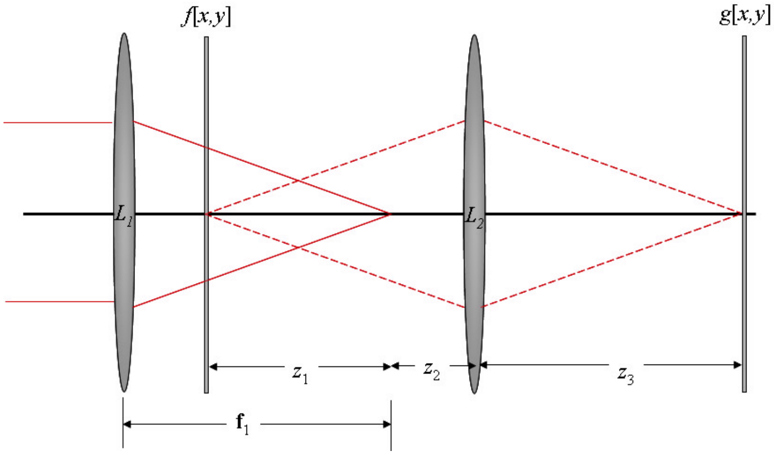
1051-738-20083 Solution Set #5

Select 4 of the 6 problems, may do either or both of the other 2 for extra credit (SPECIFY which)

1. Virtually all of the optical systems considered in this class consisted of a single optical element with the object distance z_1 and observation plane located at z_2; the only exceptions were the cascades of such systems to produce the so-called "4f" and "6f" optical correlators. We found that we could evaluate an impulse response (and thus a transfer function) of the system at an image point. Consider the two-lens system shown in the figure below, that is illuminated by "collimated" monochromatic radiation (i.e., plane waves). In words, the object $f[x,y]$ is illuminated with a converging wave from the first lens. The second lens produces an image of $f[x,y]$ at the plane labeled by $g[x,y]$. The pupil functions of the two lenses are respectively $p_1[x,y]$ and $p_2[x,y]$.



   (a) Write down the mathematical equation for the amplitude located at the distance z_1 from the object.

   I'm going to ignore the leading factors of $(i\lambda z)^{-1}$ *and the constant phase factors. I will call the amplitude immediately after first lens:*

$$w[x,y;0] = p_1[x,y]\exp\left[-i\pi\frac{x^2+y^2}{\lambda \mathbf{f}_1}\right]$$

   Amplitude at input plane is this convolved with the impulse response for the propagation distance $\mathbf{f}_1 - z_1$*:*

$$\begin{aligned} w[x,y;\mathbf{f}_1 - z_1] &= g[x,y;0] * \exp\left[+i\pi\frac{x^2+y^2}{\lambda(\mathbf{f}_1 - z_1)}\right] \\ &= p_1[x,y]\exp\left[-i\pi\frac{x^2+y^2}{\lambda \mathbf{f}_1}\right] * \exp\left[+i\pi\frac{x^2+y^2}{\lambda(\mathbf{f}_1 - z_1)}\right] \end{aligned}$$

Find amplitude at object using method of stationary phase (paraxial approximation) is a scaled replica of the pupil function converging to the focal plane:

$$w\left[x,y;z=\mathbf{f}_1-z_1\right]\cong\frac{A_0}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)},\frac{y}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\cdot\exp\left[-i\pi\frac{x^2+y^2}{\lambda z_1}\right]$$

where A_0 is the amplitude at the lens (which may be assumed to be unity). Write in 1-D to save space:

$$\begin{aligned}w\left[x;z=\mathbf{f}_1-z_1\right] &\simeq \frac{1}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]\\ &= \frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]\end{aligned}$$

After the object $f[x]$, the amplitude is:

$$w\left[x;f_1-z_1\right]\cdot f[x]\cong f[x]\cdot\frac{\mathbf{f}_1}{z_1}p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]$$

Amplitude at focal plane of first lens is:

$$\left(f[x]\cdot\frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\cdot\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]\right)*\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]$$

$$\left(f[x]\cdot\frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\cdot\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]\right)*\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]$$

$$=\frac{\mathbf{f}_1}{z_1}\int_{-\infty}^{+\infty}\left(f[\alpha]\cdot p_1\left[\frac{\alpha}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\right)\cdot\exp\left[-i\pi\frac{\alpha^2}{\lambda z_1}\right]\exp\left[+i\pi\frac{(x-\alpha)^2}{\lambda z_1}\right]d\alpha$$

$$=\frac{\mathbf{f}_1}{z_1}\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]\int_{-\infty}^{+\infty}\left(f[\alpha]\cdot p_1\left[\frac{\alpha}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\right)\exp\left[\frac{-2i\pi x\alpha}{\lambda z_1}\right]d\alpha$$

$$=\frac{\mathbf{f}_1}{z_1}\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]\mathcal{F}\left\{f[x]\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\right\}\Bigg|_{\xi\to\frac{x}{\lambda z_1}}$$

$$=\frac{\mathbf{f}_1}{z_1}\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]F\left[\frac{x}{\lambda z_1}\right]*\frac{z_1}{\mathbf{f}_1}P_1\left[\left(\frac{z_1}{\mathbf{f}_1}\right)\frac{x}{\lambda z_1}\right]\ \textit{(via scaling property of F.T.)}$$

$$=\exp\left[+i\pi\frac{x^2}{\lambda z_1}\right]\left(F\left[\frac{x}{\lambda z_1}\right]*P_1\left[\frac{x}{\lambda\mathbf{f}_1}\right]\right)$$

The amplitude at the focal plane of L_1 is proportional to the convolution of the Fourier transform of $f[x,y]$ and the F.T. of the pupil function. The 2-D version is:

$$g[x,y] = \exp\left[+i\pi\frac{x^2+y^2}{\lambda z_1}\right]\left(F\left[\frac{x}{\lambda z_1},\frac{y}{\lambda z_1}\right] * P_1\left[\frac{x}{\lambda \mathbf{f}_1},\frac{y}{\lambda \mathbf{f}_1}\right]\right)$$

(b) Evaluate the amplitude of the output image $g[x,y]$ in terms of $f[x,y]$ and the parameters of the system.

