1.1 1-D Spectra via Method of Stationary Phase

Now we consider a different approximation of the Fourier transform that is valid for certain 1-D functions at large values of $|\xi|$. The process is an application of the *method of stationary phase*, which was developed by Lord Kelvin in the 1800s to solve integrals encountered in the study of hydrodynamics. In turn, this is a variation of the *method of steepest descents* for evaluating path integrals of complex functions. The method of stationary phase provides useful estimates of integrals of oscillating functions, and thus of integrands with imaginary-valued exponents. This method is particularly applicable to superchirp functions $e^{\pm i\pi x^n}$, for which no closed form of the spectrum has been derived. The results obtained for these functions will be applied in several contexts later in the book. More detailed descriptions are available in Erdelyi (1956) and Copson (1965).

The governing principle behind the method of stationary phase will be introduced by example. Consider an integral of the general form:

$$I[k] = \int_{-\infty}^{+\infty} r[x] \; e^{ik\cdot\mu[x]} \; dx \tag{1}$$

where $r[x]$ and $\mu[x]$ are real-valued functions and k is a selectable real-valued parameter. The exponential function $e^{ik\cdot\mu[x]}$ oscillates at a rate that depends on both k and the functional form of $\mu[x]$. If k is large, the rate of oscillation of the exponential term must also be large in all regions of the domain where $\mu[x] \neq 0$. In such cases, the contribution to the oscillating function to the area will be small in any region where the exponential term oscillates more rapidly than the variation of $r[x]$, because the areas of the adjacent positive and negative lobes will approximately cancel. Conversely, in those regions of the domain where $k \cdot \mu[x]$ is small, the amplitude of the exponential term will approximate the unit constant. The area of those regions will be determined by the width of this region of the domain and the amplitude of the real-valued function $r[x]$. The real and imaginary parts of a sample integrand in these two regions are shown in Figure 1c,d. This shows that the integral in Eq.(1) may be estimated by evaluating the integrand *only* in those regions where the exponential term oscillates slowly, i.e., wherever the derivative of the phase function is approximately 0, $\frac{d\mu}{dx} \cong 0$; these are the *stationary* points of $\mu[x]$. The "semi-infinite" integrals of the real and imaginary parts over the domain $-\infty < x \leq a$ are shown in Figure 1e,f. These illustrate that the primary contribution to the area in both cases arise from the integrand in the region of the stationary point.

The example in Figure 1 also shows that the requirement that $r[x]$ be real valued in Eq.(1) creates no problem when evaluating the integral of a complex-valued function, because the linearity of integration allows the integrals of the individual parts to be

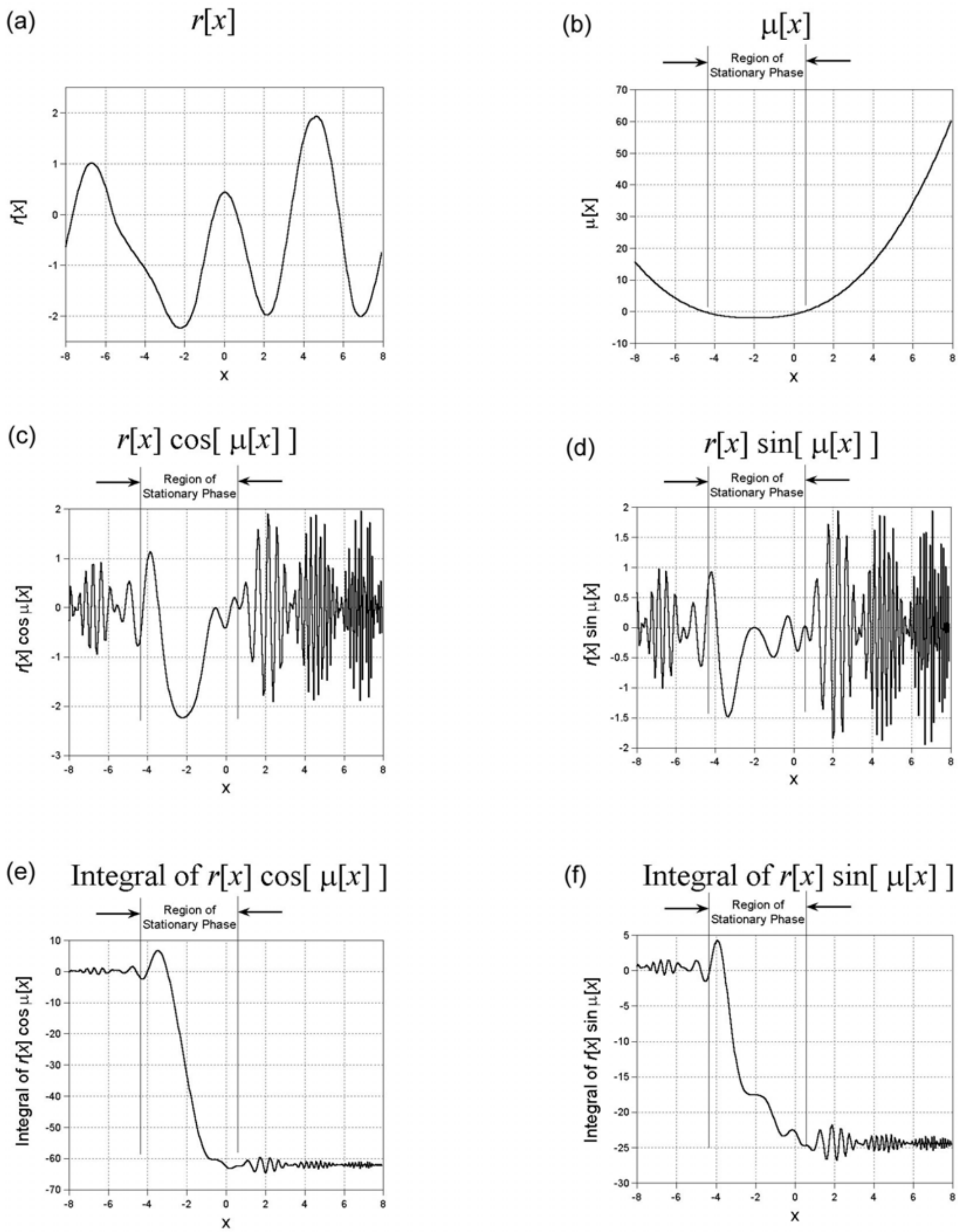

Figure 1.1: Principle of the Method of Stationary Phase: (a) real-valued modulation $r\,[x]$; (b) phase function $m\,[x]$, which is approximately stationary in vicinity of $x = -2$; (c), (d) Real and imaginary parts of $r\,[x]\,e^{i\mu[x]}$, showing rapid oscillations away from stationary point; (e), (f) Real and imaginary parts of $\int_{-\infty}^{x} r\,[\alpha]\; e^{i\mu[\alpha]}\,d\alpha$, showing that the primary contribution to the area is from $r\,[x]$ in the vicinity of the stationary point.

performed separately and summed.

$$\begin{aligned} I[k] &= \int_{-\infty}^{+\infty} \left(\Re\{r[x]\} + i\,\Im\{r[x]\}\right)\, e^{ik\,\mu[x]}\, dx \\ &= \left(\int_{-\infty}^{+\infty} \Re\{r[x]\}\, e^{ik\,\mu[x]}\, dx\right) + i\left(\int_{-\infty}^{+\infty} \Im\{r[x]\}\, e^{ik\,\mu[x]}\, dx\right) \end{aligned} \quad (2)$$

Under some circumstances, other and more subtle aspects of the method of stationary phase require special treatment. Since these occur rarely in the cases of greatest interest in imaging, they will be mentioned only in passing and not considered in detail. Interested readers should consult sources that concentrate on this subject, especially the work of Erdelyi and of Friedman.

