds up" in a way that enables us to compute c_n by

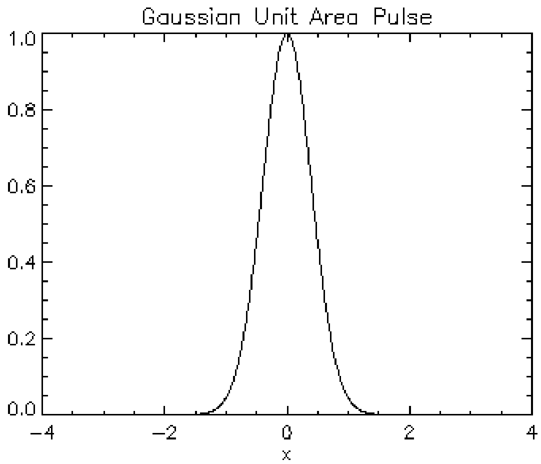
$$
c_n = \frac{1}{L} F(\frac{n}{L})
$$

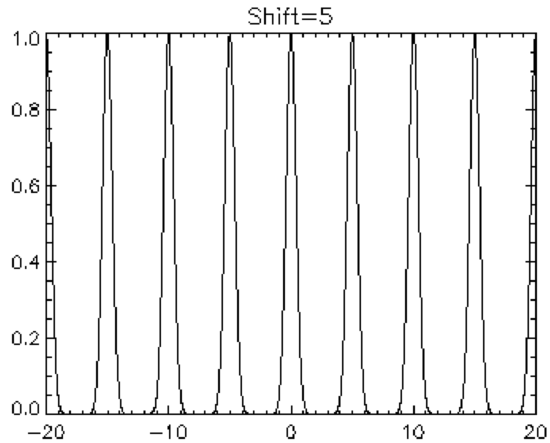
To see this, write out the expression for c_n from its definition.

$$
\begin{aligned}
c_n &= \frac{1}{L} \int_{-L/2}^{L/2} f_p(x) e^{-i2\pi nx/L} dx \\
&= \frac{1}{L} \int_{-L/2}^{L/2} \sum_{k=-\infty}^{\infty} e^{-\pi(x-kL)^2} e^{-i2\pi nx/L} dx \\
&= \frac{1}{L} \int_{-L/2}^{L/2} \sum_{k=-\infty}^{\infty} e^{-\pi(x-kL)^2} e^{-i2\pi n(x-kL)/L} dx \quad \text{since } e^{-i2\pi x/L} = e^{-i2\pi n(x-kL)/L} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{L} \int_{-L/2}^{L/2} e^{-\pi(x-kL)^2} e^{-i2\pi n(x-kL)/L} dx = \sum_{k=-\infty}^{\infty} \frac{1}{L} \int_{-L/2-kL}^{L/2-kL} e^{-\pi t^2} e^{-i2\pi nt/L} dt \quad \text{with } t = x - kL \\
&= \frac{1}{L} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi nx/L} dx = \frac{1}{L} F(\frac{n}{L}) = \frac{1}{L} e^{-\frac{n^2\pi}{L^2}}
\end{aligned}
$$

This is a general result because any $f(x)$ could be used in place of $exp(-\pi x^2)$ in the above analysis. (You were expected to use the relationship, but not prove it, so this is just a tutorial extra.)

A look at some plots of these functions is helpful. The graphs below show the Gaussian pulse on the left and the periodic function created by adding up shifted versions with a shift of $L = 5$. One clearly expects a number of fairly strong harmonics of the fundamental frequency.