The output image is the image of the object $f[x,y]$ seen through L_2, the amplitude at the object is:

$$f[x]\cdot\frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]$$

The image is:

$$\begin{aligned} &g[x] \\ &= \left(\left(\left(f[x]\cdot\frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2}{\lambda z_1}\right]\right) * \exp\left[+i\pi\frac{x^2}{\lambda(z_1+z_2)}\right]\right)\cdot p_2[x]\exp\left[-i\pi\frac{x^2}{\lambda \mathbf{f}_2}\right]\right) \\ &\qquad * \exp\left[+i\pi\frac{x^2}{\lambda z_3}\right] \end{aligned}$$

(c) Use the result of part (b) to find a condition on the distances z_1, z_2, and z_3 that must be satisfied for an image to be formed at the plane where $g[x,y]$ is observed.

Since lens L_2 forms an image with object distance (z_1+z_2) and image distance z_3, we select z_3 to satisfy the imaging equation::

$$\frac{1}{(z_1+z_2)}+\frac{1}{z_3}=\frac{1}{\mathbf{f}_2}$$

then we know that we have a "shift-invariant" system and the output is:

$$\left(f[x,y]\cdot\frac{\mathbf{f}_1}{z_1}\cdot p_1\left[\frac{x}{\left(\frac{z_1}{\mathbf{f}_1}\right)},\frac{y}{\left(\frac{z_1}{\mathbf{f}_1}\right)}\right]\exp\left[-i\pi\frac{x^2+y^2}{\lambda z_1}\right]\right) * P_2\left[\frac{x}{\lambda z_3},\frac{y}{\lambda z_3}\right]$$

(d) Locate the point in this system where the Fourier transform of the object exists and may be modified by introducing a multiplicative transfer function for the system.

We already showed that the FT of the object is located at the distance z_1 from the first lens. A transparency may be introduced there to modify the input spectrum and produce a filtered output at $g[x,y]$.

2. The MTF is defined for nonnegative sinusoidal functions and specifies how well the modulation is transferred as a function of spatial frequency. The analogous metric for square-wave functions is the *contrast transfer function* (CTF), which is measured for square waves with 50% duty cycle (50% "on" and 50% "off"). Derive an expression for the 1-D CTF $C[\xi]$ in terms of 1-D MTF $M[\xi]$ (hint: spectrum of square wave) (J.W. Coltman, *"The specification of imaging properties by response to a sine wave input." JOSA* **44**(6), *1954.*

   Solution: *the biased sinusoidal wave with modulation m has the form:*

$$f[x;\xi_0] = \frac{1}{2} + \frac{m}{2}\cos[2\pi\xi_0 x]$$

where the bias is $\frac{1}{2}$ *and the amplitude is* $\frac{m}{2}$. *The modulation is trivial to calculate:*

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}} = \frac{\left(\frac{1}{2}+\frac{m}{2}\right) - \left(\frac{1}{2}-\frac{m}{2}\right)}{\left(\frac{1}{2}+\frac{m}{2}\right) + \left(\frac{1}{2}-\frac{m}{2}\right)} = m$$

Its spectrum is:

$$F[\xi] = \frac{1}{2}\delta[x] + \frac{m}{4}\left(\delta[\xi+\xi_0] + \delta[\xi-\xi_0]\right)$$

Given the output spectrum $G[\xi]$, *the transfer function is*

$$\begin{aligned} M[\xi] &= \frac{G[\xi]}{F[\xi]} \\ \Longrightarrow \quad M[\xi_0] = \frac{G[\xi_0]}{F[\xi_0]} &= \frac{M[\xi=+\xi_0]\cdot\frac{m}{4}\delta[\xi-\xi_0]}{\frac{m}{4}\delta[\xi-\xi_0]} = M[\xi=+\xi_0] \end{aligned}$$

For the square wave with period $X_0 = \xi_0^{-1}$, *we can write the function as a biased thresholded cosine:*

$$sf[x;\xi_0] = \frac{1}{2} + \frac{m}{2}\cdot STEP\{\cos[2\pi\xi_0 x]\}$$

If the step function evaluates to ± 1, *the amplitude is* $\frac{1}{2}\pm\frac{m}{2}$, *so the contrast is:*

$$\frac{s_{\max} - s_{\min}}{s_{\max} + s_{\min}} = \frac{\left(\frac{1}{2}+\frac{m}{2}\right) - \left(\frac{1}{2}-\frac{m}{2}\right)}{\left(\frac{1}{2}+\frac{m}{2}\right) + \left(\frac{1}{2}-\frac{m}{2}\right)} = m$$

We can write the square wave in other forms:

$$\begin{aligned} s[x;\xi_0] &= \frac{1}{2} + \frac{m}{2}\cdot STEP\{\cos[2\pi\xi_0 x]\} \\ &= \frac{1}{2} + \frac{m}{2}\left(2\cdot RECT[2\xi_0 x] * \xi_0 COMB[\xi_0 x] - 1\right) \\ &= \frac{1}{2} + m\cdot\left(RECT[2\xi_0 x] * \xi_0 COMB[\xi_0 x] - \frac{1}{2}\right) \\ &= \frac{1}{2} + m\cdot RECT[2\xi_0 x] * \xi_0 COMB[\xi_0 x] - \frac{m}{2} \\ &= \frac{1}{2}(1-m) + m\cdot\left(\xi_0 COMB[\xi_0 x] * RECT[2\xi_0 x]\right) \end{aligned}$$

The spectrum of the nonnegative square wave is:

$$\begin{aligned}
S[\xi;\xi_0] &= \frac{1}{2}(1-m)\cdot\delta[\xi]+m\cdot\left(COMB\left[\frac{\xi}{\xi_0}\right]\cdot\frac{1}{2|\xi_0|}SINC\left[\frac{\xi}{2\xi_0}\right]\right)\\
& \frac{1}{2}(1-m)\cdot\delta[\xi]+\frac{m}{2|\xi_0|}\cdot\left(COMB\left[\frac{\xi}{\xi_0}\right]\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\right)\\
&= \frac{1}{2}(1-m)\cdot\delta[\xi]+\frac{m}{2|\xi_0|}\cdot\sum_{k=-\infty}^{+\infty}\delta\left[\frac{\xi}{\xi_0}-k\right]\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\\
&= \frac{1}{2}(1-m)\cdot\delta[\xi]+\frac{m}{2|\xi_0|}\cdot\sum_{k=-\infty}^{+\infty}\delta\left[\frac{\xi-k\xi_0}{\xi_0}\right]\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\\
&= \frac{1}{2}(1-m)\cdot\delta[\xi]+\frac{m}{2}\cdot\sum_{k=-\infty}^{+\infty}\delta[\xi-k\xi_0]\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\\
&= \frac{1}{2}(1-m)\cdot\delta[\xi]+\frac{m}{2}\cdot\sum_{k=-\infty}^{+\infty}\delta[\xi-k\xi_0]\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\\
&= \left(\frac{1}{2}(1-m)+\frac{m}{2}\right)\cdot\delta[\xi]+\sum_{k=1}^{+\infty}(\delta[\xi+k\xi_0]+\delta[\xi-k\xi_0])\cdot SINC\left[\frac{\xi}{2\xi_0}\right]\\
&= \frac{1}{2}\cdot\delta[\xi]+\sum_{k=1}^{+\infty}(\delta[\xi+k\xi_0]+\delta[\xi-k\xi_0])\cdot SINC\left[\frac{k\xi_0}{2\xi_0}\right]\\
&= \frac{1}{2}\cdot\delta[\xi]+\sum_{k=1}^{+\infty}(\delta[\xi+k\xi_0]+\delta[\xi-k\xi_0])\cdot SINC\left[\frac{k}{2}\right]\\
&= \frac{1}{2}\cdot\delta[\xi]+\sum_{\ell=1}^{+\infty}(\delta[\xi+(2\ell-1)\xi_0]+\delta[\xi-(2\ell-1)\xi_0])\cdot\left((-1)^{\ell-1}\cdot\frac{2}{(2\ell-1)\pi}\right)
\end{aligned}$$

