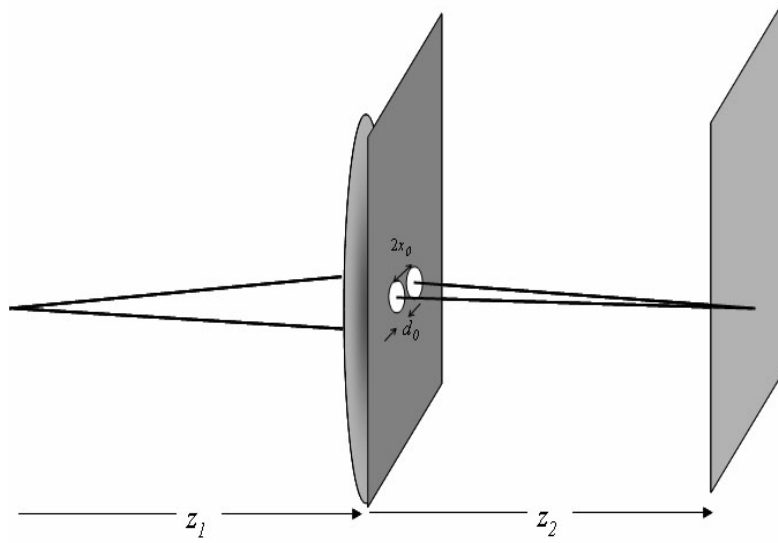


1. The pupil function of the lens in an imaging system has the form:

$$p[x, y] = CYL \left(\frac{\sqrt{x^2 + y^2}}{d_0} \right) * (\delta[x + x_0, y] + \delta[x - x_0, y])$$

where $d_0 \ll x_0, y_0$. Light through each small opening interferes to create a fringe pattern at the image plane.



The Fourier transform of the pupil function is:

$$\begin{aligned} P[\xi, \eta] &= \frac{\pi d_0^2}{4} \text{SOMB} \left(d_0 \sqrt{\xi^2 + \eta^2} \right) \cdot 2 \cos [2\pi \xi \cdot x_0] \\ P \left[\frac{x}{\lambda_0 z_2}, \frac{y}{\lambda_0 z_2} \right] &= \frac{\pi d_0^2}{4} \text{SOMB} \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot 2 \cos \left[2\pi \frac{x_0}{\lambda_0 z_2} \cdot x \right] \\ &= \frac{\pi d_0^2}{4} \text{SOMB} \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot 2 \cos \left[2\pi \frac{x}{\left(\frac{\lambda_0 z_2}{x_0} \right)} \right] \end{aligned}$$

If the light source is coherent with wavelength λ_0 , then the impulse response is:

$$\begin{aligned} h[x, y; z_2, \lambda_0] &= \frac{1}{i\lambda_0 z_2} \exp \left[+2\pi i \frac{z_0}{\lambda_0} \right] P \left[\frac{x}{\lambda_0 z_2}, \frac{y}{\lambda_0 z_2} \right] \\ &= \frac{1}{i\lambda_0 z_2} \exp \left[+2\pi i \frac{z_0}{\lambda_0} \right] \cdot \frac{\pi d_0^2}{2} \left(\text{SOMB} \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot \cos \left[2\pi \frac{x}{\left(\frac{\lambda_0 z_2}{x_0} \right)} \right] \right) \end{aligned}$$

If the light source is incoherent with mean wavelength λ_μ , then the impulse response is:

$$\begin{aligned} \mathfrak{h}[x, y; z_2, \lambda_\mu] &\propto \left| P \left[\frac{x}{\lambda_\mu z_2}, \frac{y}{\lambda_\mu z_2} \right] \right|^2 \\ &= \left(\frac{\pi d_0^2}{2} \right)^2 \left(\text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot \cos^2 \left[2\pi \frac{x_0}{\lambda_0 z_2} \cdot x \right] \right) \end{aligned}$$

Recall that

$$\cos^2[\theta] = \frac{1}{2} (1 + \cos[2\theta])$$

so the phase angle (and thus the spatial frequency) increases with increasing separation $2x_0$:

$$\begin{aligned} \mathfrak{h}[x, y; z_2, \lambda_\mu] &\propto \frac{\pi^2 d_0^4}{8} \left(\text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot \left(1 + \cos \left[2\pi \cdot 2 \cdot \frac{x}{\frac{\lambda_0 z_2}{x_0}} \right] \right) \right) \\ &= \frac{\pi^2 d_0^4}{8} \left(\text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot \left(1 + \cos \left[2\pi \left(\frac{2x_0}{\lambda_0 z_2} \right) x \right] \right) \right) \end{aligned}$$

- (a) Determine the spatial frequency of this fringe in terms of the center-to-center spacing $2 \cdot x_0$ between the openings, the wavelength λ_0 , and the image distance z_2 .