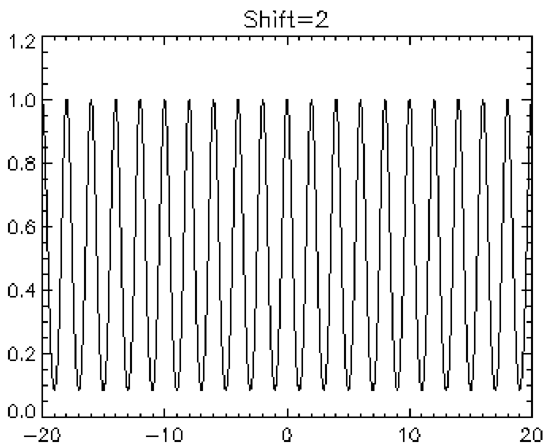


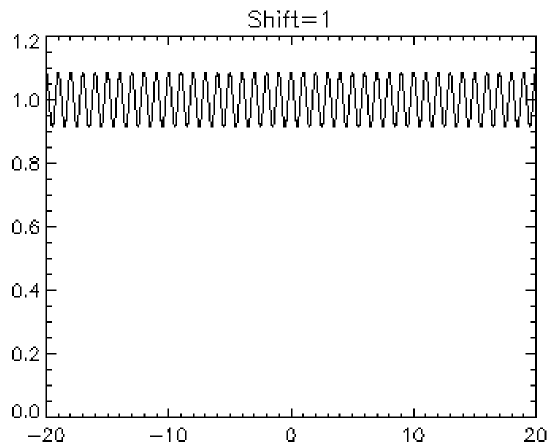


Original Gaussian Pulse

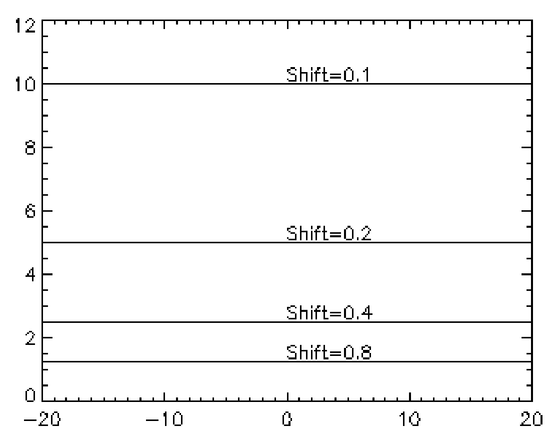
Periodic repetitions with $L = 5$

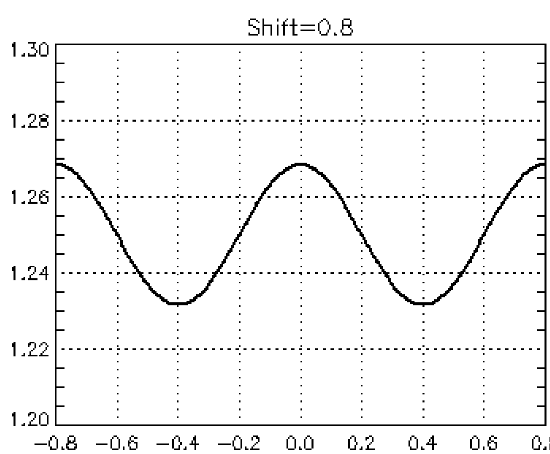
Shorter repetition periods will cause the pulses to overlap and add, which leads to a periodic waveform that resembles constant level with a ripple on top. The effect is shown for $L = 2$ and $L = 1$ in the graphs below.



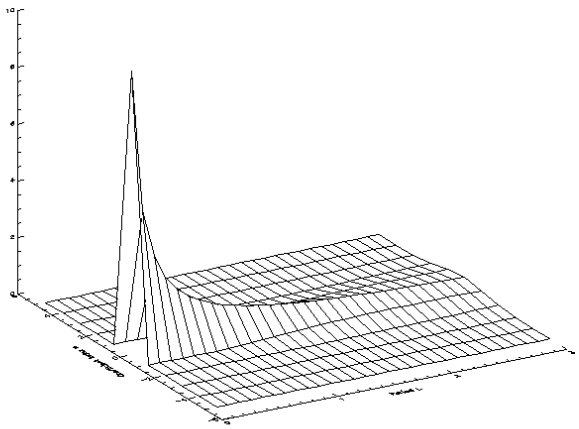


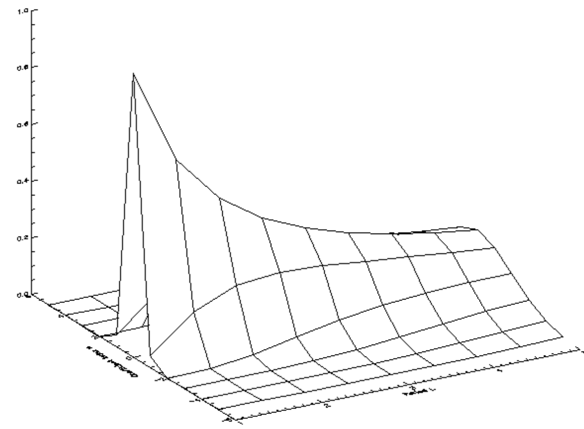
The effect becomes stronger as L is decreased. The graphs on the left below show $f_p(x)$ for small values of L. Because L is so small several pulses combine at each point to produce a larger constant level. At $L = 0.1$ about 10 pulse peaks get added up to produce a level of 10.0. Even at $L = 0.8$ the ripple is hard to see in the graph on the left. The figure on the right is a zoomed section of the $L = 0.8$ graph, where we see the ripple with period 0.8 around an average value of 1/0.8=1.25.





