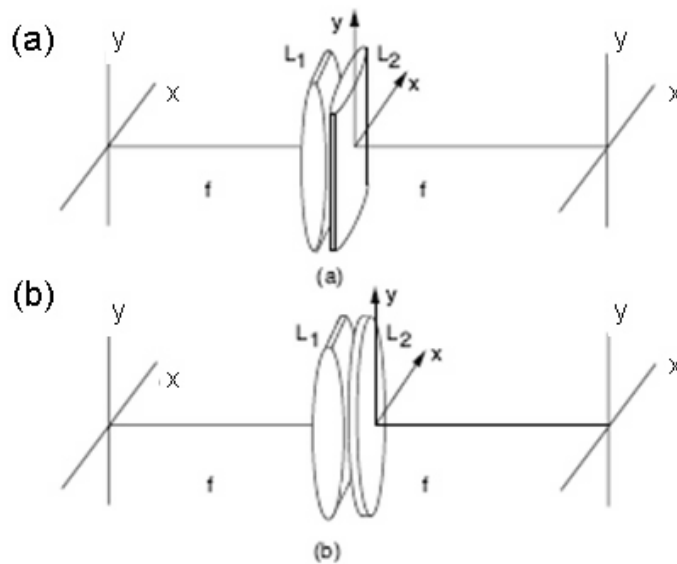


1. An array of one-dimensional input functions can be represented by $f_o[x, y_n]$, where y_n are N fixed coordinates in the input plane. Your task is to perform 1-D Fourier transforms on all N 1-D functions along the x -direction to produce the array of transforms:

$$F_o[\xi; y_n] = \int_{-\infty}^{+\infty} f_o[x, y_n] \exp[-2\pi i x \xi] dx$$

After neglecting the finite extent of the apertures of the object and the lens, use the Fourier transforming and imaging properties of lenses derived in this chapter to show how to compute the squared magnitude $|F_o[\xi; y_n]|^2$ with the lenses available in two cases:

- (a) two cylindrical lenses with different focal lengths



To make an imaging system that produces an image along the y -direction and a Fourier transform along x in the same plane, we need different focal lengths along the two directions. To get an image along y , we need a focal length that satisfies the relation:

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{\mathbf{f}_y}$$

where z_2 is the distance between lens and image plane. The focal length along the x direction must be the same as the distance z_2 . If we choose to have transverse magnification $M_T = -1$, then $z_2 = \mathbf{f}_x = 2\mathbf{f}_y$. To do this with two cylindrical lenses L_1 and L_2 with focal lengths \mathbf{f}_1 and \mathbf{f}_2 , consult part (a) of the figure. As drawn, L_1 focuses light (i.e., “has power”) only in the y direction, so the phase it adds to the light is:

$$\Phi[x, y] = -\pi \frac{y^2}{\lambda_0 \mathbf{f}}$$

so the transmittance of the L_1 is:

$$t_1[x, y] = p[x, y] \exp \left[-i\pi \frac{y^2}{\lambda_0 \mathbf{f}_1} \right]$$

Similarly the transmittance function for L_2 is:

$$t_2[x, y] = p[x, y] \exp \left[-i\pi \frac{x^2}{\lambda_0 \mathbf{f}_2} \right]$$

The transmittance of the collection of the two thin lenses is:

$$\begin{aligned} t_1[x, y] \cdot t_2[x, y] &= p[x, y] \exp \left[-i\pi \frac{y^2}{\lambda_0 \mathbf{f}_1} \right] \cdot p[x, y] \exp \left[-i\pi \frac{x^2}{\lambda_0 \mathbf{f}_2} \right] \\ &= p[x, y] \exp \left[-i\pi \left(\frac{x^2}{\lambda_0 \mathbf{f}_2} + \frac{y^2}{\lambda_0 \mathbf{f}_1} \right) \right] \end{aligned}$$

