

1. Goodman 9-1: A hologram is recorded using a spherical reference wave that is diverging from the point  $[x_r, y_r, z_r]$  and the images from that hologram are played back with a reconstruction beam that is diverging from the point  $[x_p, y_p, z_p]$ . The wavelength used for both recording and reconstruction is  $\lambda_1$ . The hologram is taken to be circular with diameter  $D$ . It is claimed that the image of an arbitrary three-dimensional object obtained by this method is entirely equivalent to that obtained by a lens of the same diameter and the same distance from the object and a prism, where again the wavelength is  $\lambda_1$ . What are the two possible focal lengths for the lens that produce equivalence?

*A complicated but correct solution to the problem would write all the fields incident on the film, and the intensity, and find the fields transmitted by the hologram. A much simpler solution is based on Eq. (9-38)*

$$z_i = \left( \frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1}$$

after setting  $\lambda_2 = \lambda_1$ :

$$\frac{1}{z_i} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o}$$

*This equation should now be compared with the lens law, which must be adapted to the sign convention used in the discussion of holographic image locations ( $z_o < 0$  for an object to the left of the hologram or lens, which implies that  $z_o = -|z_o|$ ). The lens law is:*

$$\frac{1}{z_o} + \frac{1}{z_i} = \frac{1}{\mathbf{f}} \implies \frac{1}{z_i} = \frac{1}{\mathbf{f}} - \frac{1}{z_o} = \frac{1}{\mathbf{f}} + \frac{1}{|z_o|}$$

*Equate these equations for  $z_i^{-1}$ :*

$$\frac{1}{z_p} \pm \frac{1}{z_r} \mp \left( -\frac{1}{|z_o|} \right) = \frac{1}{\mathbf{f}} + \frac{1}{|z_o|} \implies \frac{1}{\mathbf{f}} = \frac{1}{z_p} \pm \frac{1}{z_r} \pm \frac{1}{|z_o|} - \frac{1}{|z_o|}$$

*which yields two focal lengths. Use the upper sign to obtain:*

$$\begin{aligned} \frac{1}{\mathbf{f}_1} &= \frac{1}{z_p} + \frac{1}{z_r} + \frac{1}{|z_o|} - \frac{1}{|z_o|} = \frac{1}{z_p} + \frac{1}{z_r} \\ \implies \mathbf{f}_1 &= \left( \frac{1}{z_p} + \frac{1}{z_r} \right)^{-1} \end{aligned}$$

*The lower sign yields*

$$\begin{aligned} \frac{1}{\mathbf{f}_2} &= \frac{1}{z_p} - \frac{1}{z_r} - \frac{1}{|z_o|} - \frac{1}{|z_o|} = \frac{1}{z_p} - \frac{1}{z_r} - \frac{2}{|z_o|} \\ \implies \mathbf{f}_2 &= \left( \frac{1}{z_p} - \frac{1}{z_r} - \frac{2}{|z_o|} \right)^{-1} \end{aligned}$$

*Note that one of the two lenses has a focal length that depends on  $z_o$ , the location of the object.*

2. Goodman 9-3: A hologram is recorded and its images reconstructed with the same wavelength  $\lambda$ . Assuming that  $z_o < 0$ , show that when  $z_p = z_r$ , there results a virtual image with unity transverse magnification, whereas with  $z_p = -z_r$ , there results a real image with unity transverse magnification. What is the transverse magnification of the twin image in each case?

Take first the case of  $z_p = z_r$ . The image distance is:

$$\frac{1}{z_i} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o} \implies \frac{1}{z_i} = \frac{1}{z_p} \pm \frac{1}{z_p} \mp \frac{1}{z_o}$$

The upper-sign solution is:

$$z_i = \left( \frac{1}{z_p} + \frac{1}{z_p} - \frac{1}{z_o} \right)^{-1} = \left( \frac{2}{z_p} - \frac{1}{z_o} \right)^{-1} = \left( \frac{2z_o - z_p}{z_p z_o} \right)^{-1} = \frac{z_p z_o}{2z_o - z_p}$$

The lower-sign solution is:

$$z_i = \left( \frac{1}{z_p} - \frac{1}{z_p} + \frac{1}{z_o} \right)^{-1} = z_o$$

Since the object is to the left of the hologram, then  $z_o < 0$  and the lower-sign solution is a virtual image. The upper-sign solution depends on both  $z_o$  and  $z_p$  and so may be real or virtual.