So the function may be written in this form

$$\begin{aligned}
s[x;\xi_0] &= \frac{1}{2}+\sum_{\ell=1}^{+\infty}(2\cos[(2\ell-1)\xi_0 x])\cdot\left((-1)^{\ell-1}\cdot\frac{2}{(2\ell-1)\pi}\right)\\
&= \frac{1}{2}+\frac{4}{\pi}\sum_{\ell=1}^{+\infty}\frac{(\cos[(2\ell-1)\xi_0 x])}{2\ell-1}
\end{aligned}$$

The output sinusoids in the input function are scaled by the corresponding values of the MTF:

$$\begin{aligned}
g[x;\xi_0] &= \frac{1}{2}+\frac{4}{\pi}\sum_{\ell=1}^{+\infty}(-1)^{\ell-1}\cdot\frac{(\cos[(2\ell-1)\xi_0 x])}{2\ell-1}\cdot M[(2\ell-1)\xi_0]\\
&= \frac{1}{2}+\sum_{\ell=1}^{+\infty}\frac{4}{\pi}(-1)^{\ell-1}\cdot\frac{M[(2\ell-1)\xi_0]}{2\ell-1}\cdot(\cos[(2\ell-1)\xi_0 x])
\end{aligned}$$

which implies that the CTF is the sum of the scale factors:

$$\begin{aligned} C\left[\xi\right] &= \sum_{\ell=1}^{+\infty} \frac{4}{\pi} \left(-1\right)^{\ell-1} \cdot \frac{M\left[\left(2\ell-1\right)\xi_0\right]}{2\ell-1} \\ &= \frac{4}{\pi} \left(M\left[\xi_0\right] - \frac{M\left[3\xi_0\right]}{3} + \frac{M\left[5\xi_0\right]}{5} - \frac{M\left[7\xi_0\right]}{7} + \cdots \right) \end{aligned}$$

3. Consider the following optical imaging systems that operate in both monochromatic (coherent) and "quasimonochromatic" (incoherent) light centered at wavelength $\lambda_0 = 500\,\text{nm}$. The systems operate at "equal conjugates" so that the object and image distances are identically $2\mathbf{f}_1$, where $\mathbf{f}_1 = +200\,\text{mm}$. In other words, the complex amplitude transmittance of the lens is

$$t[x,y] = p[x,y] \cdot \exp\left[-i\pi\left(\frac{x^2+y^2}{\lambda_0 \mathbf{f}_1}\right)\right]$$

where $p[x,y]$ may be complex valued. Evaluate and plot graphical profiles of the impulse responses of the following optical imaging systems along the x- and y-axes AND of the transfer functions along the ξ- and η-axes.

(a) $p[x,y]$ is a square aperture with sides of length 50 mm. Find the spatial frequency where the MTF is 50% and find the "cutoff" frequency where the MTF first reaches 0 along both x- and y-axes.

$$p[x,y] = RECT\left[\frac{x}{50\,\text{mm}}, \frac{y}{50\,\text{mm}}\right]$$

$$P[\xi,\eta] = (50\,\text{mm})^2 \cdot SINC[50\,\text{mm}\,\xi, 50\,\text{mm}\,\eta] = 2500\,\text{mm}^2\; SINC\left[\frac{\xi}{\left(\frac{1}{50\,\text{mm}}\right)}, \frac{\eta}{\left(\frac{1}{50\,\text{mm}}\right)}\right]$$

$$|P[\xi,\eta]|^2 = 2500^2\,\text{mm}^4\; SINC^2\left[\frac{\xi}{\left(\frac{1}{50\,\text{mm}}\right)}, \frac{\eta}{\left(\frac{1}{50\,\text{mm}}\right)}\right]$$

$$\frac{|P[\xi,\eta]|^2}{|P[0,0]|^2} = SINC^2\left[\frac{\xi}{\left(\frac{1}{50\,\text{mm}}\right)}, \frac{\eta}{\left(\frac{1}{50\,\text{mm}}\right)}\right]$$

$$\frac{p[x,y] \bigstar p[x,y]}{\iint_{-\infty}^{+\infty} p[x,y] \bigstar p[x,y]\; dx\, dy} = TRI\left[\frac{x}{50\,\text{mm}}, \frac{y}{50\,\text{mm}}\right]$$

Coherent transfer function is scaled replica of square pupil function, coherent impulse response is proportional to SINC function:

$$\begin{aligned} H[\xi,\eta] &= p[-\lambda_0 z\xi, -\lambda_0 z\eta] \to RECT\left[\frac{-\lambda_0\cdot(2\mathbf{f})\cdot\xi}{50\,\text{mm}}, \frac{-\lambda_0\cdot(2\mathbf{f})\cdot\eta}{50\,\text{mm}}\right] \\ &= RECT\left[\frac{\lambda_0\cdot 2\mathbf{f}}{50\,\text{mm}}\xi, \frac{-\lambda_0\cdot 2\mathbf{f}}{50\,\text{mm}}\eta\right] \end{aligned}$$

$$\frac{2\lambda_0\mathbf{f}}{50\,\text{mm}} = \frac{2\cdot(500\cdot 10^{-6}\,\text{mm})\cdot 200\,\text{mm}}{50\,\text{mm}} = \frac{1}{250}\,\text{mm}$$

$$\Longrightarrow H[\xi,\eta] = RECT\left[-\frac{1}{250}\,\text{mm}\cdot\xi, -\frac{1}{250}\,\text{mm}\cdot\eta\right] = RECT\left[\frac{\xi}{250\,(\text{mm}^{-1})}, \frac{\eta}{250\,(\text{mm}^{-1})}\right]$$

for ξ,η *measured in cycles per mm.*

The corresponding coherent impulse response is:

$$h[x,y] = \mathcal{F}_2^{-1}\{H[\xi,\eta]\} = 250^2\, SINC[250x, 250y] = 250^2\, SINC\left[\frac{x}{\left(\frac{1}{250}\right)\text{mm}}, \frac{y}{\left(\frac{1}{250}\right)\text{mm}}\right]$$