To get the lens to “image” along y at the same time as it evaluates the Fourier transform along x , we need to select the focal lengths to be $\mathbf{f}_1 = \mathbf{f}$ along x and $\mathbf{f}_2 = \frac{\mathbf{f}}{2}$ along y , so if placed at the distance \mathbf{f} from the object, the amplitude at the plane at the distance \mathbf{f} will be two focal lengths away in y (and thus form an image with transverse magnification $M_T = -1$) and one focal length away in x (and thus evaluate the Fourier transform

$$t[x, y] = p[x, y] \exp \left[-i\pi \left(\frac{x^2}{\lambda_0 \mathbf{f}} + \frac{y^2}{\lambda_0 \frac{\mathbf{f}}{2}} \right) \right]$$

(b) a cylindrical and a spherical lens with the same focal length.

In part (b) of the figure, a spherical lens and a cylindrical lens are in contact and both focal lengths are identically \mathbf{f} . If we orient the lenses so that the power of the cylindrical lens is oriented along y , then the transmittance of the pair of lenses is:

$$\begin{aligned} t[x, y] &= \left(p[x, y] \exp \left[-i\pi \left(\frac{x^2 + y^2}{\lambda_0 \mathbf{f}} \right) \right] \right) \cdot \left(p[x, y] \exp \left[-i\pi \frac{y^2}{\lambda_0 \mathbf{f}} \right] \right) \\ &= p[x, y] \exp \left[-i\pi \left(\frac{x^2 + y^2}{\lambda_0 \mathbf{f}} + \frac{y^2}{\lambda_0 \mathbf{f}} \right) \right] \\ &= p[x, y] \exp \left[-i\pi \left(\frac{x^2}{\lambda_0 \mathbf{f}} + \frac{y^2}{\lambda_0 \frac{\mathbf{f}}{2}} \right) \right] \end{aligned}$$

which is the same as that in part (a):

$$\Phi[x, y] = -\pi \frac{y^2}{\lambda_0 \mathbf{f}}$$

So the image and Fourier planes are located at the distance \mathbf{f} from the lens pair.

2. A normally incident unit-amplitude monochromatic plane wave with $\lambda_0 = 633 \text{ nm}$ illuminates a converging lens with diameter $d_0 = 50 \text{ mm}$ and $\mathbf{f} = 2 \text{ m}$. An object with transmittance

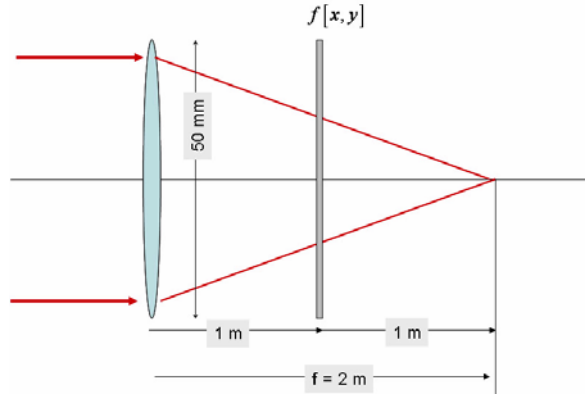
$$t[x, y] = \frac{1}{2} (1 + \cos [2\pi \cdot 10 \text{ mm}^{-1} \cdot x]) \text{RECT} \left[\frac{x}{10 \text{ mm}}, \frac{y}{10 \text{ mm}} \right]$$

is placed one meter *behind* the lens (i.e., on the “output” side). Sketch the irradiance across the x -axis in the focal plane; label the numerical values of the distance between the diffracted components and the width of the individual components between the first zeros.