To illustrate use of the method of stationary phase in Fourier analysis, consider the Fourier transform of a 1-D complex-valued function $f[x]$ expressed in terms of its magnitude $|f[x]|$ and phase $\Phi\{f[x]\}$:

$$\begin{aligned} \mathcal{F}_1\{f[x]\} &= \int_{-\infty}^{+\infty} f[x]\; e^{-2\pi i\xi x}\, dx \\ &= \int_{-\infty}^{+\infty} \left(|f[x]|\; e^{i\boldsymbol{\Phi}\{f[x]\}}\right)\, e^{-2\pi i\xi x}\, dx \\ &= \int_{-\infty}^{+\infty} |f[x]|\; e^{i(\boldsymbol{\Phi}\{f[x]\} - 2\pi\xi x)}\, dx \end{aligned} \quad (3)$$

Note that the form of this integral is somewhat less general than that in Eq.(1) because the modulation function $|f[x]|$ is not only real valued, but also nonnegative, whereas $r[x]$ in Eq.(1) may be negative.

The Fourier integral in Eq.(3) may be rewritten in the form of Eq.(1) by defining:

$$\mu[x] = \frac{1}{\xi}\boldsymbol{\Phi}\{f[x]\} - 2\pi x \quad (4)$$

and substituting the spatial frequency ξ for the parameter k and $|f[x]|$ for $r[x]$:

$$F[\xi] = \int_{-\infty}^{+\infty} |f[x]|\; e^{i\xi\cdot\mu[x]}\, dx \quad (5)$$

Note that the phase function $\mu[x]$ includes a factor of ξ^{-1}, but this is reasonable since ξ is a parameter (rather than a variable) in the integrand. Thus the result will be a function of this parameter, as it must be.

If the phase function $\mu[x]$ has no singularities (i.e., if all of its derivatives are finite), then $\mu[x]$ may be expanded into a Taylor series about any arbitrary location x_0:

$$\mu[x] = \mu[x_0] + \left((x - x_0) \left. \frac{d\mu}{dx} \right|_{x=x_0} \right) + \left(\frac{(x-x_0)^2}{2} \left. \frac{d^2\mu}{dx^2} \right|_{x=x_0} \right) + \cdots$$
$$= \mu[x_0] + (x - x_0)\ \mu'[x_0] + \frac{(x-x_0)^2}{2}\ \mu''[x_0]$$
$$+ \cdots + \frac{(x-x_0)^n}{n!}\ \mu^{(n)}[x_0] + \cdots \tag{6}$$

where the common "multiple-prime" shorthand notation for derivatives has been substituted in the second expression for simplicity. We now select x_0 to be a stationary point of the phase function, so that $\mu'[x_0] = 0$. For now, assume that $\mu[x]$ has only one such stationary point; extension to cases with multiple stationary points is straightforward and will be considered later. The first-order term in the Taylor series vanishes:

$$\mu[x] = \mu[x_0] + 0 + \frac{(x-x_0)^2}{2!}\mu''[x_0] + \frac{(x-x_0)^3}{3!}\mu'''[x_0] + \cdots \tag{7}$$

and the Fourier integral in Eq.(3) may be rewritten in terms of this series:

$$F[\xi] = \int_{-\infty}^{+\infty} |f[x]|\ \exp\left[+i\xi\ \left(\mu[x_0] + \mu''[x_0]\ \frac{(x-x_0)^2}{2} + \cdots\right)\right]\ dx$$
$$= \int_{-\infty}^{+\infty} |f[x]| \left(\exp[+i\xi\cdot\mu[x_0]]\cdot\exp\left[+i\xi\ \left(\mu''[x_0]\frac{(x-x_0)^2}{2}\right)\right]\cdots\right)\ dx$$
$$= \int_{-\infty}^{+\infty} |f[x]|\ \exp[+i\xi\cdot\mu[x_0]] \prod_{n=2}^{+\infty}\left(\exp\left[+i\xi\ \left(\mu^{(n)}[x_0]\frac{(x-x_0)^2}{2}\right)\right]\right)\ dx \tag{8}$$

Of course, the zero-order term in the Taylor series is a constant with respect to x and may be extracted from the integral:

$$F[\xi] = \exp[+i\xi\cdot\mu[x_0]] \int_{-\infty}^{+\infty} |f[x]| \prod_{n=2}^{+\infty}\left(\exp\left[+i\xi\ \left(\mu^{(n)}[x_0]\frac{(x-x_0)^2}{2}\right)\right]\right)\ dx \tag{9}$$

If the spatial frequency ξ is assumed to be sufficiently large that the exponential term oscillates many times over the scale of variation of $|f[x]|$, then the integral may be approximated to evaluate the spectrum. Under this condition, the magnitude $|f[x]|$ contributes significant area to the Fourier integral only in the vicinity of the stationary point x_0, thus allowing the varying magnitude $|f[x]|$ to be approximated by the constant $|f[x_0]|$. In addition, the infinite limits of the integral may be changed to finite limits in the vicinity of the single stationary point. Finally, only the first nonzero term in the Taylor series of order two or larger is significant because $(x-x_0)^n << (x-x_0)^2$ for $n \geq 3$ when x is in the neighborhood of x_0. The resulting asymptotic form of the

Fourier integral is:

$$\hat{F}[|\xi| >> 0] \cong |f[x_0]| \ \exp[+i\xi \cdot \mu[x_0]] \int_{x_0-\epsilon}^{x_0+\epsilon} \exp\left[+i\xi \cdot \mu''[x_0] \frac{(x-x_0)^2}{2}\right] dx \quad (10)$$

where ϵ is a small positive number. Note that this expression assumes that $\mu''[x_0] \neq 0$. If the derivatives of second or larger order of $\mu[x]$ also are zero at x_0, then the derivative with the smallest order (other than 1) that does not vanish is used in the approximation. The remainder of the derivation must be appropriately modified; details are presented by Friedman.

In words, Eq.(10) demonstrates that the Fourier integral of an oscillating function that includes a single stationary point in the infinite domain may be evaluated as the product of some easily evaluated constants and the finite integral of a quadratic-phase exponential. Since the area of the quadratic-phase factor also is concentrated in the vicinity of the stationary point, little additional error is incurred by substituting its area over the infinite domain in Eq.(10).

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \exp\left[+i\xi \cdot \mu''[x_0] \frac{(x-x_0)^2}{2}\right] dx \cong \int_{-\infty}^{+\infty} \exp\left[+i\xi \left(\mu''[x_0] \frac{(x-x_0)^2}{2}\right)\right] dx \quad (11)$$

In short, the finite integral is approximated by the total area of the quadratic-phase exponential, which is easy to evaluate by changing the integration variable to $u \equiv \frac{1}{2}\mu''[x_0](x-x_0)^2$ and applying the central ordinate theorem from Eq.(9.110). The area of the quadratic-phase term is:

$$\begin{aligned}\int_{x=-\infty}^{x=+\infty} \exp\left[+i\xi \left(\mu''[x_0] \frac{(x-x_0)^2}{2}\right)\right] dx &= \left(\sqrt{\frac{2\pi}{\xi \ \mu''[x_0]}}\right) \int_{u=-\infty}^{u=+\infty} \exp\left[+i\pi u^2\right] du \\ &= \left(\sqrt{\frac{2\pi}{\xi \cdot \mu''[x_0]}}\right) \exp\left[+i\frac{\pi}{4}\right] \quad (12)\end{aligned}$$

This result is substituted into Eq.(10) to obtain the approximation for the spectrum that is valid in those cases where the phase of the integrand of the Fourier transform is stationary at a single coordinate:

$$\hat{F}[|\xi| >> 0] \cong |f[x_0]| \left(\sqrt{\frac{2\pi}{\xi \cdot \mu''[x_0]}}\right) \exp\left[+i\frac{\pi}{4}\right] \exp[+i\xi \cdot \mu[x_0]] \quad (13a)$$

