1. A hologram is recorded with light from an argon laser at $\lambda_0 = 488\,\text{nm}$ and reconstructed with light from a He:Ne laser at $\lambda_1 = 632.8\,\text{nm}$.

   (a) Assume that the distance from the reference source to the hologram plane is $z_r = \infty$, the distance from the object to the hologram is $z_o = 100\,\text{mm}$, and the distance of the reconstructing source to the hologram is $z_p = \infty$. Determine the image distance z_i.

   This is a generalization of the derivations in §23.2.5 of the notes with different wavelengths. The hologram transparency is recorded with wavelength λ_0 and the reconstruction is created with λ_1. From Eq.(23.58) we know that the transmittance of the hologram includes three terms:

$$t[x,y] \propto \frac{1}{2} - \frac{1}{2}\left(\frac{\alpha_1}{1+\alpha_1^2}\,e^{-i\phi_1}\right) e^{+i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{-i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}} - \frac{1}{2}\left(\frac{\alpha_1}{1+\alpha_1^2}\,e^{+i\phi_1}\right) e^{-i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{+i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}} \quad (23.58)$$

   The hologram is illuminated with λ_1 so Eq.(23.60) is generalized to:

$$\begin{aligned} t[x,y]\cdot e^{+i\pi\frac{r^2}{\lambda_1 z_p}} &= \frac{1}{2}e^{+i\pi\frac{r^2}{\lambda_1 z_p}} \\ &- \frac{1}{2}\left(\frac{\alpha_1}{1+\alpha_1^2}\,e^{-i\phi_1}\right) e^{+i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{-i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} \\ &- \frac{1}{2}\left(\frac{\alpha_1}{1+\alpha_1^2}\,e^{+i\phi_1}\right) e^{-i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{+i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} \end{aligned}$$

   The spatial part of the term in the second line simplifies to:

$$\begin{aligned} e^{+i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{-i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} &= e^{+i\pi\frac{r^2}{\lambda_0 z_r}}\,e^{-i\pi\frac{r^2-2xx_0+x_0^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} \\ &= \exp\left[+i\pi r^2\left(\frac{1}{\lambda_0}\left(\frac{1}{z_r}-\frac{1}{z_o}\right)+\frac{1}{\lambda_1}\left(\frac{1}{z_p}\right)\right)\right] \\ &= \exp\left[+i\pi\frac{r^2}{\lambda_1}\left(\frac{\lambda_1}{\lambda_0}\left(\frac{1}{z_r}-\frac{1}{z_o}\right)+\frac{1}{z_p}\right)\right] \\ &= \exp\left[+i\pi\frac{r^2}{\lambda_1 z_i}\right] \end{aligned}$$

$$z_i = \left(\frac{\lambda_1}{\lambda_0}\left(\frac{1}{z_r}-\frac{1}{z_o}\right)+\frac{1}{z_p}\right)^{-1}$$

   The spatial part of the third term simplifies to:

$$\begin{aligned} e^{-i\pi\frac{(x^2+y^2)}{\lambda_0 z_r}}\,e^{+i\pi\frac{(x-x_0)^2+y^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} &= e^{-i\pi\frac{r^2}{\lambda_0 z_r}}\,e^{+i\pi\frac{r^2-2xx_0+x_0^2}{\lambda_0 z_o}}\,e^{+i\pi\frac{r^2}{\lambda_1 z_p}} \\ &= \exp\left[+i\pi\frac{r^2}{\lambda_1}\left(\frac{\lambda_1}{\lambda_0}\left(-\frac{1}{z_r}+\frac{1}{z_o}\right)+\frac{1}{z_p}\right)\right] \\ &= \exp\left[+i\pi\frac{r^2}{\lambda_1 z_i}\right] \end{aligned}$$

$$z_i = \left(\frac{\lambda_1}{\lambda_0}\left(-\frac{1}{z_r}+\frac{1}{z_o}\right)+\frac{1}{z_p}\right)^{-1}$$

So we can combine the two expressions for the image distance z_i in terms of the distance of the reference z_r, the distance of the object z_o, and the distance of the reconstruction source z_p:

$$\boxed{z_i = \left(\frac{1}{z_p} \pm \frac{\lambda_1}{\lambda_0}\frac{1}{z_r} \mp \frac{\lambda_1}{\lambda_0}\frac{1}{z_o}\right)^{-1}}$$