The transverse magnifications are:

$$\begin{aligned} \text{lower sign} & : M_T = \left| \frac{z_i}{z_o} \right| = +1 \\ \text{upper sign} & : M_T = \left| \frac{z_p}{2z_o - z_p} \right| \end{aligned}$$

Now consider the case  $z_p = -z_r$ . The two solutions for image distance are:

$$\begin{aligned} z_i & = \left( \frac{1}{z_p} + \frac{1}{z_r} - \frac{1}{z_o} \right)^{-1} = \left( \frac{1}{z_p} - \frac{1}{z_p} - \frac{1}{z_o} \right)^{-1} = -z_o \\ M_T & = |-1| = +1 \\ z_i & = \left( \frac{1}{z_p} - \frac{1}{z_r} + \frac{1}{z_o} \right)^{-1} = \left( -\frac{1}{z_r} - \frac{1}{z_r} + \frac{1}{z_o} \right)^{-1} = \left( -\frac{2}{z_r} + \frac{1}{z_o} \right)^{-1} \\ & = \frac{z_r z_o}{2z_o - z_r} \\ M_T & = \left| -\frac{z_r}{2z_o - z_r} \right| \end{aligned}$$

Since  $z_o < 0$ , the first image is real with unit magnification.

3. Goodman 9-6: It is proposed to record an X-ray hologram using coherent radiation of wavelength  $\lambda = 0.1 \text{ nm}$  and to reconstruct the images optically using light with  $\lambda = 600 \text{ nm}$ . The object is a square transparency with a pattern of absorption at the X-ray wavelength. The lensless Fourier transform recording geometry is chosen (Goodman Figure 9.14 p. 322). The width of the object is  $100 \mu\text{m}$  and the minimum distance between the object and the reference is to be  $200 \mu\text{m}$  to assure that the twin images will be separated from “on-axis” interference. The X-ray film is placed  $20 \text{ mm}$  from the object.

- (a) What is the maximum spatial frequency (cycles per mm) in the interference pattern falling on the film?

*The maximum spatial frequency is the maximum distance from the reference point to any point on the object ( $d_{\text{max}} = \sqrt{(100 \mu\text{m} + 200 \mu\text{m})^2 + (50 \mu\text{m})^2} \cong 304 \mu\text{m}$ ) in this case), divided by  $\lambda_0 z$  :*

$$\rho_{\text{max}} = \frac{d_{\text{max}}}{\lambda_0 z} \cong \frac{304 \mu\text{m}}{0.1 \text{ nm} \cdot 20 \text{ mm}} \cong 152 \frac{\text{cycles}}{\mu\text{m}} = 152,000 \frac{\text{cycles}}{\text{mm}}$$

- (b) Assume that the film has sufficient resolution to record all of the incident intensity variations. It is proposed to reconstruct the images in the usual manner, i.e., by looking in the rear focal plane of a Fourier transforming lens. What will this experiment fail?

*The experiment will fail because the periods of all components of the holographic grating are smaller than  $d_{\text{max}}$ , which is MUCH smaller than  $\lambda = 600 \text{ nm}$  for the reconstruction source. As a consequence, all diffraction orders will be evanescent, and there will be no way to form an image.*

4. Goodman 9-15: A certain emulsion has a nonlinear curve of amplitude transmittance  $t_A$  vs. exposure  $E$ :

$$t_A = t_b + \beta E_1^3$$

where  $E_1$  is the range of exposure variation about the reference exposure and  $t_b$  is the uniform “bias” transmittance established by the constant reference exposure.

- (a) Assuming a planar reference wave  $A \exp[-2\pi i \alpha x]$  and an object wave:

$$a[x, y] \exp[-i\phi[x, y]]$$

at the film, find an expression for that portion of the transmitted field that generates twin first-order images.

*The exposure on the film is:*

$$\begin{aligned} E &= A^2 + a^2 + 2Aa \cos[2\pi\alpha x - \phi] \\ &\equiv A^2 + E_1 \end{aligned}$$

*where  $E_1$  is the variation of exposure about the bias contributed by the reference:*

$$E_1 = a^2 + 2Aa \cos[2\pi\alpha x - \phi]$$

*Evaluate the cube of the exposure variation; recall that:*

$$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$\begin{aligned} E_1^3 &= (a^2 + 2Aa \cos[2\pi\alpha x - \phi])^3 \\ &= (a^2)^3 + 3 \cdot (a^2)^2 \cdot 2Aa \cos[2\pi\alpha x - \phi] + 3 \cdot (a^2) \cdot (2Aa \cos[2\pi\alpha x - \phi])^2 \\ &\quad + (2Aa \cos[2\pi\alpha x - \phi])^3 \end{aligned}$$

$$\implies E_1^3 = a^6 + 6a^5 A \cos[2\pi\alpha x - \phi] + 12a^4 A^2 \cos^2[2\pi\alpha x - \phi] + 8A^3 a^3 \cos^3[2\pi\alpha x - \phi]$$