This complex amplitude may be expressed as magnitude and phase:

$$\left|\hat{F}\left[|\xi| >> 0\right]\right| \cong |f\left[x_0\right]| \sqrt{\frac{2\pi}{\mu''\left[x_0\right]}}\ \left|\xi^{-\frac{1}{2}}\right| \tag{13b}$$

$$\mathbf{\Phi}\left\{\hat{F}[|\xi| >> 0]\right\} \cong +\xi \cdot \mu[x_0] + \frac{\pi}{4} \tag{13c}$$

The approximation for the more general form of the Fourier integral with a real-valued bipolar modulation is obtained by a direct substitution of the real-valued modulation $r[x]$ for the nonnegative magnitude $|f[x]|$:

$$f[x] = r[x]\ e^{i\mathbf{\Phi}\{f[x]\}} \Longrightarrow \hat{F}\left[|\xi| >> 0\right] \cong r\left[x_0\right] \sqrt{\frac{2\pi}{\xi \cdot \mu''\left[x_0\right]}}\ \exp\left[+i\frac{\pi}{4}\right]\ \exp\left[+i\xi \cdot \mu\left[x_0\right]\right] \tag{14}$$

If there are two (or more) points of stationary phase in the integrand, the approximation is evaluated at each and the results are summed to obtain the asymptotic solution. An example of such a case is considered in the next section.

Obviously, an integral in the frequency domain similar to that in Eq.(3) may be constructed for the inverse Fourier transform, which will allow the asymptotic evaluation of $f[x]$ from a spectrum with an oscillating exponential. Such a development will be used in Chapter 16.

1.1.1 Examples of Spectra via Stationary Phase

13.2.1.1 Unit-Magnitude Linear-Phase Exponential

The formulation in Eq.(14) will be used to evaluate the asymptotic form of the Fourier integral for a few phase functions. Consider first the spectrum of the linear-phase exponential $f[x] = e^{+2\pi i\xi_0 x}$. The Fourier integral is:

$$\begin{aligned} F_1[\xi] &= \int_{-\infty}^{+\infty} \left(1[x]\ e^{+2\pi i\xi_0 x}\right)\left(e^{-2\pi i\xi x}\right) dx \\ &= \int_{-\infty}^{+\infty} e^{-2\pi i(\xi-\xi_0)x}\ dx \end{aligned} \tag{15}$$

which may be recast into the form of Eq.(5) by changing the integration variable ξ to $\zeta = -(\xi - \xi_0)$ and identifying the phase function $\mu[x]$ to be $2\pi x$:

$$\begin{aligned} F_1[\zeta] &= \int_{-\infty}^{+\infty} e^{+i\zeta\ \mu[x]}\ dx \\ &= \int_{-\infty}^{+\infty} e^{+i\zeta\ (2\pi x)}\ dx \end{aligned} \tag{16}$$

The derivative of the phase function is the positive constant $\mu'[x] = 2\pi$, which confirms the observation that the integrand oscillates at the same rate over the entire domain; in other words, there is no point of stationary phase. Since the criterion for stationary phase is not fulfilled, the asymptotic solution for $F[\xi]$ does not exist. This result demonstrates that the method of stationary phase may be applied only if the phase of $f[x]$ includes terms of order 2 or higher.

13.2.1.2 Modulated Quadratic-Phase Exponential

Perhaps the most useful application of the method of stationary phase to Fourier analysis is the determination of the asymptotic form of the spectrum of a function $f_2[x]$ with a quadratic phase scaled by the factor α and modulated by $|f_2[x]|$:

$$f_2[x] = |f_2[x]| \; e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \tag{17}$$

The scale factor α has units of length to ensure that the exponent is dimensionless. The Fourier integral may be recast into the form of Eq.(5):

$$F_2[\xi] = \int_{-\infty}^{+\infty} |f_2[x]| \; e^{+i\pi\left(\frac{x^2}{\alpha^2}-2\xi x\right)} dx = \int_{-\infty}^{+\infty} |f_2[x]| \; e^{+i\xi\,\left(\frac{\pi x^2}{\alpha^2\xi}-2\pi x\right)} dx \tag{18}$$

The phase function and its derivatives are easy to evaluate:

$$\mu[x] = \frac{\pi x^2}{\alpha^2\xi} - 2\pi x \tag{19a}$$

$$\mu'[x] = 2\pi\left(\frac{x}{\alpha^2\xi} - 1\right) \tag{19b}$$

$$\Longrightarrow \mu'[x_0] = 0 = 2\pi\left(\frac{x_0}{\alpha^2\xi} - 1\right) \tag{19c}$$

$$\Longrightarrow x_0 = +\alpha^2\xi \tag{19d}$$

Note that $x_0 = +a^2\xi$ in Eq.(19d) has the required dimensions of length for a coordinate in the space domain. The phase function and its derivatives evaluated at this stationary point are:

$$\mu[x_0] = \frac{\pi\alpha^4\xi^2}{\alpha^2\xi} - 2\pi\alpha^2\xi = -\pi\alpha^2\xi \tag{20a}$$

$$\mu'[x_0] = 0 \tag{20b}$$

$$\mu''[x] = \frac{2\pi}{\alpha^2\xi} \Longrightarrow \mu''[x_0] = \frac{2\pi}{\alpha^2\xi} \tag{20c}$$

$$\mu^{(n)}[x_0] = 0 \; for \; n \geq 3 \tag{20d}$$

These results are substituted into the stationary-phase solution in Eq.(13) to estimate the amplitude of the spectrum at spatial frequencies distant from the origin:

$$\begin{aligned}\hat{F}\left[|\xi| >> 0\right] &= |f_2\left[x_0\right]| \sqrt{\frac{2\pi}{\xi \cdot \mu''\left[x_0\right]}}\; e^{+i\frac{\pi}{4}}\; e^{+i\xi\;\mu[x_0]} \\ &= \left|f_2\left[\alpha^2\xi\right]\right| \sqrt{\frac{2\pi}{\xi\;\left(\frac{2\pi}{\alpha^2\xi}\right)}}\; e^{+i\frac{\pi}{4}}\; e^{+i\xi\;\left(-\pi\alpha^2\xi\right)} \\ &= \left|f_2\left[\alpha^2\xi\right]\right|\;\sqrt{\alpha^2}\; e^{+i\frac{\pi}{4}}\; e^{-i\pi\alpha^2\xi^2} \\ &= \left(|\alpha|\;\;\left|f_2\left[\alpha^2\xi\right]\right|\right)\; e^{+i\frac{\pi}{4}}\; e^{-i\pi\alpha^2\xi^2}\end{aligned} \tag{21}$$

The validity of this expression may be confirmed by by setting the chirp rate α to unity and $|f_2\left[x\right]| = 1$ to produce an unmodulated chirp. The result may be compared to the known spectrum of the quadratic-phase exponential in Eq.(9.92):

$$\mathcal{F}_1\left\{\exp\left[+i\pi x^2\right]\right\} \cong \hat{F}\left[|\xi| >> 0\right] = 1\left[\xi\right]\;\;\exp\left[+i\frac{\pi}{4}\right]\exp\left[-i\pi\xi^2\right] = F\left[\xi\right] \tag{22}$$

In the case of a unit-magnitude quadratic-phase function, the asymptotic and exact forms of the spectrum are identical. This is because $\mu^{(n)}\left[x\right] = 0$ for $n > 3$, and thus no error in the phase function is incurred by truncating the Taylor series at the second-order term.