The Fourier series coefficients provide information about the strength of the various frequency components of the periodic function. The surface plot below left shows how the coefficients vary as a function of the period L for $L < 3$ the c_0 coefficient dominates and takes on a value close to $1/L$. For $L > 1$ the harmonic nature takes over and more coefficients become significant, as seen in the surface plot on the right, which covers the interval $1 \leq L \leq 5$.





(d) A graph of $F(u)$ (which is the same graph as $f(x)$ above with x replaced by u) shows that the bandwidth is about $W \approx 1$. Hence, the number of terms required for repetition with period L is $N = 2N_0 - 1 \approx 2WL \approx 2L$ Therefore, $N_0 \approx L + \frac{1}{2}$. Since N_0 must be an integer, let's use N_0 as the first integer greater than $L + \frac{1}{2}$. The approximation to f_p using a finite number of terms is then

$$\hat{f}_p(x) = \sum_{n=-N_0+1}^{N_0-1} c_n e^{i2\pi nx/L} = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{N_0-1} c_n \cos\frac{2\pi nx}{L} = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{N_0-1} e^{-\frac{\pi n^2}{L^2}} \cos\frac{2\pi nx}{L}$$

Notice that the coefficients decrease rapidly for $n > L$. For very small L, such as $L = 0.1$, the constant value $1/L$ is the only one of significance. For large L a number of cosine terms have to be included (up to about $N_0 = L$ of them). This is illustrated in the surface plots above.

(e) The value $|c_n|^2$ represents the power at frequency n/L. The sum over the power spectrum represents the average power of the periodic waveform, $f_p(x)$.

Ex. 2 — We want to represent $f(x)$ of problem 1 by a list of sample values $f_n = f(nd)$, $n = -N, -N+1, \ldots, N-1, N$.

(a) Determine values for N and d such that the error

$$\mathcal{E} = \frac{1}{E_f}\left| E_f - d\sum_{n=-N}^{N} f_n^2 \right| \leq 0.01$$

Relate the quantities N, W, L, d where W is a measure of the bandwidth and L is a measure of the duration.

(b) Calculate the DFT from the sample set that you choose. This will produce a set of Fourier coefficients. Compare the coefficient values to those you found in Problem 1.

Answer (ex. 2) — (Sampling a Gaussian pulse)

(a) From problem 1 we know that $E_f = 1/\sqrt{2}$. Then

$$\mathcal{E} = \sqrt{2}\left| \frac{1}{\sqrt{2}} - d\sum_{n=-N}^{N} f(nd)^2 \right|$$

Since $W \approx 1.1$ and $L \approx 2.2$ we know that $N \approx 2WL = 4.84$. Choose $N = 5$. Then the sample interval must be about $d \approx 2.2/5 = .44$. With this sample interval, we establish sample points at $\mathbf{x} = [-0.88, -0.44, 0, 0.44, 0.88]$ which produces the sample values $\mathbf{f} = \begin{bmatrix} 0.0878 & 0.5443 & 1.0000 & 0.5443 & 0.0878 \end{bmatrix}$ The error that results from these samples is $\mathcal{E} = 0.000576$.

(b) Now, if you take the FFT of these samples in **f** you will get the DFT coefficients. The result is

$$\mathbf{c} = \begin{bmatrix} 0.4528 & 0.2389 & 0.0347 & 0.0347 & 0.2389 \end{bmatrix}$$

These should equal samples from the spectrum $\frac{1}{L}F(\frac{n}{L})$. Let $u_n = n/L$ then

$$\mathbf{u} = \begin{bmatrix} -0.9091 & -0.4545 & 0.0000 & 0.4545 & 0.9091 \end{bmatrix}$$

and

$$\frac{1}{L}F(\mathbf{u}) = \begin{bmatrix} 0.0339 & 0.2375 & 0.4545 & 0.2375 & 0.0339 \end{bmatrix}$$

This is close to the values of **c**, except for a "rotation" due to the shift of frequency domain origin. Note also that **c** is one period of the DFT while $\frac{1}{L}F(\mathbf{u})$ are samples from the non-periodic function.