(pretty skinny!)
Incoherent impulse response is squared magnitude of $h[x,y]$*:*

$$\mathfrak{h}[x,y] = \left(250^2\right)^2\; SINC^2[250x, 250y] = 250^4 \cdot SINC^2\left[\frac{x}{\left(\frac{1}{250}\right)\text{mm}}, \frac{y}{\left(\frac{1}{250}\right)\text{mm}}\right]$$

and the incoherent transfer function is the 2-D Fourier transform:

$$\mathfrak{H}[\xi,\eta] = \mathcal{F}_2\{\mathfrak{h}[x,y]\} = TRI\left[\frac{\xi}{250}, \frac{\eta}{250}\right] \quad \textit{for } \xi,\eta \textit{ measured in cycles per mm.}$$

$$\begin{aligned}\mathfrak{H}[\xi,\eta] &= 50\% \implies \xi = \pm 125\frac{\text{cycles}}{\text{mm}} \\ \mathfrak{H}[\xi,\eta] &= 0 \implies \xi = \pm 250\frac{\text{cycles}}{\text{mm}}\end{aligned}$$

(b) $p[x,y]$ is a square aperture with sides of length 50 mm where the left half of the aperture (where $x < 0$) is covered with a sheet of glass of refractive index $n = 1.5$ and thickness $\tau_0 = 1000\,\text{nm}$.

First find the phase difference due to that piece of glass:

$$\Delta\varphi = \frac{2\pi}{\left(\frac{\lambda_0}{n}\right)} \cdot \tau_0 = \frac{2\pi}{\left(\frac{500\,\text{nm}}{1.5}\right)} \cdot 1000\,\text{nm} = 6\pi$$

so the phase difference is an integer multiple of 2π *radians and thus has no effective impact on the emerging light. The impulse response and transfer function are identical to that in part (a).*

(c) The pupil function consists of four square apertures with sides of 10 mm whose centers are arranged to form a square with sides 30 mm that is symmetrically placed about the optical axis.

$$\begin{aligned} p[x,y] &= RECT\left[\frac{x}{10\,\text{mm}}, \frac{y}{10\,\text{mm}}\right] \\ &\quad * (\delta[x+30\,\text{mm}, y+30\,\text{mm}] + \delta[x-30\,\text{mm}, y+30\,\text{mm}]) \\ &\quad + (\delta[x+30\,\text{mm}, y-30\,\text{mm}] + \delta[x-30\,\text{mm}, y-30\,\text{mm}]) \\ &= RECT\left[\frac{x}{10\,\text{mm}}, \frac{y}{10\,\text{mm}}\right] \\ &\quad * (\delta[x+30\,\text{mm}, y] + \delta[x-30\,\text{mm}, y]) \cdot (\delta[x, y+30\,\text{mm}] + \delta[x, y-30\,\text{mm}]) \end{aligned}$$

$$\begin{aligned} P[\xi,\eta] &= (10\,\text{mm})^2 \cdot SINC[10\,\text{mm}\cdot\xi, 10\,\text{mm}\cdot\eta] \cdot (2\cos[2\pi\xi\cdot 30\,\text{mm}]) \cdot (2\cos[2\pi\eta\cdot 30\,\text{mm}]) \\ &= 400\,\text{mm}^2 \cdot \cos[2\pi\xi\cdot 30\,\text{mm}] \cdot \cos[2\pi\eta\cdot 30\,\text{mm}] \cdot SINC[10\,\text{mm}\cdot\xi, 10\,\text{mm}\cdot\eta] \end{aligned}$$

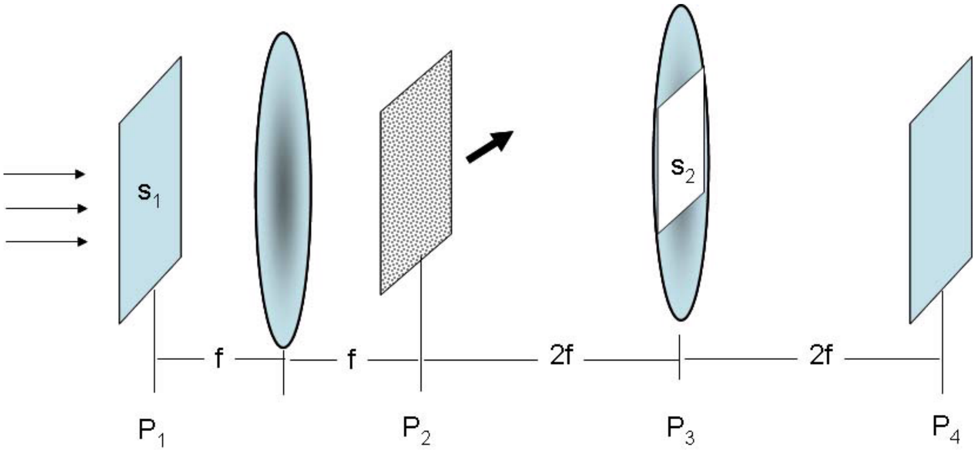
The coherent transfer function is:

$$\begin{aligned} H[\xi,\eta] &= p[-\lambda_0\cdot 2\mathbf{f}\cdot\xi, -\lambda_0\cdot 2\mathbf{f}\cdot\eta] = p[-500\,\text{nm}\cdot 200\,\text{mm}\cdot\xi, -500\,\text{nm}\cdot 200\,\text{mm}\cdot\eta] \\ &= p\left[-0.1\,\text{mm}^2\cdot\xi, -0.1\,\text{mm}^2\cdot\eta\right] \\ &= RECT\left[\frac{\xi}{\frac{1}{100\,\text{mm}}}, \frac{\eta}{\frac{1}{100\,\text{mm}}}\right] \\ &\quad * \left(\delta\left[0.1\,\text{mm}^2\cdot\xi - 30\,\text{mm}, \eta\right] + \delta\left[0.1\,\text{mm}^2\cdot\xi + 30\,\text{mm}, \eta\right]\right) \\ &\quad \cdot \left(\delta\left[\xi, 0.1\,\text{mm}^2\cdot\eta - 30\,\text{mm}\right] + \delta\left[\xi, 0.1\,\text{mm}^2\cdot\eta + 30\,\text{mm}\right]\right) \end{aligned}$$

(d) The pupil function is identical to that in part (c) but one of the four apertures is overlaid with the same sheet of glass in part (c).

