

Review Fourier transforms (§6 in the notes for *Fourier Methods* and/or Goodman §1-§2 and/or Gaskill §7)

1. Evaluate $\delta[SINC[x]] = \delta\left[\frac{\sin[\pi x]}{\pi x}\right]$

2. Prove the relation:

$$\mathcal{F}_2\{RECT[x, y]\} = SINC[\xi] \cdot SINC[\eta]$$

3. Evaluate the 2-D Fourier transform:

$$\mathcal{F}_2\{\nabla^2 f[x, y]\} = \mathcal{F}_2\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f[x, y]\right\}$$

4. Given that:

$$\mathcal{F}_1\{\exp[+i\pi x^2]\} = \exp\left[+i\frac{\pi}{4}\right] \exp[-i\pi\xi^2]$$

(a) Use the scaling theorem of the 1-D Fourier transform to show that:

$$\mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right\} = |\alpha| \exp\left[+i\frac{\pi}{4}\right] \exp[-i\pi(\alpha\xi)^2]$$

where α is some numerical constant with units of length.

(b) Sketch or plot the chirp function $\exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]$ as real and imaginary parts AND as magnitude and phase $\alpha = +1, +2$.

(c) Sketch or plot the spectrum $\mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right\}$ as real and imaginary parts AND as magnitude and phase $\alpha = +1, +2$.

5. For the frequency-domain function

$$H_1[\xi] = RECT\left[\frac{\xi}{2\xi_{\max}}\right] e^{+i\pi(\alpha\xi)^2}$$

where α is some numerical constant with units of length.

(a) Use the filter theorem to evaluate $h_1[x] = \mathcal{F}_1^{-1}\{H_1[\xi]\}$

(b) Given that $\mathcal{F}\{f[x]\} \equiv F[\xi]$, substitute

$$-2\xi x = \left(\frac{x}{\alpha} - \alpha\xi\right)^2 - \left(\frac{x}{\alpha}\right)^2 - (\alpha\xi)^2$$

in the formula for the 1-D Fourier transform to derive a relationship between $f[\alpha^2\xi]$ and $F[\xi]$. Because α has units of length, the representations have the proper dimensionality.

(c) Use this result to find a DIFFERENT expression for $H_1[\xi]$ that includes a *SINC* function in the frequency domain.

(d) Evaluate the impulse response resulting from (c) – it will include a *RECT* function – and compare with the result of (a); graphs will be helpful.

(e) Show that these expressions yield the expected results in the limiting cases of $\xi_{\max} \rightarrow \infty$ and $\xi_{\max} \rightarrow 0$ for a finite α .

6. In this problem, you can use some imaging or computing “toolbox” (e.g., IDL or MatLab "SignalShow" by Juliet Bernstein, or even my DOS program "Signals") to generate and operate on a 1-D array based on the 1-D chirp function:

$$f[x] = \exp \left[\pm i \left(\pi \frac{x^2}{\alpha^2} - \phi_0 \right) \right]$$

The array should have N samples (N is even and could be a power of 2) indexed by n such that $x = n \cdot \Delta x$ and $n = -\frac{N}{2}, -\frac{N}{2} + 1, \dots, -1, 0, +1, \dots, +\frac{N}{2} - 1$.

- (a) Derive the relation between α and the array size N such that the 1-D sampled function is “just” aliased at the edges of the 1-D array.
- (b) Print “images” of the real and imaginary parts of the sampled complex function $f[x]$ using the value of α that satisfies the constraint of part (a) for an array of size N . I suggest selecting $N = 256$ or 512 .
- (c) Evaluate the fast Fourier transform (FFT) or the discrete Fourier transform (DFT) of the sampled complex chirp (make sure that the sample corresponding to zero frequency is “centered” in the array, and not at the edges as is often the case). Print out “images” of the function and its FFT as real part, imaginary part, magnitude, and phase. Note that since $f[x, y]$ is an even function, so should be its FFT.
- (d) Now construct a “modulated” chirp functions by multiplying $f[n]$ by some functions $s_\ell[n]$. The first modulation function should be “centered” with the axis of symmetry at the origin and binary (e.g., a rectangle); the second should be the same function binary function after translation by an arbitrary distance. For example, if $N = 512$, you might select the modulation functions to be:

$$\begin{aligned} s_1[n] &= \text{RECT} \left[\frac{n}{128} \right] \\ s_2[n] &= \text{RECT} \left[\frac{n - 128}{128} \right] \end{aligned}$$

- (e) Repeat part (d) using a modulation function of your choice that is not binary and may even be bipolar (e.g., a Triangle or a SINC function). Submit “images” of the modulation function, of the real and imaginary parts of the modulated chirp functions, and of their FFTs as real part, imaginary part, magnitude, and phase.
7. (Optional) Repeat parts (b-e) of the last problem for a 2-D array of size $N \times N$ derived from the 2-D function:

$$f[x, y] = \exp \left[\pm i \left(\pi \left(\frac{x^2 + y^2}{\alpha^2} \right) - \phi_0 \right) \right]$$