A more general quadratic-phase function is obtained by replacing the nonnegative modulation $|f\left[x\right]|$ with a real-valued bipolar function $m\left[x\right]$ after scaling by b and translating by x_1:

$$f_3\left[x\right] = m\left[\frac{x - x_1}{b}\right]\;\; e^{\pm i\pi\left(\frac{x}{\alpha}\right)^2} \tag{23}$$

Eq.(14) may be applied directly to evaluate the asymptotic form of the spectrum:

$$\begin{aligned}\hat{F}_3\left[|\xi| >> 0\right] &\cong |\alpha|\;\; m\left[\frac{\alpha^2\xi \mp x_1}{b}\right]\;\; e^{\pm i\frac{\pi}{4}}\; e^{\mp i\pi\alpha^2\xi^2} \\ &= |\alpha|\;\; m\left[\frac{\xi \mp \left(\frac{x_1}{\alpha^2}\right)}{\left(\frac{b}{\alpha^2}\right)}\right]\;\; e^{+i\frac{\pi}{4}}\; e^{-i\pi\alpha^2\xi^2}\end{aligned} \tag{24a}$$

The estimate of the phase transfer function of the modulated quadratic-phase filter is identical to that of the unmodulated signal:

$$\Phi\left\{\hat{F}_3\left[\xi\right]\right\} = -\pi\left(\alpha^2\xi^2 - \frac{1}{4}\right) \tag{24b}$$

Eq.(24) has some very interesting (and perhaps unexpected) features. Note that the magnitude of the spectrum is the *same* real-valued modulation m exhibited by the space-domain function, after translating and scaling by the respective factors $\frac{x_1}{\alpha^2}$ and

$\frac{b}{\alpha^2}$. Both of these factors have dimensions of "reciprocal length," as appropriate for functions in the frequency domain. In words, increasing the width parameter of the modulation of $f_3[x]$ increases the the width parameter of the spectrum estimate $\hat{F}_3[\xi]$ by a proportional factor. Similarly, a translation of the modulation of $f_3[x]$ produces a proportional translation of the modulation of $\hat{F}_3[\xi]$. These features of the spectrum may seem to violate the scaling and shifting theorems of the Fourier transform, but in fact are artifacts of the quadratic-phase function that will be discussed in more detail in Chapter 17.

Because decreasing the scale factor α of the quadratic phase has the effect of increasing the oscillation rate of $f_3[x]$, this condition improves the accuracy of stationary-phase solution at a particular spatial frequency.

The prediction of Eq.(24) will be tested for the specific case of a scaled and translated *SINC* function modulated by a quadratic-phase function with $\alpha = 2$:

$$f[x] = SINC\left[\frac{x-2}{4}\right]\ e^{+i\pi\left(\frac{x}{2}\right)^2} \tag{25}$$

which is graphed both as real-and-imaginary parts and as magnitude-and-phase in Figure 2; note the bipolar modulation present in $f[x]$. The stationary-phase solution for the spectrum is obtained by direct substitution into Eq.(24):

$$\hat{F}[|\xi| >> 0] \cong 2\ SINC\left[\frac{\xi-\frac{1}{2}}{1}\right]\ e^{+i\frac{\pi}{4}}\ e^{-i\pi(2\xi)^2} \tag{26}$$

where the scale factor $\alpha = 1$ in the argument of the *SINC* function corrects the dimensions of both the translation and scale factor. The spectrum is compared to a discrete calculation in Figure 3. The differences between the approximations and the computed spectra are more apparent in the magnified views for $1 \leq \xi \leq 2$ cycles per unit length. Note in the approximate magnitude spectrum is zero at $\xi = 1$, while the "exact" computed spectrum is not.

13.2.1.3 Stationary-Phase Approximation for Symmetric Superchirps

We now briefly consider asymptotic forms of the spectra of unmodulated superchirp functions that were introduced in Eq.(6.143). We defined two flavors of superchirps that differ in behavior for odd values of the order n: the intrinsically symmetric form $\cos[\pi|x|^n] \pm i\ \sin[\pi|x|^n]$ and the Hermitian variety $\cos[\pi|x|^n] \pm i\ SGN[x]\ \sin[\pi|x|^n]$. We consider the symmetric form first. Based upon the symmetry arguments developed in Chapter 9.1, we expect the spectra of these complex-valued and symmetric functions to be complex and symmetric. The Fourier integral is:

$$\mathcal{F}_1\left\{e^{+i\pi\left|\frac{x}{\alpha}\right|^n}\right\} = \int_{-\infty}^{+\infty}\left(e^{+i\pi\left|\frac{x}{\alpha}\right|^n}\right)\ e^{-2\pi i\xi x}\ dx \tag{27}$$

The phase function and its first derivative are:

$$\mu[x] = +\frac{\pi}{\xi}\left(\frac{x}{\alpha}\right)^n - 2\pi x \tag{28}$$

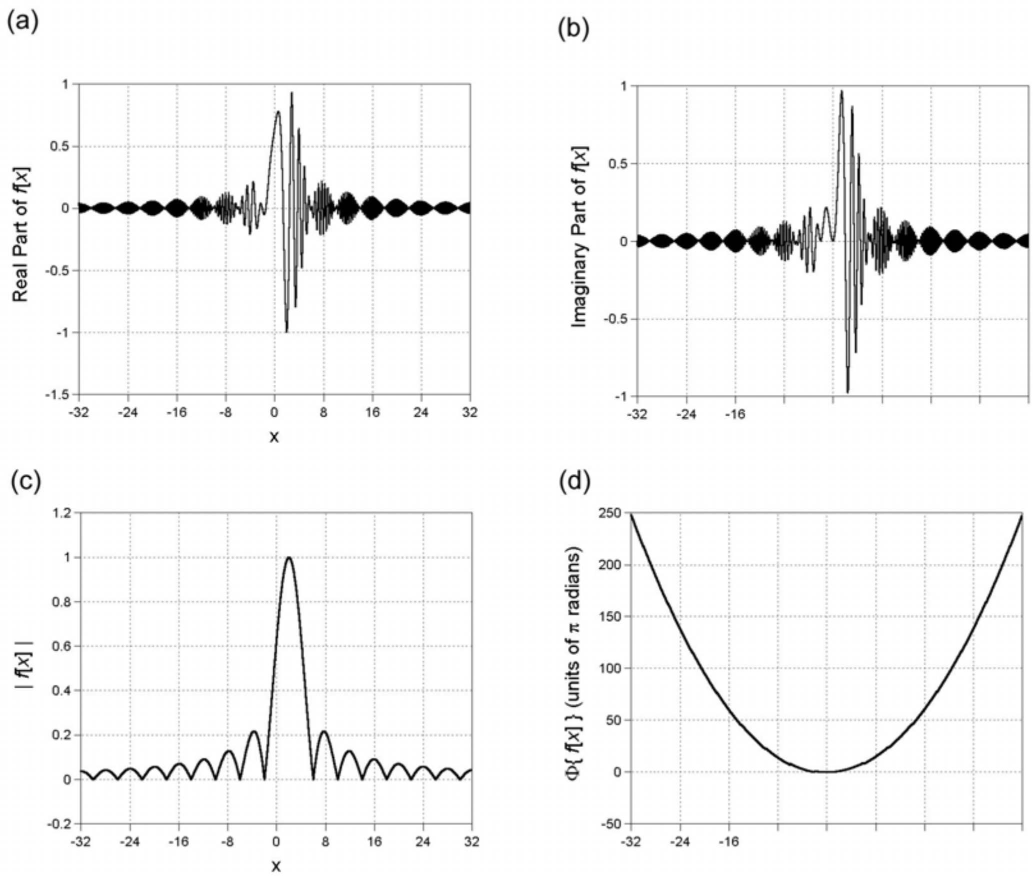

Figure 1.2: $f[x] = SINC\left[\frac{x-2}{4}\right] \exp\left[+i\pi\left(\frac{x}{2}\right)^2\right]$ as (a) real part; (b) imaginary part; (c) magnitude; (d) phase. This function will be used to demonstrate the approximation of the Fourier transform via the method of stationary phase.