Again, no change

4. Consider the optical system shown in the figure. A transparency with a real nonnegative *amplitude* transmittance (NOT transmittance of squared magnitude) $s_1[x,y]$ is placed in plane P_1 and illumination by a monochromatic, unit-intensity, normally incident plane wave. Lenses L_1 and L_2 are spherical with the same focal length $\mathbf{f}$. In plane P_2, which is the focal plane of L_1, a *moving* diffuser is placed. The effect of the moving diffuser can be considered to be the conversion of spatially coherent incident light into spatially incoherent transmitted light without changing the intensity distribution of light in plane P_3, in contact with L_2, is placed a second transparency, this one with *amplitude* transmittance $s_2[x,y]$. Find an expression for the intensity distribution on plane P_4.



The system up to the moving diffuser is purely coherent. We recognize that the amplitude at plane P_2 is the Fourier transform of $s_1[x,y]$:

$$g[x,y;P_2] = \frac{1}{i\lambda_0\mathbf{f}} \cdot S_1\left[\frac{x}{\lambda_0\mathbf{f}}, \frac{y}{\lambda_0\mathbf{f}}\right]$$

The moving diffuser disrupts the spatial coherence of the light, so everything that happens between planes P_2 and P_4 is modeled as an incoherent system. The lens at P_3 creates a unit-magnitude image at P_4. The "input" intensity at plane P_2 is the squared magnitude of the amplitude just given:

$$\begin{aligned} I[x,y;P_2] &= |g[x,y;P_2]|^2 = \left|\frac{1}{i\lambda_0\mathbf{f}} \cdot S_1\left[\frac{x}{\lambda_0\mathbf{f}}, \frac{y}{\lambda_0\mathbf{f}}\right]\right|^2 \\ &= \frac{1}{\lambda_0^2\mathbf{f}^2} \cdot \left|S_1\left[\frac{x}{\lambda_0\mathbf{f}}, \frac{y}{\lambda_0\mathbf{f}}\right]\right|^2 \end{aligned}$$

The amplitude transmittance of the pupil of the incoherent system is:

$$t[x,y] \cdot s_2[x,y]$$

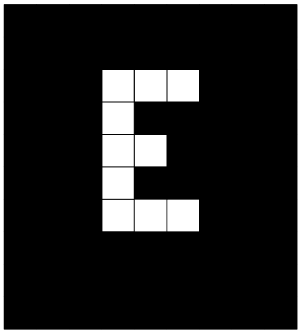
he intensity point spread function of the imaging system between planes P_2 and P_4 is:

$$\begin{aligned} \mathfrak{h}[x,y] &= |h[x,y]|^2 = \left|\frac{1}{i\lambda_0(2\mathbf{f})} \cdot \mathcal{F}_2\{s_2[x,y]\}|_{\rho\to\frac{r}{\lambda_0\mathbf{f}}}\right|^2 \\ &= \left(\frac{1}{\lambda_0(2\mathbf{f})}\right)^2 \left|S_2\left[\frac{x}{\lambda_0(2\mathbf{f})}, \frac{y}{\lambda_0(2\mathbf{f})}\right]\right|^2 \end{aligned}$$

so the intensity distribution at the output plane P_4 is:

$$\begin{aligned}
I\left[x,y;P_4\right] &= I\left[x,y;P_2\right] * \mathfrak{h}\left[x,y\right] \\
&= \left(\frac{1}{\lambda_0^2\mathbf{f}^2}\cdot\left|S_1\left[\frac{x}{\lambda_0\mathbf{f}},\frac{y}{\lambda_0\mathbf{f}}\right]\right|^2\right) * \left(\left(\frac{1}{\lambda_0\left(2\mathbf{f}\right)}\right)^2\left|S_2\left[\frac{x}{\lambda_0\left(2\mathbf{f}\right)},\frac{y}{\lambda_0\left(2\mathbf{f}\right)}\right]\right|^2\right) \\
&= \frac{1}{4\lambda_0^4\mathbf{f}^4}\left(\left|S_1\left[\frac{x}{\lambda_0\mathbf{f}},\frac{y}{\lambda_0\mathbf{f}}\right]\right|^2 * \left|S_2\left[\frac{x}{\lambda_0\left(2\mathbf{f}\right)},\frac{y}{\lambda_0\left(2\mathbf{f}\right)}\right]\right|^2\right)
\end{aligned}$$

5. The bitonal transparency $f[x,y]$ of an upper-case letter "E"(transparent character on opaque background, as shown in the figure) is placed with its center at the origin of coordinates (there is no specific "reference" source). The transparency is illuminated by light with wavelength λ_0 and the transmitted light travels down the z-axis into the Fraunhofer diffraction region located at distance z_1. The irradiance pattern is recorded on a photographic emulsion and processed to obtain a "linear" response and replaced at its original location. The transparency is reilluminated by light with the same wavelength from a point source located at the origin. The diffracted light propagates the distance z_2 to an observation plane.



Derive the amplitude and irradiance patterns that are reconstructed at the observation plane.

The object consists of 10 pixels and may be written either as the sum of those samples with appropriate shifts or as the difference of three rectangle functions:

$$e[x,y] = RECT\left[\frac{x}{3},\frac{y}{5}\right] + RECT[x,y] - RECT\left[\frac{x-\frac{1}{2}}{2},\frac{y}{3}\right]$$

$$\begin{aligned}
E[\xi,\eta] &= 15\cdot SINC[3\xi,5\eta] + SINC[\xi,\eta] - 6\cdot SINC[2\xi,3\eta]\exp[-i\pi\xi] \\
E\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right] &= 15\cdot SINC\left[\frac{3x}{\lambda_0 z_1},\frac{5y}{\lambda_0 z_1}\right] + SINC\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right] - 6\cdot SINC\left[\frac{2x}{\lambda_0 z_1},\frac{3y}{\lambda_0 z_1}\right]\exp \\
\left|E\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right]\right|^2 &= 225\cdot SINC^2\left[\frac{3x}{\lambda_0 z_1},\frac{5y}{\lambda_0 z_1}\right] \\
&+SINC^2\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right] \\
&-36\cdot SINC\left[\frac{2x}{\lambda_0 z_1},\frac{3y}{\lambda_0 z_1}\right] \\
&+30\cdot SINC\left[\frac{3x}{\lambda_0 z_1},\frac{5y}{\lambda_0 z_1}\right] SINC\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right] \\
&-12\cdot SINC\left[\frac{x}{\lambda_0 z_1},\frac{y}{\lambda_0 z_1}\right] SINC\left[\frac{2x}{\lambda_0 z_1},\frac{3y}{\lambda_0 z_1}\right]\cos\left[\pi\frac{x}{\lambda_0 z_1}\right] \\
&-180\cdot SINC\left[\frac{3x}{\lambda_0 z_1},\frac{5y}{\lambda_0 z_1}\right]\cos\left[\pi\frac{x}{\lambda_0 z_1}\right]
\end{aligned}$$