First note that the spectrum of the input transmittance is:

$$\begin{aligned} T[\xi, \eta] \\ = \frac{1}{2} \left(\delta[\xi, \eta] + \frac{1}{2} \delta[\xi + 10 \text{ mm}^{-1}] + \frac{1}{2} \delta[\xi - 10 \text{ mm}^{-1}] \right) * (10 \text{ mm})^2 \text{SINC}[10 \text{ mm} \cdot \xi, 10 \text{ mm} \cdot \eta] \end{aligned}$$

The sketch of the system is:



The light at the back side of the lens converges at the distance \mathbf{f} and so has the form at the back of the lens:

$$\exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \mathbf{f}} \right]$$

This field propagates the distance $\frac{\mathbf{f}}{2}$ to the transparency, so the field at the back(output) side of the transparency is:

$$\left(\exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \mathbf{f}} \right] * \exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \right) \cdot t[x, y]$$

This propagates the distance $\frac{\mathbf{f}}{2}$ to the output plane, where the amplitude is:

$$g[x, y] = \left(\left(\exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \mathbf{f}} \right] * \exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \right) \cdot t[x, y] \right) * \exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right]$$

Now back to the illumination of the object transparency. You can simplify the expression either in the space or frequency domains – I chose to do the latter:

$$\begin{aligned}
& \exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \mathbf{f}} \right] * \exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \\
&= \mathcal{F}_2^{-1} \left\{ \left(e^{-i\frac{\pi}{4}} \right)^2 \cdot (\lambda_0 \mathbf{f}) \exp \left[+i\pi \cdot \lambda_0 \mathbf{f} \cdot (\xi^2 + \eta^2) \right] \cdot \left(e^{+i\frac{\pi}{4}} \right)^2 \cdot \left(\lambda_0 \frac{\mathbf{f}}{2} \right) \exp \left[-i\pi \lambda_0 \cdot \frac{\mathbf{f}}{2} \cdot (\xi^2 + \eta^2) \right] \right\} \\
&= \frac{(\lambda_0 \mathbf{f})^2}{2} \frac{2}{\lambda_0 \mathbf{f}} \left(\exp \left[+i\frac{\pi}{4} \right] \right)^2 \exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] = i \lambda_0 \mathbf{f} \cdot \exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right]
\end{aligned}$$

so the light at the output side of the transparency is:

$$i \lambda_0 \mathbf{f} \cdot \exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \cdot t[x.y]$$

The light at the focal plane is:

$$g[x,y] = i \lambda_0 \mathbf{f} \cdot \left(\exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \cdot t[x.y] \right) * \left(\exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \right)$$

Note that this expression includes two-thirds of the M-C-M chirp Fourier transform:

$$\left(\left(t[x,y] \cdot \exp \left[-i\pi \frac{x^2 + y^2}{\alpha^2} \right] \right) * \exp \left[+i\pi \frac{x^2 + y^2}{\alpha^2} \right] \right) \cdot \exp \left[-i\pi \frac{x^2 + y^2}{\alpha^2} \right] = T \left[\frac{x}{\alpha^2}, \frac{y}{\alpha^2} \right]$$

where $\alpha = \sqrt{\lambda_0 \cdot \frac{\mathbf{f}}{2}}$. We can cross multiply to rewrite the output in terms of the spectrum of the input:

$$\Rightarrow \left(\left(t[x,y] \cdot \exp \left[-i\pi \frac{x^2 + y^2}{\alpha^2} \right] \right) * \exp \left[+i\pi \frac{x^2 + y^2}{\alpha^2} \right] \right) = T \left[\frac{x}{\alpha^2}, \frac{y}{\alpha^2} \right] \cdot \exp \left[+i\pi \frac{x^2 + y^2}{\alpha^2} \right]$$

$$\begin{aligned}
\Rightarrow g[x,y] &= i \lambda_0 \mathbf{f} \cdot \left(\exp \left[-i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \cdot t[x.y] \right) * \left(\exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \right) \\
&= i \lambda_0 \mathbf{f} \cdot \left(T \left[\frac{x}{\lambda_0 \cdot \frac{\mathbf{f}}{2}}, \frac{y}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \cdot \exp \left[+i\pi \frac{x^2 + y^2}{\lambda_0 \cdot \frac{\mathbf{f}}{2}} \right] \right)
\end{aligned}$$