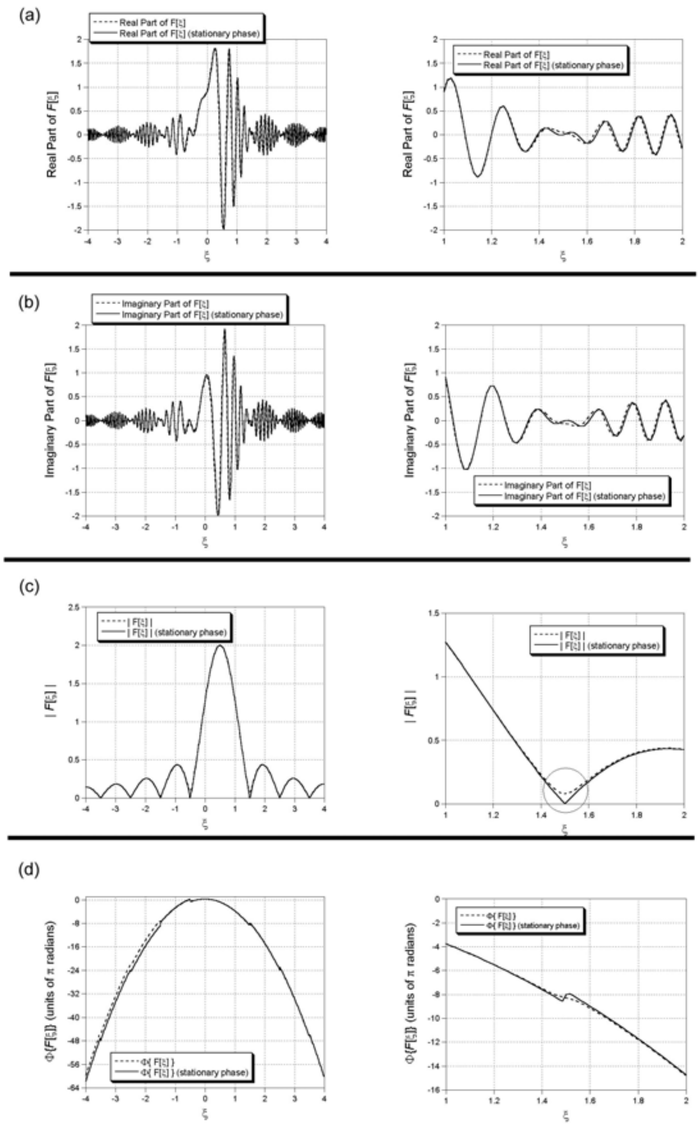

Figure 1.3: The stationary phase approximation to the Fourier transform of $f[x] = SINC\left[\frac{x-2}{4}\right] \exp\left[+i\pi\left(\frac{x}{2}\right)^2\right]$. The approximation is $\hat{F}[\xi] = 2\,SINC\left[\xi + \frac{1}{2}\right] \exp\left[-i\pi\left(2\xi\right)^2\right] \exp\left[+i\frac{\pi}{4}\right]$ (a) real part; (b) imaginary part; (c) magnitude; and (d) phase. The differences between the approximation and the computed spectra are more visible in the magnified views for $1 \leq \xi \leq 2$ cycles per unit length.

$$\mu'[x] = +\frac{\pi}{\alpha\xi}\; n\; \left(\frac{x}{\alpha}\right)^{n-1} - 2\pi \tag{29}$$

It is convenient to consider the cases of even and odd values of n separately. When n is even, the exponent $n-1$ in $\mu'[x]$ is odd. The point(s) of stationary phase (if any) are the solutions to $\mu'[x_0] = 0$, which are determined by the spatial frequency ξ in the Fourier integral:

$$x_0 = \alpha \left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{30}$$

Note that x_0 has the required units of length. The stationary points of an even-order symmetric superchirp are the real-valued solutions of Eq.(30), and thus are proportional to odd-order roots of the selected spatial frequency ξ. For example, if $n = 4$, the stationary points are proportional to the real-valued solutions of $\xi^{\frac{1}{3}}$. For $\xi > 0$, a single real-valued solution for $\xi^{\frac{1}{3}} \propto x_0$ exists and it is positive. When $\xi < 0$, the single real-valued solution for x_0 is negative.

When n is odd, the phase function of the symmetric superchirp may be decomposed into two functions, one each for positive and negative x:

$$\mu[x \geq 0] = +\frac{\pi}{\xi}\left(+\frac{x}{\alpha}\right)^{n} - 2\pi x \tag{31a}$$

$$\mu[x \leq 0] = +\frac{\pi}{\xi}\left(-\frac{x}{\alpha}\right)^{n} - 2\pi x \tag{31b}$$

The first derivatives of the phase in these two regions are:

$$\mu'[x \geq 0] = +\frac{\pi}{\alpha\xi}\; n\; \left(+\frac{x}{\alpha}\right)^{n-1} - 2\pi \tag{32a}$$

$$\mu'[x \leq 0] = +\frac{\pi}{\alpha\xi}\; n\; \left(-\frac{x}{\alpha}\right)^{n-1} - 2\pi \tag{32b}$$

Because n is odd, $n-1$ is even. For positive values of x, the stationary point(s) must satisfy:

$$(x_0)_+ = +\alpha\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{33a}$$

Because it is proportional to an even-order root of ξ, this expression is real valued only for $\xi > 0$. In other words, the stationary-phase estimate of the spectrum of an odd-order superchirp is nonzero only for positive frequencies.

Similarly, the stationary point for negative x must satisfy the condition:

$$(x_0)_- = -\alpha\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{33b}$$

The real-valued even-order root of ξ again exists only for $\xi > 0$, and thus $(x_0)_- < 0$. In short, the integrand has a single stationary point if n is symmetric. An example is shown in Figure 4 for $n = 4$, $\alpha = 1$ unit, and $\xi = +4$ cycles per unit length. The stationary point is located at $x_0 = +\left(\frac{2\cdot 1\cdot 4}{4}\right)^{\frac{1}{3}} \cong 1.26$.