Now substitute for the cases given:

$$\begin{aligned} z_r &= \infty, z_o = 100\,\text{mm} \\ z_i &= \left(\frac{1}{\infty} \pm \frac{632.8\,\text{nm}}{488\,\text{nm}}\frac{1}{\infty} \mp \frac{632.8\,\text{nm}}{488\,\text{nm}}\frac{1}{100\,\text{mm}}\right)^{-1} \\ & \boxed{z_i = \pm 77.12\,\text{mm}} \text{ (virtual and real)} \end{aligned}$$

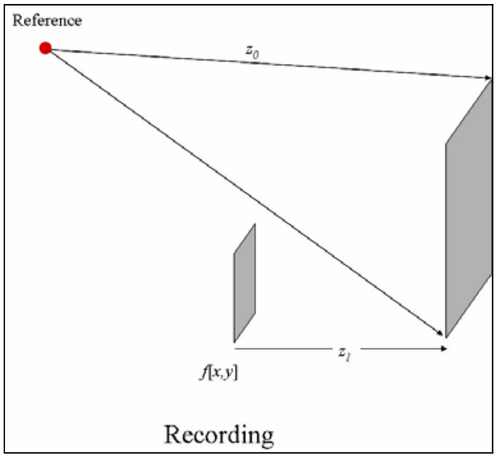
(b) Assume that $z_p = \infty$, $z_o = 100\,\text{mm}$, $z_r = 200\,\text{mm}$ and find z_i and the transverse magnification M_T.

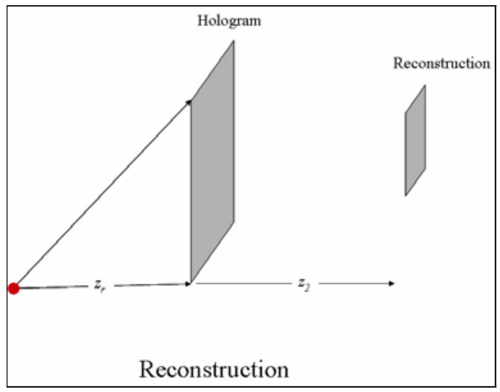
$$\begin{aligned} z_p &= \infty, z_o = 100\,\text{mm}, z_r = 200\,\text{mm} \\ z_i &= \left(\frac{1}{\infty} \pm \frac{632.8\,\text{nm}}{488\,\text{nm}}\frac{1}{200\,\text{mm}} \mp \frac{632.8\,\text{nm}}{488\,\text{nm}}\frac{1}{100\,\text{mm}}\right)^{-1} \\ & \boxed{z_i = \pm 154.24\,\text{mm}} \text{ (virtual and real)} \end{aligned}$$

The image distance is twice that in part (a). As for imaging lenses, the transverse magnification is related by the ratio of the object and image distances, but the wavelengths also must be considered:

$$M_T = \left|\frac{\lambda_1 z_i}{\lambda_0 z_0}\right| = \left|\frac{632.8\,\text{nm}}{488\,\text{nm}} \cdot \frac{\pm 154.24\,\text{mm}}{100\,\text{mm}}\right| = \boxed{M_T = 2}$$

2. Consider a holographic display that will project a real image of a planar transparency object. The recording and reconstruction geometries are shown in the figures. The reference and object are constrained to lie to the left of the hologram as shown, and the projected image must be placed on the right as shown. The hologram is placed exactly as recorded, i.e., it is not flipped to make the real image. A He:Ne laser is used for recording at $\lambda_0 = 632.8\,\text{nm}$ and an Argon laser for reconstruction at $\lambda_1 = 488\,\text{nm}$. The object transparency is a square of side 20 mm and the size of the image is to be 40 mm. The axial distance of the reconstruction source to the hologram is constrained to be $z_r = 500\,\text{mm}$. Subject to the constraints given, specify all possible axial object and reference distances that together produce the desired image.