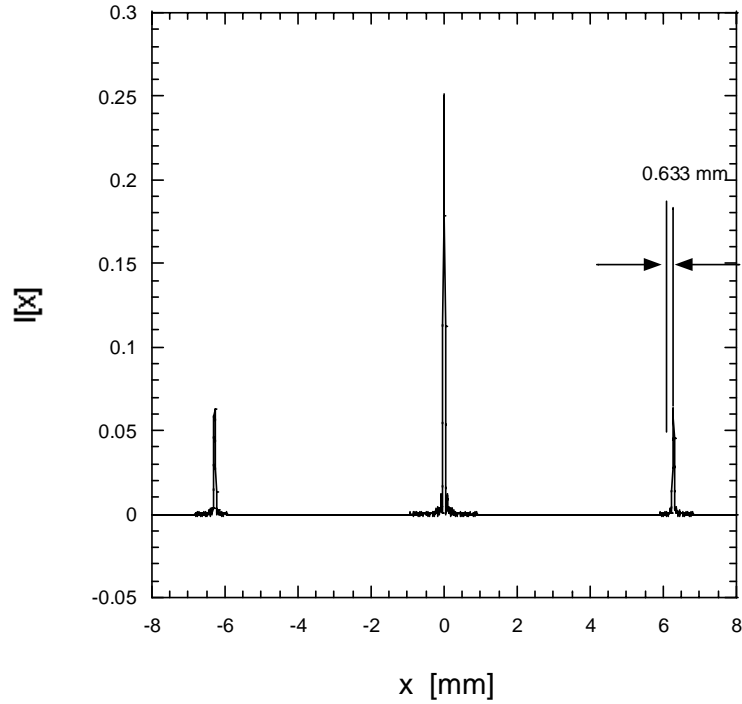
We substitute the parameters:

$$\begin{aligned}
\lambda_0 &= 633 \text{ nm} \\
\mathbf{f} &= 2 \text{ m} \Rightarrow \frac{\mathbf{f}}{2} = 1 \text{ m} \\
\lambda_0 \cdot \frac{\mathbf{f}}{2} &= 633 \text{ nm} \cdot 1 \text{ m} = 6.33 \times 10^{-7} \text{ m}^2
\end{aligned}$$

$$\begin{aligned} &\Rightarrow T \left[\frac{x}{\lambda_0 \cdot \frac{f}{2}}, \frac{y}{\lambda_0 \cdot \frac{f}{2}} \right] \\ &= \frac{1}{2} \left(\delta \left[\frac{x}{6.33 \times 10^{-7} \text{ m}^2}, \frac{y}{6.33 \times 10^{-7} \text{ m}^2} \right] + \frac{1}{2} \delta \left[\frac{x + 6.33 \text{ mm}}{6.33 \times 10^{-7} \text{ m}^2} \right] + \frac{1}{2} \delta \left[\frac{x - 6.33 \text{ mm}}{6.33 \times 10^{-7} \text{ m}^2} \right] \right) \\ &\quad * (10 \text{ mm})^2 \text{SINC} \left[\frac{x}{0.0633 \text{ mm}}, \frac{y}{0.0633 \text{ mm}} \right] \end{aligned}$$

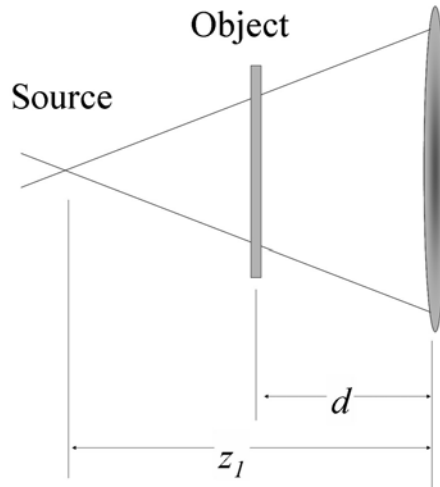
$$\begin{aligned} T \left[\frac{x}{\lambda_0 \cdot \frac{f}{2}}, \frac{y}{\lambda_0 \cdot \frac{f}{2}} \right] &\propto \frac{1}{2} \text{SINC} \left[\frac{x}{0.0633 \text{ mm}}, \frac{y}{0.0633 \text{ mm}} \right] \\ &+ \frac{1}{4} \text{SINC} \left[\frac{x + 6.33 \text{ mm}}{0.0633 \text{ mm}}, \frac{y}{0.0633 \text{ mm}} \right] + \frac{1}{4} \text{SINC} \left[\frac{x - 6.33 \text{ mm}}{0.0633 \text{ mm}}, \frac{y}{0.0633 \text{ mm}} \right] \end{aligned}$$