The transparency is illuminated by light from a point source and propagates a distance z_2 *to the image plane, which evaluates the autocorrelation of the signal:*

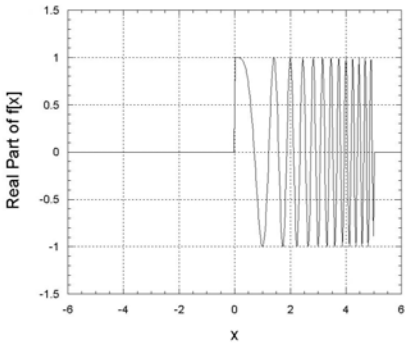
$$g[x,y] = \mathcal{F}_2^{-1}\left\{\left|E\left[\frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1}\right]\right|^2\right\} = e[x,y] \bigstar e[x,y]\Big|_{x=\frac{x}{\left(\frac{z_2}{z_1}\right)},\, y=\frac{y}{\left(\frac{z_2}{z_1}\right)}}$$

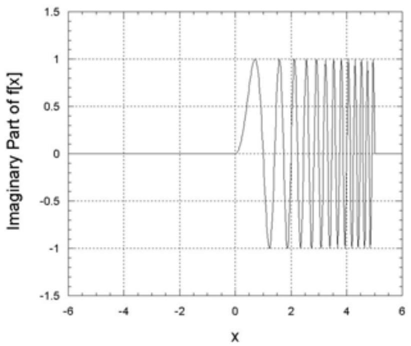
6. This problem demonstrates the utility of the stationary-phase approximation in imaging applications. Consider the 1-D modulated quadratic-phase function:

$$f[x] = RECT\left[\frac{x-2.5}{5}\right]\ \exp\left[+i\pi x^2\right]$$

(a) Try to evaluate $f[x] * f[x]$ (I dare you!)

As usual, I suggest graphing the function first:





The direct integral form for the convolution is:

$$g[x] = f[x] * f[x] = \left(RECT\left[\frac{x-2.5}{5}\right]\exp\left[+i\pi x^2\right]\right) * \left(RECT\left[\frac{x-2.5}{5}\right]\exp\left[+i\pi x^2\right]\right)$$

$$= \int_{-\infty}^{+\infty}\left(RECT\left[\frac{\alpha-2.5}{5}\right]\exp\left[+i\pi\alpha^2\right]\right)\left(RECT\left[\frac{(x-\alpha)-2.5}{5}\right]\exp\left[+i\pi(x-\alpha)^2\right]\right)d\alpha$$

If $x < 0$ *or* $x > 10$*, the rectangles do not overlap so the convolution* $g[x] = 0.$

If $0 \leq x \geq 5:$

$$\begin{aligned} g[x] &= \int_0^x \exp\left[+i\pi\alpha^2\right]\exp\left[+i\pi(x-\alpha)^2\right]d\alpha \\ &= \int_0^x \exp\left[+i\pi\alpha^2\right]\cdot\left(\exp\left[+i\pi x^2\right]\cdot\exp\left[+i\pi\alpha^2\right]\cdot\exp\left[-2\pi i x\alpha\right]\right)d\alpha \\ &= \exp\left[+i\pi x^2\right]\cdot\int_0^x \exp\left[+i\pi\left(\sqrt{2}\alpha\right)^2\right]\cdot\exp\left[-2\pi i x\alpha\right]\ d\alpha \end{aligned}$$

If $5 \leq x \leq 10:$

$$g[x] = \int_{x-5}^{5}\exp\left[+i\pi\alpha^2\right]\exp\left[+i\pi(x-\alpha)^2\right]d\alpha$$

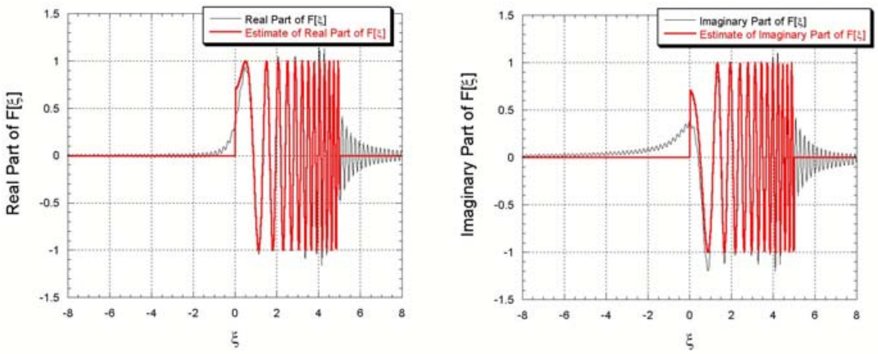
But we don't have the analytic solution to these integrals...

(b) Evaluate the stationary-phase approximation $\hat{F}[\xi]$ of the Fourier transform of $f[x]$ (easy)