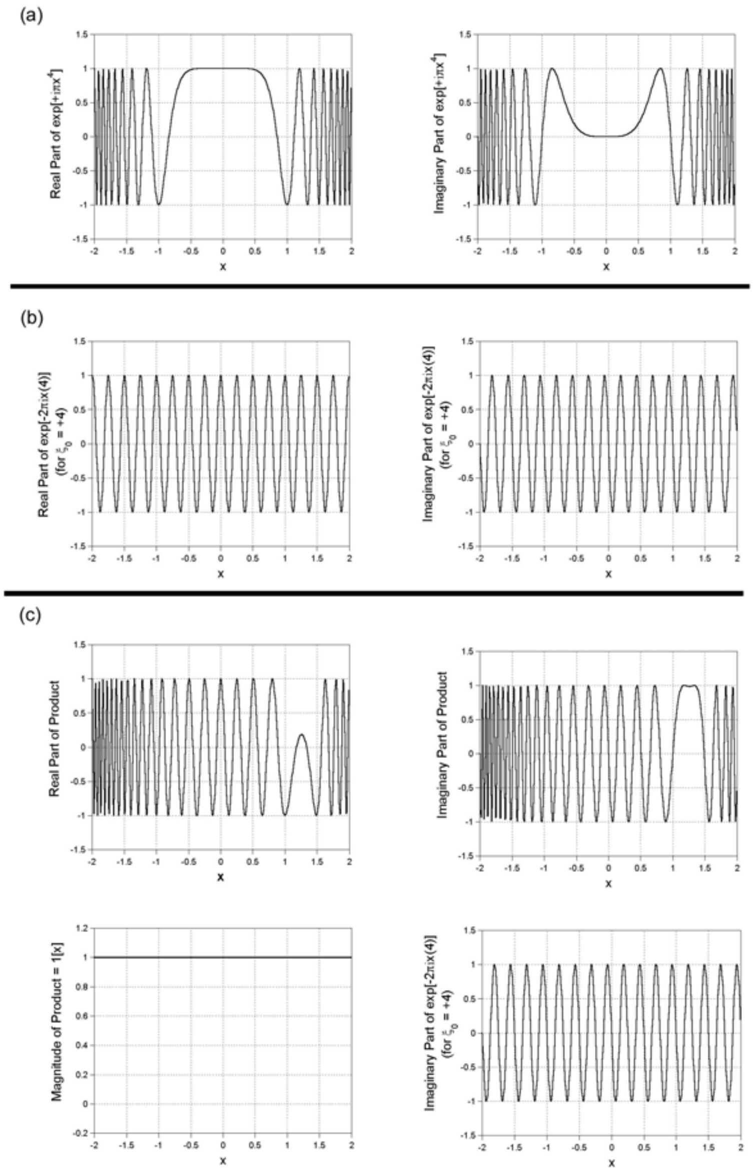

Figure 1.4: Stationary point of $\exp\left[+i\pi x^4\right] \cdot \exp\left[-2\pi i x \cdot 4\right]$: (a) real and imaginary parts of $\exp\left[+i\pi x^4\right]$; (b) real and imaginary parts of $\exp\left[-2\pi i \cdot x \cdot 4\right]$; (c) real part, imaginary part, unit magnitude, and phase of $\exp\left[+i\pi x^4\right] \cdot \exp\left[-2\pi i \cdot x \cdot 4\right]$, showing the stationary point at $x_0 = \sqrt[3]{2} \cong 1.26$.

It may be convenient to recast the integral in Eq.(27) into the form of Eq.(5) by changing variables:

$$\int_{-\infty}^{+\infty} \left(e^{+i\pi \frac{|x|^n}{\alpha}}\right) e^{-2\pi i\xi x} dx = \int_{-\infty}^{+\infty} e^{+iv(u^n - u)} \ \left(\frac{v}{\pi}\right)^{\frac{1}{n}} \ \alpha \ du \tag{34a}$$

where:

$$x = \alpha u \ \left(\frac{v}{\pi}\right)^{\frac{1}{n}} \tag{34b}$$

$$\xi = \frac{1}{2\alpha}\left(\frac{v}{\pi}\right)^{\frac{n-1}{n}} \frac{1}{2\alpha} \tag{34c}$$

The phase function and its derivatives are easily evaluated in terms of u and v:

$$\mu\left[u\right] = u^n - u \tag{35a}$$

$$\mu'[u] = nu^{n-1} - 1 \Longrightarrow u_0 = n^{\left(\frac{1}{n-1}\right)} \tag{35b}$$

$$\mu''\left[u\right] = n \ \ (n-1) \ \ u^{n-2} \tag{35c}$$

$$\mu[u_0] = (1-n)\left(n^{\left(\frac{n}{1-n}\right)}\right) \tag{35d}$$

$$\mu''\left[u_0\right] = n\left(n-1\right)\left(n^{-\left(\frac{n-2}{n-1}\right)}\right) \tag{35e}$$

Substitution of these terms into the stationary-phase solution of Eq.(13) yields the approximate solution for the spectrum that is valid for large $|\nu|$:

$$\hat{F}\left[|v| >> 0\right] = \left(\sqrt{\frac{2\pi}{v \ \mu''\left[u_0\right]}}\right) \ e^{+i\frac{\pi}{4}} \ e^{+iv\cdot\mu[u_0]} \ \alpha \left(\left(\frac{v}{\pi}\right)^{\frac{1}{n}}\right) \tag{36}$$

which may be written in the desired form by substituting the form of ν from Eq.(34b). The stationary-phase solution for the spectrum of the unmodulated symmetric superchirp function is the complex-valued symmetric function:

$$\hat{F}\left[\xi\right] = \alpha\left(\frac{2}{n\left(n-1\right)}\right)^{\frac{1}{2}}\left(\frac{2\alpha\left|\xi\right|}{n}\right)^{-\left(\frac{n-2}{2n-2}\right)} \ \exp\left[+i\frac{\pi}{4}\right] \ \left(\exp\left[+i\pi\left(1-n\right)\left(\frac{2\alpha\left|\xi\right|}{n}\right)^{\left(\frac{n}{n-1}\right)}\right]\right) \tag{37}$$

To check (if not confirm) the validity of this expression, substitute $n = 2$ and compare to the known spectrum of the quadratic-phase function:

$$\begin{aligned}\mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\} &\Longrightarrow \hat{F}\left[\xi\right] = \alpha^1\sqrt{1}\left(\alpha\left|\xi\right|\right)^0 \ \exp\left[+i\frac{\pi}{4}\right] \ \exp\left[-i\pi\left(\alpha\left|\xi\right|\right)^2\right] \\ &= \alpha \ \exp\left[+i\frac{\pi}{4}\right] \ \exp\left[-i\pi\left(\alpha\left|\xi\right|\right)^2\right]\end{aligned} \tag{38}$$

which we know to be the correct spectrum for all values of ξ from the application of the known transform and the scaling theorem.

It is perhaps instructive to examine the functional forms of the magnitude and phase of the stationary-phase solution to the superchirp spectrum. The magnitude spectrum is the even function:

$$\left|\hat{F}[\xi]\right| = \left|\alpha\sqrt{\frac{2}{n(n-1)}}\left(\frac{2\alpha|\xi|}{n}\right)^{-\left(\frac{n-2}{2n-2}\right)}\right| \tag{39a}$$

The fact that the magnitude spectrum is constant for $n = 2$ confirms the observation that chirp functions are composed of sinusoids with identical magnitudes at all spatial frequencies. However, for $n > 2$, the magnitude of the spectrum is *not* constant, but rather *decreases* with increasing frequency. The rate of decline in the magnitude spectrum is $|\xi|^0 = 1$, $|\xi|^{-\frac{1}{4}}$, $|\xi|^{-\frac{1}{3}}$, and $|\xi|^{-\frac{3}{8}}$ for $n = 2 - 5$. In words, the magnitude falls off more quickly with increasing order n. This behavior is consistent with the observation that the superchirp amplitude is approximately constant over larger regions and that the spatial frequency changes more quickly with x as the order n is increased.

The phase spectrum of the symmetric superchirp is:

$$\mathbf{\Phi}\left\{\mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^n}\right\}\right\} \cong \pi\left(\frac{1}{4} + (1-n)\left(\frac{2\alpha|\xi|}{n}\right)^{\frac{n}{n-1}}\right) \tag{39b}$$

which varies as $-|\xi|^2$, $-|\xi|^{\frac{3}{2}}$, $-|\xi|^{\frac{4}{3}}$, and $-|\xi|^{\frac{5}{4}}$ for $n = 2 - 5$, respectively. In words, the variation in phase of the spectrum over ξ *decreases* as the order of the superchirp is *increased*. Examples are shown in Figure 5.

The initial phase of the stationary-phase approximation of the spectrum of all superchirps of the form $e^{+i\pi|x|^n}$ is $+\frac{\pi}{4}$ radians, though we know from the moment calculation in Eq.(39) that the initial phase actually is $+\frac{\pi}{2n}$ radians. This again reminds us that the stationary-phase calculation is valid for $|\xi| >> 0$.