The spatial frequency of the amplitude fringe is $\boxed{\frac{x_0}{\lambda_0 z_2}}$ and that of the irradiance fringe is $\boxed{\frac{2x_0}{\lambda_0 z_2}}$, both of which increase with the separation $2x_0$,

- (b) Specify the envelope of the fringe pattern due to the finite diameter d_0 of the apertures.

The envelope is the square of the sombrero function:

$$\text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right)$$

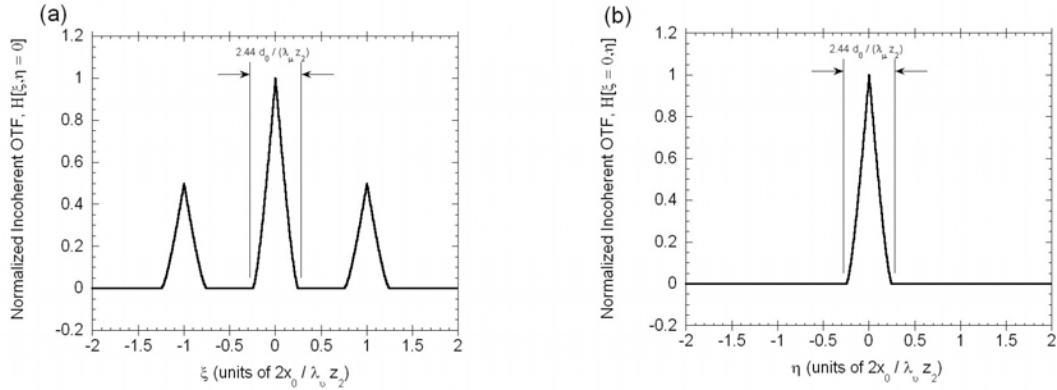
whose central lobe increases in width with decreasing aperture size d_0 .

2. For the pupil function in the previous problem, sketch the cross sections of the incoherent transfer function. Label the various cutoff frequencies and center frequencies on the graphs.

The incoherent transfer function is the Fourier transform of the incoherent impulse response:

$$\begin{aligned}
 \mathcal{F}_2 \{h[x, y; z_2, \lambda_\mu]\} &= \mathcal{F}_2 \left\{ \frac{\pi^2 d_0^4}{8} \left(\text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \cdot \left(1 + \cos \left[2\pi \left(\frac{2x_0}{\lambda_0 z_2} \right) x \right] \right) \right) \right\} \\
 &= \frac{\pi^2 d_0^4}{8} \cdot \mathcal{F}_2 \left\{ \text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \right\} * \mathcal{F}_2 \left\{ \left(1 + \cos \left[2\pi \left(\frac{2x_0}{\lambda_0 z_2} \right) x \right] \right) \right\} \\
 &= \frac{\pi^2 d_0^4}{8} \cdot \mathcal{F}_2 \left\{ \text{SOMB}^2 \left(\frac{\sqrt{x^2 + y^2}}{\left(\frac{\lambda_0 z_2}{d_0} \right)} \right) \right\} * \left(\delta[\xi, \eta] + \frac{1}{2} \delta \left[\xi + \frac{2x_0}{\lambda_0 z_2}, \eta \right] + \frac{1}{2} \delta \left[\xi - \frac{2x_0}{\lambda_0 z_2}, \eta \right] \right)
 \end{aligned}$$

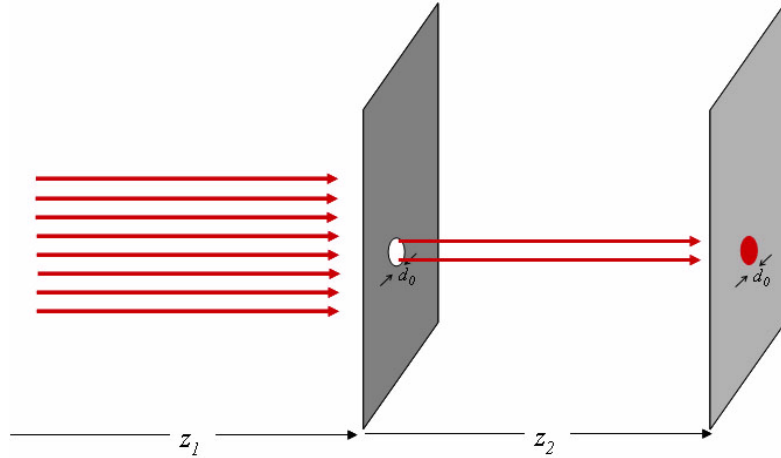
The Fourier transform of the square of the sombrero is the autocorrelation of the cylinder function, which is the “circular triangle.” The width of the circular triangle is approximately $2.44 \frac{d_0}{\lambda_\mu z_2}$ and the centers are located at $[\xi, \eta] = [0, 0], \left[\pm \frac{2x_0}{\lambda_0 z_2}, 0 \right]$



3. Consider a pinhole camera where the pupil function (circular pinhole of diameter d_0) is small compared to the object distance z_1 , which may be approximated as infinite. Assume that the object is illuminated by quasimonochromatic light with mean wavelength λ_μ .