$$
\begin{aligned}
F[\xi] &= \int_{-\infty}^{+\infty} \left(RECT\left[\frac{x-2.5}{5}\right] \exp\left[+i\pi x^2\right] \right) \exp\left[-2\pi i \xi x\right] \, dx \\
&= \int_{-\infty}^{+\infty} RECT\left[\frac{x-2.5}{5}\right] \cdot \exp\left[i\xi \left(\frac{\pi}{\xi}x^2 - 2\pi x\right)\right] \, dx \\
\mu[x] &= \frac{\pi}{\xi}x^2 - 2\pi x \\
\mu'[x] &= 2\pi \left(\frac{x}{\xi} - 1\right) \\
\mu'[x_0] &= 0 \Longrightarrow x_0 = \xi \\
\mu[x_0] &= \frac{\pi}{\xi}x^2 - 2\pi x \bigg|_{x_0=\xi} = \frac{\pi}{\xi}\xi^2 - 2\pi\xi = -\pi\xi \\
\mu''[x] &= \frac{2\pi}{\xi} = \mu''[x_0]
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}[\xi] &= |f[x_0]| \cdot \exp\left[+i\xi\mu[x_0]\right] \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \sqrt{\frac{2\pi}{\xi\mu''[x_0]}} \\
&= RECT\left[\frac{\xi-2.5}{5}\right] \cdot \exp\left[+i\xi(-\pi\xi)\right] \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \sqrt{\frac{2\pi}{\xi \cdot \frac{2\pi}{\xi}}} \\
&= RECT\left[\frac{\xi-2.5}{5}\right] \cdot \exp\left[-i\pi\xi^2\right] \cdot \exp\left[+i\frac{\pi}{4}\right] \\
&= RECT\left[\frac{\xi-2.5}{5}\right] \cdot \exp\left[-i\pi\left(\xi^2 - \frac{1}{4}\right)\right]
\end{aligned}
$$

which is complex valued and has both even and odd parts.



(c) Use the result of part (b) to find an approximation for expression in part (a)

Apply the filter theorem to $\hat{F}[\xi]$*:*

$$\begin{aligned}
\mathcal{F}_1\{f[x]*f[x]\} &= (F[\xi])^2 \simeq \left(\hat{F}[\xi]\right)^2 \\
&= \left(RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[-i\pi\xi^2\right]\cdot\exp\left[+i\frac{\pi}{4}\right]\right)^2 \\
\left(\hat{F}[\xi]\right)^2 &= RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\cdot\exp\left[+i\frac{\pi}{2}\right]
\end{aligned}$$

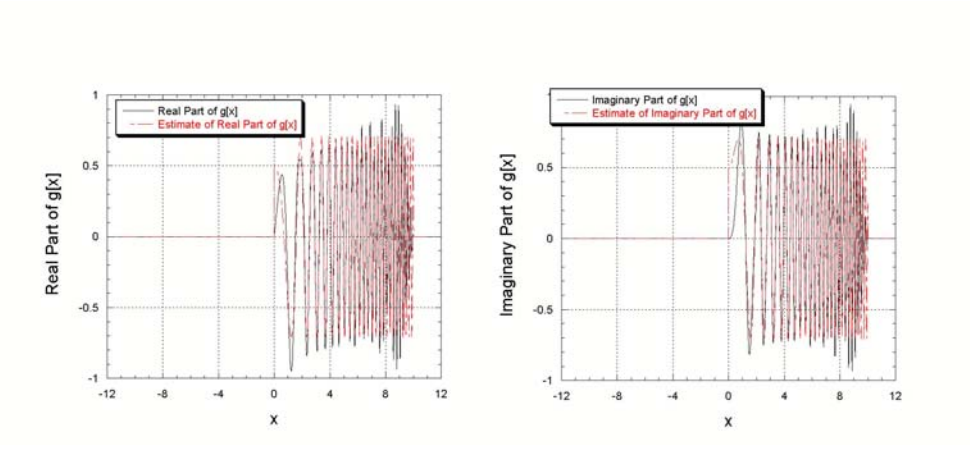
$$\begin{aligned}
g[x] &\simeq \mathcal{F}_1^{-1}\left\{\left(\hat{F}[\xi]\right)^2\right\} \\
&= \mathcal{F}_1^{-1}\left\{RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\cdot\exp\left[+i\frac{\pi}{2}\right]\right\} \\
&= \exp\left[+i\frac{\pi}{2}\right]\int_{-\infty}^{+\infty} RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\cdot\exp\left[+2\pi i\xi x\right]\;d\xi \\
&= \exp\left[+i\frac{\pi}{2}\right]\int_{-\infty}^{+\infty} RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[+ix\left(-\frac{\pi}{x}\left(\sqrt{2}\xi\right)^2+2\pi\xi\right)\right]\;d\xi \\
&= \exp\left[+i\frac{\pi}{2}\right]\int_{-\infty}^{+\infty} R[\xi]\cdot\exp\left[+ix\mu[\xi]\right]\;d\xi \\
\textit{where } R[\xi] &= RECT\left[\frac{\xi-2.5}{5}\right] \;\textit{ and }\; \mu[\xi] = -\frac{2\pi}{x}\xi^2+2\pi\xi
\end{aligned}$$

To evaluate the convolution, apply the stationary phase approximation to compute the inverse Fourier transform:

$$\begin{aligned}
\mu'[\xi] &= -\frac{4\pi}{x}\xi+2\pi \\
\mu'[\xi_0] &= 0 \Longrightarrow -\frac{4\pi}{x}\xi_0-2\pi=0 \Longrightarrow \xi_0=+\frac{x}{2} \\
\mu[\xi_0] &= -\frac{2\pi}{x}(\xi_0)^2+2\pi\xi_0 = -\frac{2\pi}{x}\left(+\frac{x}{2}\right)^2+2\pi\left(+\frac{x}{2}\right) \\
&= \frac{\pi x}{2} \\
\mu''[\xi] &= -\frac{4\pi}{x} = \mu''[\xi_0]
\end{aligned}$$

$$\begin{aligned}
g[x] &\simeq \exp\left[+i\frac{\pi}{2}\right]\cdot\left(R[\xi_0]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[+ix\mu[\xi_0]\right]\cdot\sqrt{\frac{2\pi}{x\mu''[\xi_0]}}\right) \\
&= \exp\left[+i\frac{\pi}{2}\right]\cdot\left(RECT\left[\frac{\xi_0-2.5}{5}\right]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[+ix\left(\frac{\pi x}{2}\right)\right]\cdot\sqrt{\frac{2\pi}{x\left(-\frac{4\pi}{x}\right)}}\right) \\
&= \exp\left[+i\frac{\pi}{2}\right]\cdot\left(RECT\left[\frac{\frac{x}{2}-2.5}{5}\right]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[+ix\left(\frac{\pi x}{2}\right)\right]\right) \\
&= \sqrt{-\frac{1}{2}}\cdot\exp\left[+i\frac{\pi}{2}\right]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(RECT\left[\frac{x-5}{10}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right]\right)
\end{aligned}$$