13.2.1.4 Spectra of Hermitian Superchirp Functions via Stationary Phase

The symmetry arguments of Chapter 9.1 and the expansion in terms of moments in Eq.(48) demonstrate that the spectra of all odd-order Hermitian superchirps are real valued. The approximate forms of these spectra obtained from the method of stationary phase demonstrate the variations required when the integrand has more than one stationary point. We consider the case for odd n for simplicity. The phase function of the Fourier integral of the Hermitian function is:

$$\mu[x] = \left(+\frac{\pi}{\xi}\left(\frac{x}{\alpha}\right)^n - 2\pi x\right) \tag{40a}$$

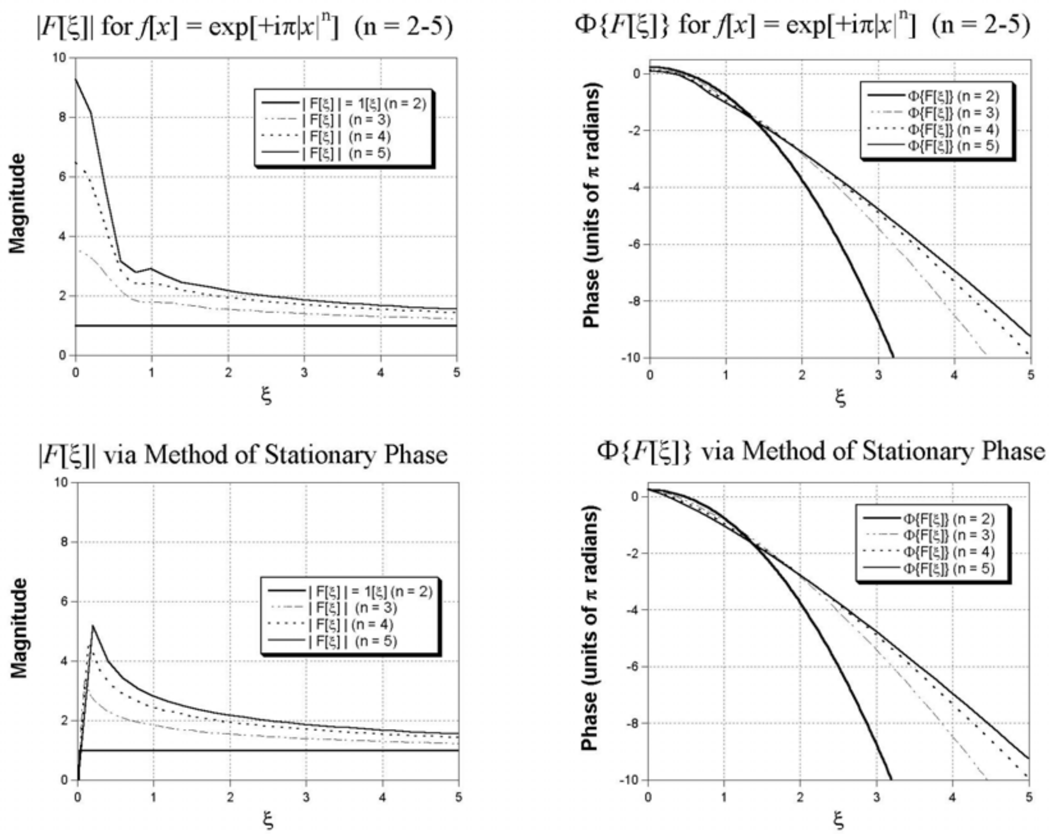

Figure 1.5: Magnitude and phase of spectra of symmetric superchirps $f[x] = \exp[+i\pi|x|^n]$ for $n = 2, 3, 4, 5$ by discrete computation and the approximation by the method of stationary phase from Eq.(39). Note that the magnitude falls off more rapidly and the phase less rapidly with ξ as n increases.

and its first and second derivatives are respectively:

$$\mu'[x] = \left(+\frac{\pi}{\alpha\xi}\right) \; n \; \left(\frac{x}{\alpha}\right)^{n-1} - 2\pi \tag{40b}$$

$$\mu''[x] = \left(+\frac{\pi}{\alpha^2\xi}\right) \; n(n-1) \; \left(\frac{x}{\alpha}\right)^{n-2} \tag{40c}$$

The stationary point(s) (if any) are the solutions to $\mu'[x_0] = 0$:

$$x_0 = \alpha\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{41}$$

Since n is odd for all Hermitian superchirps, then $(n-1)$ is even. In words, the coordinate of the stationary point is proportional to the real-valued even-order roots of the spatial frequency ξ where the Fourier integral is evaluated. When ξ is large and positive ($\xi >> 0$), there are two real-valued solutions for x_0 with identical magnitudes and opposite sign, i.e., at $x_0 = \pm|x_0|$. The contributions from the two stationary points must be added to evaluate the asymptotic form of the Fourier integral for positive frequencies.

The situation is qualitatively different for $\xi < 0$. The coordinates of the stationary points are proportional to even-order roots of negative numbers, which have NO real-valued solutions; there are no stationary points if $\xi < 0$, and thus the stationary-phase approximation of the Fourier integral is zero for all $\xi << 0$. Examples of the integrand of the Fourier integral of the cubic Hermitian superchirp are shown in Figure 6.

It remains to evaluate the phase function and its second derivative at the two stationary points of the odd-order Hermitian superchirp for $\xi > 0$. These may be inserted separately into Eq.(13) and summed to evaluate the approximation. The positive and negative stationary points are labelled x_+ and x_-, respectively:

$$0 < x_+ = +\alpha\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{42a}$$

$$0 > x_- = -\alpha\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}} \tag{42b}$$

where ξ, α, and n are all real-valued positive quantities. The corresponding phase functions are obtained by substitution of these derivatives into Eq.(40a). The phase function evaluated for x_+ is:

$$\begin{aligned}\mu[x_+] &= (+1)^n\left(\frac{\pi\left(\frac{2\alpha\xi}{n}\right)^{\frac{n}{n-1}}}{\xi} - 2\pi\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}}\right)\\ &= \pi\left(\frac{2\alpha}{n}\right)^{\frac{n}{n-1}}\xi - 2\pi\left(\frac{2\alpha}{n}\right)^{\frac{1}{n-1}}\xi^{\frac{1}{n-1}}\end{aligned} \tag{43a}$$

Since ξ is assumed to be large and n is positive, the first term must be larger than