So the pattern is a set of three SINC functions of the same narrow width ($63.3 \mu\text{m}$) separated by 6.33 mm , so this is a pretty good approximation of three Dirac delta functions.



where each individual term is the square of a narrow SINC² function.

3. A monochromatic point source is placed at a fixed distance z_1 to the left of a positive lens with focal length \mathbf{f} , where $z_1 > \mathbf{f}$. A transparent object with transmittance $t[x, y]$ is placed at a variable distance d to the left of the lens. The Fourier transform and the image of the object appear to the right of the lens.



- (a) Find d in terms of z_1 and \mathbf{f} , that assures that the *Fourier plane* and the *object* are equidistant from the lens.

As said in class, *Fourier transforms* are calculated (except possibly for an additional quadratic phase) at planes where the source is imaged. Therefore the distance z_f of the *Fourier plane* to the right of the lens must satisfy the condition:

$$\frac{1}{z_1} + \frac{1}{z_f} = \frac{1}{\mathbf{f}} \implies z_f = \frac{z_1 \mathbf{f}}{z_1 - \mathbf{f}}$$

where z_1 is the distance from the source to the lens. For the distance of the object to the left of the lens to equal the distance of the *Fourier plane* to the right of the lens:

$$d = z_f = \frac{z_1 \mathbf{f}}{z_1 - \mathbf{f}}$$

- (b) When the object is located at the distance d found in part (a), determine the location of the image and its transverse magnification.

Let z_i represent the distance from the lens to the image. The imaging equation tells us:

$$\frac{1}{d} + \frac{1}{z_i} = \frac{1}{\mathbf{f}}$$

Substitute the expression for d obtained in part (a) into this equation and solve for z_i . The result is:

$$z_i = z_1$$

The transverse magnification is given by:

$$M_T = \left| \frac{z_i}{d} \right| = \left| \frac{z_1}{d} \right|$$

4. Two lenses L_1 with focal length $\mathbf{f}_1 = -|\mathbf{f}|$ and L_2 with $\mathbf{f}_2 = +|\mathbf{f}|$ are separated by the distance \mathbf{f}_2 . The object is located at the distance $2 \cdot |\mathbf{f}|$ to the left of L_1 . Find the distances to the Fourier plane and the image plane and sketch the system.

The distance to the Fourier plane may be determined by finding the location where the source is imaged. If the object is illuminated by a plane wave (not stated, but a reasonable assumption), then we can use the imaging equation twice to find the image of the source. For the source an infinite distance away, the image created by the first lens is located at:

$$\begin{aligned}(z_2)_1 &= \frac{(z_1)_1 \mathbf{f}_1}{(z_1)_1 - \mathbf{f}_1} = \frac{\infty \cdot \mathbf{f}_1}{\infty - \mathbf{f}_1} = \mathbf{f}_1 = -\mathbf{f} \\(z_1)_2 &= d - (z_2)_1 = \mathbf{f} - (-\mathbf{f}) = 2\mathbf{f} \\(z_2)_2 &= \frac{(z_1)_2 \mathbf{f}_2}{(z_1)_2 - \mathbf{f}_2} = \frac{(2\mathbf{f}) \cdot \mathbf{f}}{2\mathbf{f} - \mathbf{f}} = \boxed{2\mathbf{f}}\end{aligned}$$

In words, the image of the source is located at $2\mathbf{f}$ after the second lens.