Ex. 3 — Suppose that an image of dimensions 4×6 inches has detail to the frequency of 300 dots per inch in each direction.

(a) How many samples are required to preserve the information in the image?

(b) How many values are contained in the DFT $F(a, b)$ of the image?

(c) Suppose that the image is sampled at a frequency that corresponds to detail up to 600 dots per inch (but in reality the detail only goes to 300 dots per inch). What is the effect on the DFT?

Answer (ex. 3) — (Image Sampling)

(a) The bandwidth is $W = 300$ in both directions, so samples must be taken at 600 dots per inch in both dimensions. Hence, a total of $4 \times 600 \times 3 \times 600 = 4,320,000$ samples are needed.

(b) The DFT contains the 4,320,000 components.

(c) The DFT will now contain 17,280,000 values. The values that correspond to frequencies above 300 will be (essentially) zero. That would be 3/4 of the samples. Making the samples closer together in the spatial domain causes the repetition interval in the frequency domain to be correspondingly larger, with the space outside the base region filled with zero values.

Ex. 4 — Here we are going to examine filtering of an image in the frequency domain. Obtain the image 'barb.png' from the images directory. Investigate the effect of filtering the image with a Butterworth filter with parameter $p = 2$ and $D_0 = 10$, 20, 40 and 80. (See lecture 12, page 20). Explain the results in terms of both frequency domain and spatial domain concepts.

Answer (ex. 4) — (Low-pass filtering)Images that are produced by this technique are shown below.

Original Barb image

Filtered with $D = 10$, $n = 2$

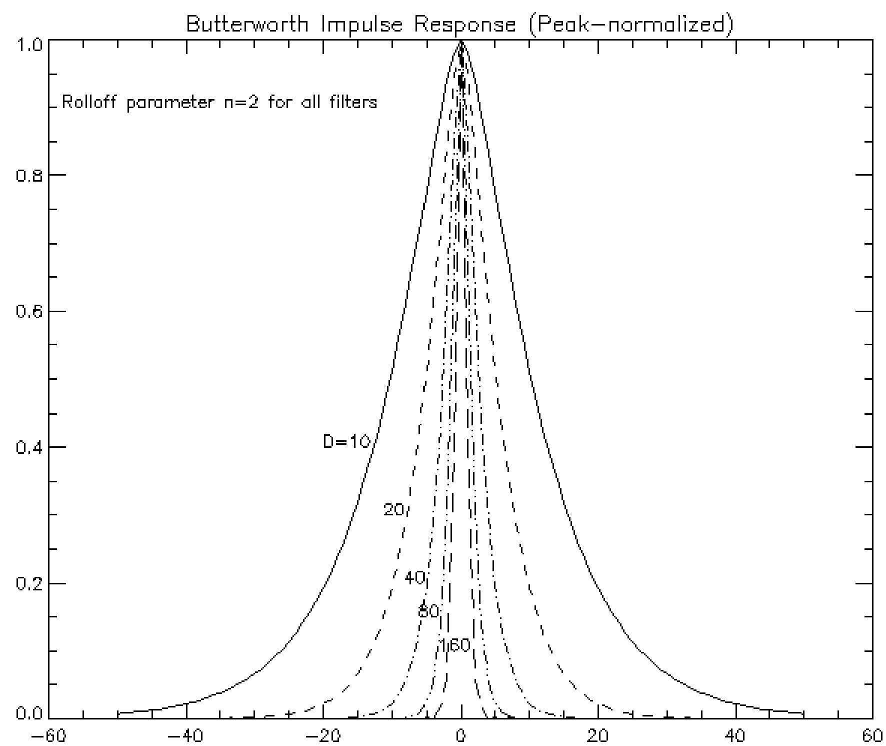
Filtered with $D = 20$, $n = 2$

Filtered with $D = 40$, $n = 2$

Filtered with $D = 80$, $n = 2$

Filtered with $D = 160$, $n = 2$

The blurring of the images can be visualized by looking at the impulse response of the filters. This is shown in a normalized form in the graph below. The impulse for the $D = 10$ filter has a width of about 80 pixels, which represents a large amount of spatial smoothing. The width of the impulse response of the filters is reduced as the bandwidth D is increased.



Cross-section of the spatial impulse response of Butterworth low-pass filters.