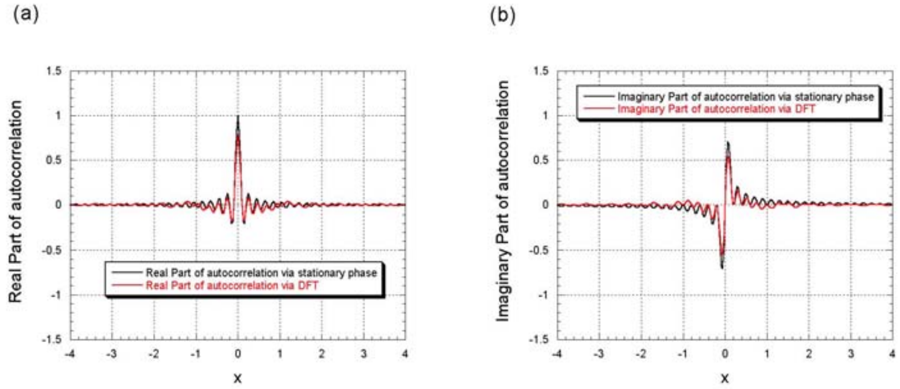
$$\boxed{g[x] \simeq \sqrt{\frac{1}{2}} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \left(RECT\left[\frac{x-5}{10}\right] \cdot \exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right]\right)}$$



(d) Use the result of part (b) to find an approximation for $f[x] \bigstar f[x] = f[x] * f^*[-x]$.

$$\begin{aligned}
f[x] \bigstar f[x] &= \mathcal{F}_1^{-1}\left\{|F[\xi]|^2\right\} \simeq \mathcal{F}_1^{-1}\left\{\left|\hat{F}[\xi]\right|^2\right\} \\
&= \mathcal{F}_1^{-1}\left\{\left| RECT\left[\frac{\xi-2.5}{5}\right] \cdot \exp\left[-i\pi\xi^2\right] \cdot \exp\left[+i\frac{\pi}{4}\right]\right|^2\right\} \\
&= \mathcal{F}_1^{-1}\left\{ RECT\left[\frac{\xi-2.5}{5}\right]\right\} \\
&= 5 \cdot SINC[5x] \cdot \exp[+2\pi i x(2.5)] \\
&= 5 \cdot SINC[5x] \cdot \exp[+5\pi i x]
\end{aligned}$$

which is a SINC function modulated by a complex sinusoid. Compare the estimate to a calculation via the DFT.



(e) Find an approximate expression for the Fourier transforms of the real and imaginary parts of $f[x]$.

$$\text{Re}\{f[x]\} = RECT\left[\frac{x-2.5}{5}\right]\cos\left[\pi x^2\right]$$

This is the product of an even function and a function with no symmetry, and so $\text{Re}\{f[x]\}$ *has even and odd parts. Thus its transform is Hermitian. We know some relations between the real and imaginary parts and of the even and odd parts of the function and the Fourier transform. For example, we can decompose* $f[x]$ *into:*

$$f[x] = \text{Re}\{f[x]\} + i\ \text{Im}\{f[x]\} = f_{even}[x] + f_{odd}[x]$$

where:

$$\text{Re}\{f[x]\} = \text{Re}\{f_{even}[x]\} + \text{Re}\{f_{odd}[x]\}$$

We also know the Fourier transforms of the even and odd parts of a real-valued function:

$$\begin{aligned}\mathcal{F}\{\text{Re}\{f_{even}[x]\}\} &= \text{Re}\{F_{even}[\xi]\} = \textit{Even part of } \text{Re}\{F[\xi]\} \\ \mathcal{F}\{\text{Re}\{f_{odd}[x]\}\} &= \text{Im}\{F_{odd}[\xi]\} = \textit{Odd part of } \text{Im}\{F[\xi]\}\end{aligned}$$

SO, the transform of the real part of $f[x]$ *is the sum of the even part of the real part of* $F[\xi]$ *and the odd part of the imaginary part of* $F[\xi]$. *In this example, we have the estimate* $\hat{F}[\xi]$

$$\hat{F}[\xi] = RECT\left[\frac{\xi-2.5}{5}\right]\cdot\exp\left[-i\pi\left(\xi^2-\frac{1}{4}\right)\right]$$

$$\begin{aligned}\text{Re}\left\{\hat{F}[\xi]\right\} &= RECT\left[\frac{\xi-2.5}{5}\right]\cdot\cos\left[-i\pi\left(\xi^2-\frac{1}{4}\right)\right] \\ &= RECT\left[\frac{\xi-2.5}{5}\right]\cdot\cos\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\end{aligned}$$

$$\begin{aligned}\text{Im}\left\{\hat{F}[\xi]\right\} &= RECT\left[\frac{\xi-2.5}{5}\right]\cdot\sin\left[-i\pi\left(\xi^2-\frac{1}{4}\right)\right] \\ &= -RECT\left[\frac{\xi-2.5}{5}\right]\cdot\sin\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\end{aligned}$$

where the cosine and sine chirps are BOTH even functions of ξ. *The even part of the real part is:*

$$\begin{aligned}\left(\text{Re}\left\{\hat{F}[\xi]\right\}\right)_{even} &= \frac{1}{2}\left(RECT\left[\frac{-\xi-2.5}{5}\right]+RECT\left[\frac{\xi-2.5}{5}\right]\right)\cdot\cos\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right] \\ &= \frac{1}{2}\left(RECT\left[\frac{\xi+2.5}{5}\right]+RECT\left[\frac{\xi-2.5}{5}\right]\right)\cdot\cos\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right] \\ &= \frac{1}{2}RECT\left[\frac{\xi}{10}\right]\cdot\cos\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\end{aligned}$$

The odd part of the imaginary part is:

$$\begin{aligned}\left(\mathrm{Im}\left\{\hat{F}\left[\xi\right]\right\}\right)_{odd} &= \frac{1}{2}\left(RECT\left[\frac{-\xi-2.5}{5}\right]-RECT\left[\frac{\xi-2.5}{5}\right]\right)\cdot-\sin\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\\ &= \frac{1}{2}\left(RECT\left[\frac{\xi+2.5}{5}\right]-RECT\left[\frac{\xi-2.5}{5}\right]\right)\cdot\sin\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\\ &= \frac{1}{2}RECT\left[\frac{\xi}{10}\right]\cdot SGN\left[\xi\right]\cdot\sin\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\end{aligned}$$

So the approximation of the transform of the real part of $f\left[x\right]$ is:

$$\begin{aligned}&\mathcal{F}\left\{\mathrm{Re}\left\{f\left[x\right]\right\}\right\}\\ \simeq\ &\frac{1}{2}RECT\left[\frac{\xi}{10}\right]\cdot\cos\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]+i\ \frac{1}{2}RECT\left[\frac{\xi}{10}\right]\cdot SGN\left[\xi\right]\cdot\sin\left[+i\pi\left(\xi^2-\frac{1}{4}\right)\right]\end{aligned}$$