The transverse magnification must be the ratio of the image and object sizes:

$$\begin{aligned} M_T &= \frac{h_i}{h_o} = \frac{40\,\text{mm}}{20\,\text{mm}} = 2 \\ &= \left|\frac{\lambda_1 z_i}{\lambda_0 z_0}\right| = \left|\frac{488\,\text{nm}}{632.8\,\text{nm}} \cdot \frac{z_i}{z_o}\right| \\ \Longrightarrow \quad & \frac{z_o}{z_i} = \frac{1}{M_T}\left|\frac{488\,\text{nm}}{632.8\,\text{nm}}\right| = \frac{1}{2}\left|\frac{488\,\text{nm}}{632.8\,\text{nm}}\right| = 0.386 \end{aligned}$$

Since no image distance was given, we can find the object distance in terms of the image distance, which must be positive since the image is real. We can use the result *for the object and image distances that was derived in Problem #1:*

$$
\begin{aligned}
z_i &= \left(\frac{1}{z_p} \pm \frac{\lambda_1}{\lambda_0}\frac{1}{z_r} \mp \frac{\lambda_1}{\lambda_0}\frac{1}{z_o}\right)^{-1} \\
z_i &= \left(\frac{1}{500\,\text{mm}} \pm \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{1}{500\,\text{mm}} \mp \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{1}{z_o}\right)^{-1} \\
&\Longrightarrow \quad 1 = \left(\frac{z_i}{500\,\text{mm}} \pm \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{z_i}{500\,\text{mm}} \mp \left[\frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{z_i}{z_o}\right]\right)^{-1} \\
1 &= \left(\frac{z_i}{500\,\text{mm}} \pm \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{z_i}{500\,\text{mm}} \mp M_T\right)^{-1} \\
1 &= z_i\left(\frac{1}{500\,\text{mm}} + \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{1}{500\,\text{mm}}\right) - 2 \Longrightarrow z_i \cong 0.847\,\text{m} \\
1 &= z_i\left(\frac{1}{500\,\text{mm}} - \frac{488\,\text{nm}}{632.8\,\text{nm}}\frac{1}{500\,\text{mm}}\right) + 2 \Longrightarrow z_i \cong -2.185\,\text{m}
\end{aligned}
$$

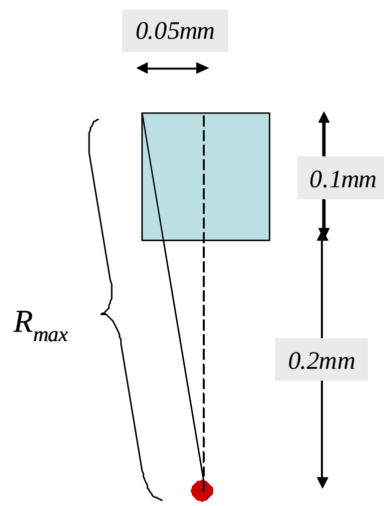
There are two solutions for the image distance: $z_i = +847\,\text{mm}$ *and* $z_i = --2.184\,\text{m}$. *Since the image must be real, we take the positive value:*

$$\boxed{z_i = +847\,\text{mm}}$$

3. A hologram is to be recorded using X rays with $\lambda_0 = 0.1\,\text{nm}$. The object is a square transparency made of lead with different thicknesses to absorb the X rays. The object and reference point source are located at the same plane. The width of the object is 0.1 mm and the minimum distance between the object and the reference is 0.2 mm. The distance to the recording X-ray film is 20 mm.

(a) Determine the maximum spatial frequency in cycles per mm in the interference pattern at the X-ray film.

At the distance and wavelength, the maximum spatial frequency is determined by the maximum separation of the object and reference. From the drawing, find the maximum distance from the reference to a point on the object:



$$R_{\text{max}} = \sqrt{(0.3\,\text{mm})^2 + (0.05\,\text{mm})^2} = 0.304\,\text{mm}$$

The maximum spatial frequency is determined from Eq.(23.4) (or other places):

$$\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_1}\right]$$

$$\xi_0 = \frac{x_0}{\lambda_0 z_1} \implies \rho_{\text{max}} = \frac{\sqrt{x_0^2+y_0^2}}{\lambda_0 z_1} = \frac{0.304\,\text{mm}}{0.1\,\text{nm}\cdot 20\,\text{mm}} = 152,000\frac{\text{cycles}}{\text{mm}}$$

(b) (OPTIONAL bonus) Assume that the film has sufficient resolution to recard all incident spatial variations in irradiance. It is proposed to reconstruct the images using visible light with $\lambda_1 = 600\,\text{nm}$. Explain why this will not work.