To find the OTF, first find the point spread function under the different conditions. If the object is a point source and the object distance z_1 is very large, then we can assume that the pinhole is illuminated by a normally-incident plane wave.

- (a) Assuming that the pinhole is small, but sufficiently large so that ray optics is valid, find the optical transfer function of the camera. Determine the spatial frequency of the first zero of the OTF (which we may call the “cutoff frequency”).



If the ray optics description is valid, then the incoherent impulse response is proportional to the projection of the pinhole:

$$\mathfrak{h}[x, y; \lambda_\mu, z_2] = A_0 \text{CYL} \left(\frac{r}{d_0} \right)$$

Note that this does NOT depend on wavelength λ_μ or propagation distance z_2 . The OTF is just the appropriately scaled (and, arguably, normalized) 2-D Fourier transform:

$$\mathcal{F}_2 \left\{ A_0 \text{CYL} \left(\frac{r}{d_0} \right) \right\} = \pi \frac{d_0^2}{4} A_0 \text{SOMB}(d_0 \rho) = \mathfrak{H}[\xi, \eta; \lambda_\mu, z_2]$$

The normalized OTF is:

$$\frac{\mathfrak{H}[\xi, \eta; \lambda_\mu, z_2]}{\mathfrak{H}[0, 0; \lambda_\mu, z_2]} = \text{SOMB}(d_0 \rho)$$

The cutoff frequency (first zero) is located at the frequencies that satisfy

$$d_0 \rho_{\text{cutoff}} \cong 1.22 \implies \boxed{\rho_{\text{cutoff}} \cong \frac{1.22}{d_0}}$$

which decreases with increasing diameter of the pinhole. This is easy to explain; as the diameter of the impulse response increases, the image is “blurrier.”

- (b) Now assume that the pinhole is so small that Fraunhofer diffraction governs the shape of the psf; calculate the spatial frequency of the first zero of the OTF.

If Fraunhofer diffraction applies, then the incoherent impulse response is proportional to the squared magnitude of the aperture function:

$$\begin{aligned}
\mathfrak{h}[x, y; \lambda_\mu, z_2] &\cong \left| \frac{1}{i\lambda_\mu z_2} \exp \left[+2\pi i \frac{z_2}{\lambda_0} \right] \cdot \mathcal{F}_2 \left\{ A_0 \text{CYL} \left(\frac{r}{d_0} \right) \right\} \Big|_{\rho \rightarrow \frac{r}{\lambda_\mu z_2}} \right|^2 \\
&= \left(\frac{1}{\lambda_\mu z_2} \right)^2 \cdot \pi \frac{d_0^2}{4} \cdot A_0^2 \cdot \text{SOMB}^2(d_0 \rho) \Big|_{\rho \rightarrow \frac{r}{\lambda_\mu z_2}} \\
&= \left(\frac{1}{\lambda_\mu z_2} \right)^2 \cdot \pi \frac{d_0^2}{4} \cdot A_0^2 \cdot \text{SOMB}^2 \left(\frac{r}{\left(\frac{\lambda_\mu z_2}{d_0} \right)} \right)
\end{aligned}$$

so the impulse response in the Fraunhofer case IS a function of the wavelength λ_μ and the propagation distance z_2 . The OTF is normalized 2-D Fourier transform:

$$\begin{aligned}
\mathfrak{H}[\xi, \eta; \lambda_\mu, z_2] &= \left(\frac{1}{\lambda_\mu z_2} \right)^2 \cdot \pi \frac{d_0^2}{4} \cdot A_0^2 \cdot \mathcal{F}_2 \left\{ \text{SOMB}^2 \left(\frac{r}{\left(\frac{\lambda_\mu z_2}{d_0} \right)} \right) \right\} \\
&= \left(\frac{1}{\lambda_\mu z_2} \right)^2 \cdot \pi \frac{d_0^2}{4} \cdot A_0^2 \cdot \text{CTRI} \left(-\frac{\lambda_\mu z_2}{d_0} \rho \right)
\end{aligned}$$

The cutoff frequencies satisfy the condition:

$$\frac{\lambda_\mu z_2}{d_0} \rho_{\text{cutoff}} \cong 1.22 \implies \boxed{\rho_{\text{cutoff}} \cong 1.22 \frac{d_0}{\lambda_\mu z_2}}$$

which INCREASES with increasing diameter d_0 .