*Recall that:*

$$\begin{aligned} \cos^2[\theta] &= \frac{1}{2}(1 + \cos[2\theta]) \\ \sin^2[\theta] &= \frac{1}{2}(1 - \cos[2\theta]) \\ e^{i(3\theta)} &= \cos[3\theta] + i \sin[3\theta] = (e^{i\theta})^3 = (\cos[\theta] + i \sin[\theta])^3 \\ &= \cos^3[\theta] + 3i \cos^2[\theta] \sin[\theta] - 3 \cos[\theta] \sin^2[\theta] - i \sin^3[\theta] \\ \implies \cos^3[\theta] &= \cos[3\theta] + 3 \cos[\theta] \sin^2[\theta] \\ &= \cos[3\theta] + 3 \cos[\theta] \cdot (1 - \cos^2[\theta]) \\ \cos^3[\theta] &= \cos[3\theta] + 3 \cos[\theta] - 3 \cos^3[\theta] \\ \implies 4 \cos^3[\theta] &= \cos[3\theta] + 3 \cos[\theta] \\ \implies \cos^3[\theta] &= \frac{1}{4} \cos[3\theta] + \frac{3}{4} \cos[\theta] \end{aligned}$$

$$\begin{aligned}\cos^3 [2\pi\alpha x - \phi] &= \frac{1}{4} \cos [3 \cdot (2\pi\alpha x - \phi)] + \frac{3}{4} \cos [2\pi\alpha x - \phi] \\ \cos^2 [2\pi\alpha x - \phi] &= \frac{1}{2} + \frac{1}{2} \cdot \cos [2 \cdot (2\pi\alpha x - \phi)]\end{aligned}$$

Collect all the terms:

$$\begin{aligned}E_1^3 &= a^6 + 6a^5 A \cos [2\pi\alpha x - \phi] + 12a^4 A^2 \cos^2 [2\pi\alpha x - \phi] + 8A^3 a^3 \cos^3 [2\pi\alpha x - \phi] \\ &= a^6 + 6a^5 A \cos [2\pi\alpha x - \phi] + 12a^4 A^2 \left( \frac{1}{2} + \frac{1}{2} \cos [2 \cdot (2\pi\alpha x - \phi)] \right) \\ &\quad + 8A^3 a^3 \left( \frac{1}{4} \cos [3 \cdot (2\pi\alpha x - \phi)] + \frac{3}{4} \cos [2\pi\alpha x - \phi] \right) \\ &= a^6 + 6a^5 A \cos [2\pi\alpha x - \phi] + 6a^4 A^2 + 6a^4 A^2 \cos [2 \cdot (2\pi\alpha x - \phi)] \\ &\quad + 2A^3 a^3 \cdot \cos [3 \cdot (2\pi\alpha x - \phi)] + 6A^3 a^3 \cos [2\pi\alpha x - \phi]\end{aligned}$$

$$\begin{aligned}E_1^3 &= (a^6 + 6a^4 A^2) + (6a^5 A + 6a^3 A^3) \cos [2\pi\alpha x - \phi] + 6a^4 A^2 \cos [2 \cdot (2\pi\alpha x - \phi)] \\ &\quad + 2A^3 a^3 \cdot \cos [3 \cdot (2\pi\alpha x - \phi)]\end{aligned}$$

The linear cosine term generates the first-order image:

$$\boxed{g[x, y] = (6(a[x, y])^5 A + 6(a[x, y])^3 A^3) \cos [2\pi\alpha x - \phi]}$$

(b) To what does this expression reduce if  $A \gg |a|$ ?

$$\boxed{g[x, y] \cong 6(a[x, y])^3 A^3 \cos [2\pi\alpha x - \phi]}$$

(c) How do the amplitude and phase modulations obtained in the previous parts of the problem compare with the ideal amplitude and phase modulations present when the film is a linear curve of  $t_A$  vs.  $E$ ?

*The phase modulation is correct, but the amplitude modulation is distorted from its ideal value of  $2A \cdot a[x, y]$*