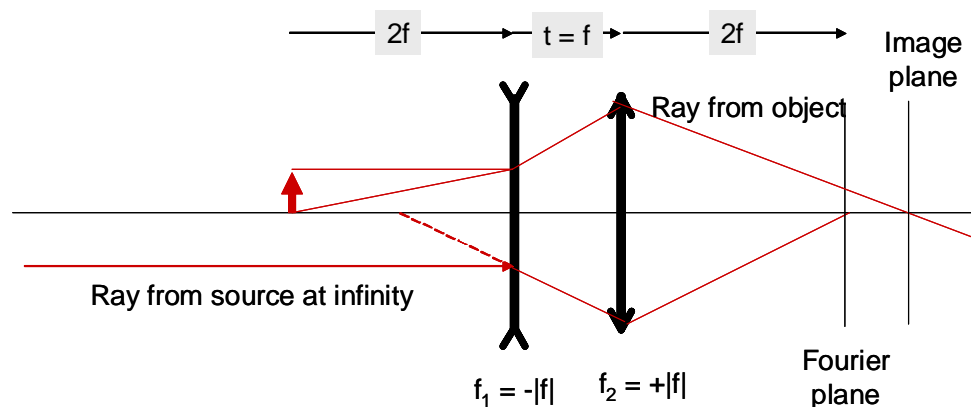
The location of the image of the object is the image if $(z_1)_1 = 2\mathbf{f}$:

$$\begin{aligned}(z_2)_1 &= \frac{(z_1)_1 \mathbf{f}_1}{(z_1)_1 - \mathbf{f}_1} = \frac{(2\mathbf{f}) \cdot (-\mathbf{f})}{2\mathbf{f} - (-\mathbf{f})} = -\frac{2}{3}\mathbf{f} \\(z_1)_2 &= d - (z_2)_1 = \mathbf{f} - \left(-\frac{2}{3}\mathbf{f}\right) = \frac{5\mathbf{f}}{3} \\(z_2)_2 &= \frac{(z_1)_2 \mathbf{f}_2}{(z_1)_2 - \mathbf{f}_2} = \frac{\left(\frac{5\mathbf{f}}{3}\right) \cdot \mathbf{f}}{\frac{5\mathbf{f}}{3} - \mathbf{f}} = \boxed{\frac{5}{2}\mathbf{f}}\end{aligned}$$

so the image appears at the distance $\frac{5}{2}\mathbf{f}$ to the right of the second lens.

The transverse magnification of the image is:

$$M_T = (M_T)_1 \cdot (M_T)_2 = \left(-\frac{-\frac{2}{3}\mathbf{f}}{2\mathbf{f}}\right) \cdot \left(-\frac{\frac{5}{2}\mathbf{f}}{\frac{5\mathbf{f}}{3}}\right) = -\frac{1}{2}$$



5. A converging spherical wave illuminates a transparency $t[x, y]$ and converges to a point at the distance \mathbf{f} to its right. A lens with focal length $+\mathbf{f}$ is placed at the distance $2 \cdot \mathbf{f}$ to the right of the focal point. Find the locations of all Fourier and image planes on both sides of the lens.

Fourier planes are found at (1) the image of the source, i.e., where the illumination beam comes to a focus, and (2) the plane where the image of the source just mentioned is reimaged.

Since the Fourier plane is located $2\mathbf{f}$ to the left of the the lens,

the image of the Fourier plane is $2\mathbf{f}$ to the right of the lens.

Since the object is $3\mathbf{f}$ from the lens, the image is located at:

$$\frac{1}{3\mathbf{f}} + \frac{1}{z_2} = \frac{1}{\mathbf{f}} \implies z_2 = \frac{3\mathbf{f} \cdot \mathbf{f}}{3\mathbf{f} - \mathbf{f}} = \frac{3}{2}\mathbf{f}$$

In words, the image is formed at the distance $\frac{3}{2}\mathbf{f}$ to the right of the lens.