Again, here we want to illuminate a grating with the period equal to the reciprocal of the spatial frequency:

$$X_0 = \frac{1}{152,000}\,\text{mm} = 6.5789\,\text{nm}$$

which is much shorter than the wavelength of the illuminating light. We would expect the diffraction angle to be:

$$\sin[\theta] = \frac{\lambda_0}{X_0} = \frac{600\,\text{nm}}{\left(\frac{1}{152000}\,\text{mm}\right)} \cong 91.2$$

For which there are no real-valued solutions; in other words, only evanescent light emerges from the grating if illuminated at $\lambda_0 = 600\,\mathrm{nm}$

4. The table lists approximate cutoff frequencies in cycles per mm for four emulsions (that may no longer be commercially available!). Assume illumination at $\lambda_0 = 632.8\,\text{nm}$ and that the point source reference is located on the optical axis at a distance of 100 mm from the emulsion (Fraunhofer diffraction model is valid!). The planar object also is located at the same distance . For each, estimate the radius of the circle about the reference point where reconstructions of object points are visible

Emulsion	$\rho_{\max}$ $\left(\frac{\text{cycles}}{\text{mm}}\right)$
Kodak TRI-X	25
Kodak High-Contrast Copy	20
Kodak SO-243	150
Agfa Agepan FF	300
Kodak 649F	2000
Agfa 8E75 HD	5000

I place the reference point on the optical axis. We have seen (many times, e.g., Eq.23.4) that the spatial frequency in the Fraunhofer region of a point object located at the off-axis distance x_0 from the on-axis reference is:

$$\xi_0 = \frac{x_0}{\lambda_0 z_1}$$

so the generalization to 2-D object is that the spatial frequency ρ is:

$$\rho_0 = \sqrt{\xi^2 + \eta^2} = \frac{\sqrt{x_0^2 + y_0^2}}{\lambda_0 z_1} = \frac{r_0}{\lambda_0 z_1}$$

Equate the cutoff frequency $\rho_{\max}$ to find expressions for the largest separation $r_{\max}$:

Emulsion	$\rho_{\max}$ $\left(\frac{\text{cycles}}{\text{mm}}\right)$	$r_{\max} = \rho_{\text{cutoff}} \cdot \lambda_0 z_1$
Kodak TRI-X	25	$25\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 1.582\,\text{mm}$
Kodak High-Contrast Copy	20	$20\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 1.266\,\text{mm}$
Kodak SO-243	150	$150\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 9.49\,\text{mm}$
Agfa Agepan FF	300	$300\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 1.898\,4\,\text{mm}$
Kodak 649F	2000	$2000\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 126.6\,\text{mm}$
Agfa 8E75 HD	5000	$5000\frac{\text{cycles}}{\text{mm}} \cdot 632.8\,\text{nm} \cdot 100\,\text{mm} \cong 316.4\,\text{mm}$

5. Consider the Fresnel hologram of the two point sources that emit the same wavelength λ_0. The hologram is processed, replaced in the original location, and illuminated by both sources. The "real-image" reconstruction of the fainter object source is measured by a detector placed at its location. The fainter source is then moved at a constant velocity in some direction while both continue to illuminate the hologram in any convenient direction. Describe the brightness measured by the detector as a function of time. Be as quantitative as possible, but a qualitative description is valuable too.

The relative locations of the two point sources was not specified, so we have (at least) three cases to consider: (a) both points in the same plane at distance z_1 from the emulsion (§23.2.1 in notes); (b) point sources on axis at different distances from the emulsion (§23.2.4); (c) point sources in different planes with the object point offset by the distance x_0 (§23.2.4).