- (c) Use the results of parts (a) and (b) to estimate an “optimum” size of the pinhole in terms of the various parameters, i.e., the diameter that produces the largest cutoff frequency (of the first zero).

For a pinhole with diameter d_0 that is sufficiently large for geometrical optics to be valid, the cutoff frequency increases with decreasing diameter. Eventually the pinhole diameter will be small enough for the Fraunhofer approximation to be valid. The approximate diameter where the two expressions are both valid is where the cutoff frequencies match:

$$\begin{aligned}
\frac{1.22}{(d_0)_{\text{optimum}}} &\cong \frac{(d_0)_{\text{optimum}}}{\lambda_\mu z_2} \\
\implies &(d_0)_{\text{optimum}}^2 \cong 1.22 \lambda_\mu z_2 \\
\implies &\boxed{(d_0)_{\text{optimum}} \cong \sqrt{1.22 \lambda_\mu z_2} \cong 1.104 \cdot \sqrt{\lambda_\mu z_2}}
\end{aligned}$$

4. A transparent object has transmittance $t[x, y] = \left(\text{RECT} \left[\frac{x}{b_0} \right] * \frac{1}{d_0} \text{COMB} \left[\frac{x}{d_0} \right] \right) \cdot 1[y]$, where $d_0 = \frac{1}{100} \text{ mm} > b_0$. The object is imaged by a lens with a circular pupil and focal length $f = 100 \text{ mm}$. The illumination is monochromatic with $\lambda_0 = 1 \mu\text{m}$.

The fundamental frequency of the square wave is the reciprocal of the period:

$$\xi_0 = 100 \frac{\text{cycles}}{\text{mm}}$$

The image distance is:

$$\frac{1}{z_2} = \frac{1}{100 \text{ mm}} - \frac{1}{z_1} \implies z_2 = \frac{z_1 \cdot 100 \text{ mm}}{z_1 - 100 \text{ mm}}$$

- (a) Determine the minimum lens diameter that will produce an image with any variation of intensity due to the object.

For coherent illumination with $\lambda_0 = 1 \mu\text{m}$, the cutoff frequency is the scaled radius of the pupil; I will call this radius R_0

$$\begin{aligned} \lambda_0 z_2 \cdot \rho_{\text{cutoff}} &\lesssim R_0 \\ \implies 1 \mu\text{m} \cdot \left(\frac{z_1 \cdot 100 \text{ mm}}{z_1 - 100 \text{ mm}} \right) \cdot 100 \frac{\text{cycles}}{\text{mm}} &\lesssim R_0 \end{aligned}$$

$$\text{diameter in coherent case} \gtrsim 2 \cdot R_0 \gtrsim \frac{200}{\text{mm}} \cdot \left(1 \mu\text{m} \cdot \left(\frac{z_1 \cdot 100 \text{ mm}}{z_1 - 100 \text{ mm}} \right) \right)^{-1}$$

If $z_1 \rightarrow \infty$, then diameter

$$2R_0 \gtrsim \frac{200}{\text{mm}} \cdot 1 \mu\text{m} \cdot 100 \text{ mm} = 20 \text{ mm}$$

- (b) Determine the minimum diameter that produces an image with the same feature if the illumination is quasimonochromatic with $\lambda_\mu = 1 \mu\text{m}$.

In the incoherent case, the cutoff frequency is twice as large so the required lens diameter is half as large:

$$\text{diameter in incoherent case} \gtrsim 2R_0 \gtrsim \frac{1}{2} \cdot \frac{200}{\text{mm}} \cdot \left(1 \mu\text{m} \cdot \left(\frac{z_1 \cdot 100 \text{ mm}}{z_1 - 100 \text{ mm}} \right) \right)^{-1}$$

Again, if $z_1 \rightarrow \infty$, then diameter

$$2R_0 \gtrsim \frac{1}{2} \frac{200}{\text{mm}} \cdot 1 \mu\text{m} \cdot 100 \text{ mm} = 10 \text{ mm}$$

5. The $f/\#$ of a lens with a circular aperture is the ratio of the focal length to the diameter. Under coherent illumination with λ_0 and for an infinite object distance, find an expression for the cutoff frequency.

(zzzzz) If the diameter of the aperture is d_0 , then the $f/\#$ is:

$$f/\# = \frac{\mathbf{f}}{d_0}$$

Since $z_1 = \infty$, $z_2 = \mathbf{f}$. The cutoff frequency is proportional to the scaled semidiameter of the lens:

$$\lambda_0 \cdot z_2 \cdot \rho_{\text{cutoff}} = \frac{d_0}{2} \implies \rho_{\text{cutoff}} = \frac{d_0}{2\lambda_0 \cdot \mathbf{f}} = \rho_{\text{cutoff}} = \frac{1}{2\lambda_0 \cdot f/\#}$$