(a) *The hologram is a biased sinusoidal wave with period determined by the separation and modulation determined by the relative brightness. The source function consists of two Dirac delta functions in the sampe plane given by Eq.(23.33):*

$$f[x,y] = \delta[x,y,z] + \left(\alpha_1 e^{+i\phi_1}\right)\ \delta[x-x_0,y,z] \tag{23.33}$$

The amplitude at the hologram plane comes from Eq.(23.35):

$$s[x,y;z_1] = K_0\left(e^{+i\pi\frac{x^2}{\lambda_0 z_1}} + \alpha_1 e^{+i\phi_1}\ e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}}\right) e^{+i\pi\frac{y^2}{\lambda_0 z_1}} \tag{23.35}$$

The transmittance function is proportional to the squared magnitude in Eq.(23.37); split it into its complex form:

$$\begin{aligned} t[x,y] &= \frac{1}{2}\left(1-\frac{2\alpha_1}{1+\alpha_1^2}\ \cos\left[2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right]\right)\ 1[y] \qquad (23.37)\\ &= \frac{1}{2}\left(1[x,y]-\frac{2\alpha_1}{1+\alpha_1^2}\ \left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}+e^{-i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}\right)\right)\end{aligned}$$

so the transmittance includes a constant average transmittance and a linear-phase cosine function. If illuminated by BOTH sources, the result includes six complex exponential terms, which then must be propagated to the observation plane; only three of these produce real images:

$$\begin{aligned} s[x,y;z_1]\cdot t[x,y] &= K_0\left(e^{+i\pi\frac{x^2}{\lambda_0 z_1}}+\alpha_1 e^{+i\phi_1}\ e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}}\right) e^{+i\pi\frac{y^2}{\lambda_0 z_1}}\\ &\quad\cdot\frac{1}{2}\left(1[x,y]-\frac{2\alpha_1}{1+\alpha_1^2}\ \left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}+e^{-i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}\right)\right)\\ &\propto e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}}\cdot\frac{1}{2}\cdot 1[x,y]+\alpha_1 e^{+i\phi_1}\ e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}}\ e^{+i\pi\frac{y^2}{\lambda_0 z_1}}\cdot\frac{1}{2}\cdot 1[x,y]\\ &\quad-\left(e^{+i\pi\frac{x^2}{\lambda_0 z_1}}+\alpha_1 e^{+i\phi_1}\ e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}}\right) e^{+i\pi\frac{y^2}{\lambda_0 z_1}}\ \frac{\alpha_1}{1+\alpha_1^2} e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}\\ &\quad-\left(e^{+i\pi\frac{x^2}{\lambda_0 z_1}}+\alpha_1 e^{+i\phi_1}\ e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}}\right) e^{+i\pi\frac{y^2}{\lambda_0 z_1}}\ \frac{\alpha_1}{1+\alpha_1^2} e^{-i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)}\end{aligned}$$

$$\propto \frac{1}{2}\cdot e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} + \frac{1}{2}\cdot \alpha_1 e^{+i\phi_1} \; e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}}$$
$$-e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} \frac{\alpha_1}{1+\alpha_1^2} e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} - e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}} \frac{\alpha_1^2}{1+\alpha_1^2} e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\phi_1}$$
$$-e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} \frac{\alpha_1}{1+\alpha_1^2} e^{-i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} - e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}} \frac{\alpha_1^2}{1+\alpha_1^2} e^{-i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\phi_1}$$

We now convolve this with the Fresnel propagator for real images so that $z_2 > 0$:
$e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}SGN[z_2]} = e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}$

$$\begin{aligned}
(s\,[x,y;z_1]\cdot t\,[x,y]) * h\,[x,y;z_2] \propto\; & \frac{1}{2}\cdot\left(e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}\right) \\
& + \frac{1}{2}\cdot \alpha_1 e^{+i\phi_1}\left(e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}}\right) * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}} \\
& - \frac{\alpha_1}{1+\alpha_1^2}\left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}\right) \\
& - \frac{\alpha_1^2}{1+\alpha_1^2} e^{+i\phi_1}\left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}\right) \\
& - \frac{\alpha_1}{1+\alpha_1^2}\left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}\right) \\
& - \frac{\alpha_1^2}{1+\alpha_1^2} e^{+i\phi_1}\left(e^{+i\left(2\pi\xi_0\left(x-\frac{x_0}{2}\right)-\phi_1\right)} e^{+i\pi\frac{(x-x_0)^2}{\lambda_0 z_1}} e^{+i\pi\frac{y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2+y^2}{\lambda_0 z_2}}\right)
\end{aligned}$$

The real images are formed where the chirp rates have equal magnitude and opposite sign.