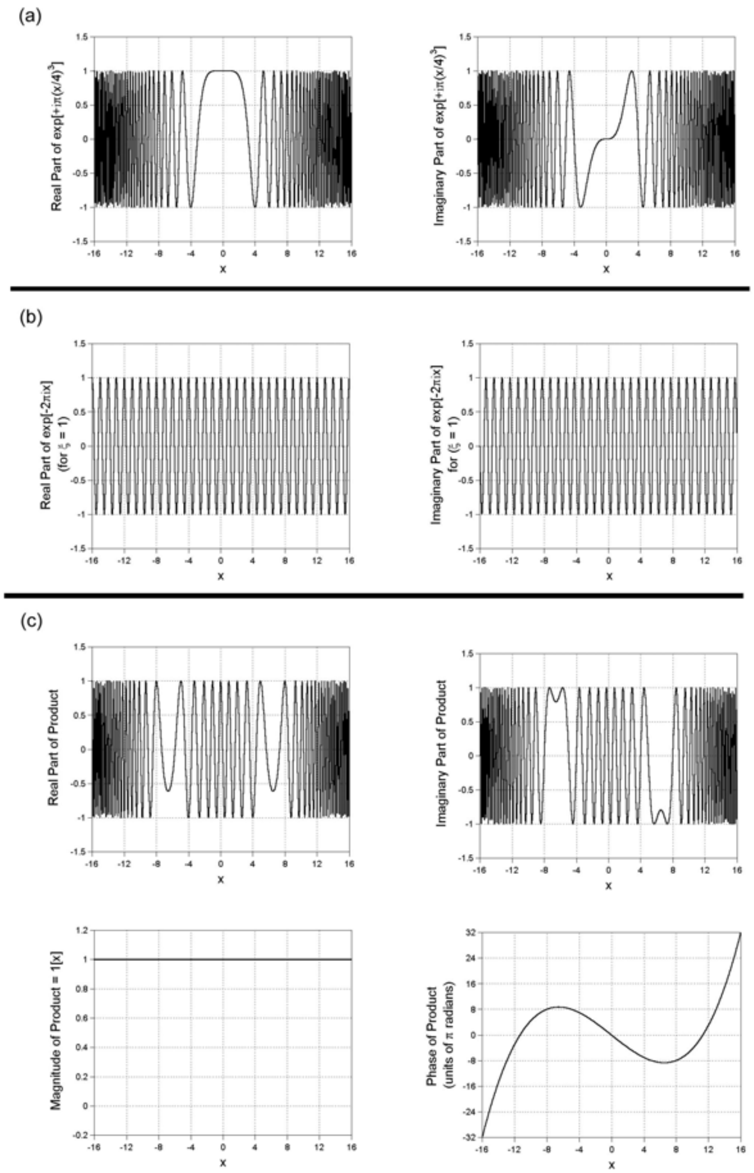

Figure 1.6: Stationary points of the product $\exp + i\pi\left(\frac{x}{4}\right)^3 \cdot \exp\left[-2\pi i x\right]$, $(n = 3, \alpha = 4, \xi = +1)$. There are two stationary points located at $x_0 = \pm 4\sqrt{\left(\frac{2\cdot 4\cdot 1}{3}\right)} \cong \pm 6.53$: (a) real and imaginary parts of $\exp\left[+i\pi\left(\frac{x}{4}\right)^3\right]$; (b) real and imaginary parts of $\exp\left[-2\pi i x\right]$; (c) real part, imaginary part, magnitude, and phase of the product, showing the two stationary points. The contributions from the two stationary points to the Fourier integral are complex conjugates, and so the imaginary parts cancel.

the second, which means that $\mu[x_+]$ must be positive. The phase function evaluated at the negative stationary point is:

$$\mu\left[x_-\right] = (-1)^n\left(\frac{\pi\left(\frac{2\alpha\xi}{n}\right)^{\frac{n}{n-1}}}{\xi}\right) - 2\pi\left(-\left(\frac{2\alpha\xi}{n}\right)^{\frac{1}{n-1}}\right)$$
$$= -\left(\pi\left(\frac{2\alpha}{n}\right)^{\frac{n}{n-1}}\xi - 2\pi\left(\frac{2\alpha}{n}\right)^{\frac{1}{n-1}}\xi^{\frac{1}{n-1}}\right) = -\mu[x_+] \tag{43b}$$

Note that the factors $e^{+i\xi\mu[x_0]}$ evaluated at these two stationary points are complex conjugates.

The corresponding solutions for the second derivative are:

$$\mu''[x_+] = -\frac{\pi}{\alpha^2\xi}\, n\,(n-1)\left(\frac{2\alpha\xi}{n}\right)^{\frac{n-2}{n-1}} \tag{44a}$$

$$\mu''\left[x_-\right] = +\frac{\pi}{\alpha^2\xi}\, n\,(n-1)\left(\frac{2\alpha\xi}{n}\right)^{\frac{n-2}{n-1}} = -\mu''\left[x_+\right] \tag{44b}$$

which also have the same magnitude and opposite sign; the second derivative evaluated at the negative stationary point x_- is positive, while that at x_+ is negative . Substitution of these results into Eq.(13) yield the asymptotic form of the Fourier integral:

$$\hat{F}\left[\xi >> 0\right] = e^{+i\frac{\pi}{4}}\left(e^{i\xi\mu[x_+]}\sqrt{\frac{2\pi}{\xi\mu''\left[x_+\right]}} + e^{i\xi\mu[x_-]}\sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\right)$$
$$= \sqrt{\frac{2\pi}{\xi}}\; e^{+i\frac{\pi}{4}}\left(\frac{e^{-i\xi\mu[x_-]}}{\sqrt{-\mu''\left[x_-\right]}} + \frac{e^{+i\xi\mu[x_-]}}{\sqrt{+\mu''\left[x_-\right]}}\right)$$
$$= \sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\; e^{+i\frac{\pi}{4}}\left(\left[\frac{e^{-i\xi\mu[x_-]}}{\sqrt{-1}}\right] + \left[\frac{e^{+i\xi\mu[x_-]}}{\sqrt{+1}}\right]\right)$$
$$= \sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\; e^{+i\frac{\pi}{4}}\left(\left[\frac{e^{-i\xi\mu[x_-]}}{+i}\right] + \left[\frac{e^{+i\xi\mu[x_-]}}{+1}\right]\right)$$
$$= \sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\;\frac{e^{+i\frac{\pi}{4}}}{e^{+i\frac{\pi}{4}}}\left(\left[\frac{e^{-i\xi\mu[x_-]}}{e^{+i\frac{\pi}{4}}}\right] + \left[\frac{e^{+i\xi\mu[x_-]}}{e^{-i\frac{\pi}{4}}}\right]\right)$$
$$= \sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\;\left(e^{-i\left(\xi\mu[x_-]+\frac{\pi}{4}\right)} + e^{+i\left(\xi\mu[x_-]+\frac{\pi}{4}\right)}\right)$$
$$= \sqrt{\frac{2\pi}{\xi\mu''\left[x_-\right]}}\left(2\;\cos\left[\xi\;\mu\left[x_-\right] + \frac{\pi}{4}\right]\right) \tag{45}$$

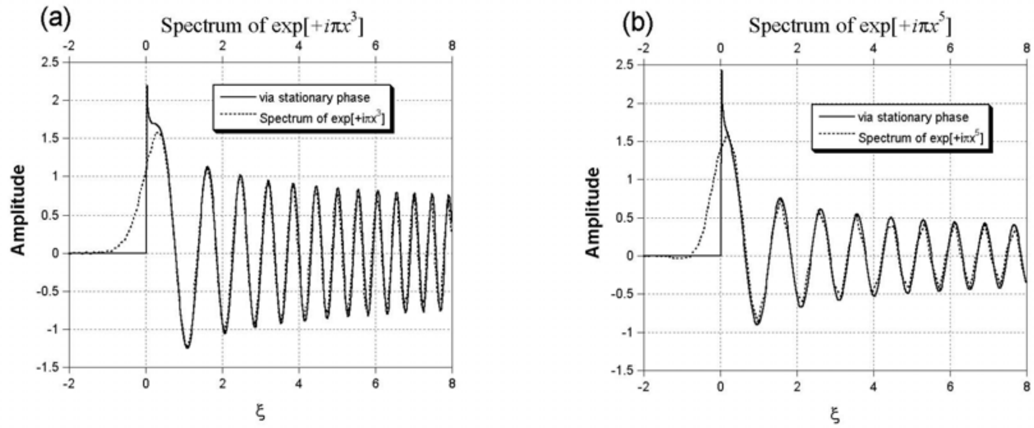

Figure 1.7: Fourier transforms of Hermitian superchirp functions compared to approximations from the method of stationary phase: (a) $\mathcal{F}\left\{\exp\left[+i\pi x^3\right]\right\}$; (b) $\mathcal{F}\left\{\exp\left[+i\pi x^4\right]\right\}$.

After substituting the expressions for μ and μ'' from Eq.(43) and Eq.(44), the asymptotic solutions for the spectra of odd-order Hermitian superchirp are obtained:

$$\mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^n}\right\} \text{ for odd } n$$
$$\cong \begin{cases} 2\alpha\sqrt{\frac{2}{n(n-1)}}\left(\frac{2\alpha\xi}{n}\right)^{-\frac{n-2}{2n-2}}\cos\left[\pi\left(1-n\right)\left(\frac{2\alpha\xi}{n}\right)^{\frac{n}{n-1}}-\frac{\pi}{4}\right] & \textit{if } \xi >> 0 \\ 0 & \textit{if } \xi << 0 \end{cases} \tag{46}$$

In words, the asymptotic spectrum of the odd-order Hermitian superchirp function is real valued and also is zero for negative frequencies. The computed forms for $n = 3$ and $n = 5$ are shown in Figure 7. The graph of Eq.(52) in Figure 6 shows that the moment expansion of the spectrum for $n = 3$ has nonzero amplitude for small negative frequencies. This behavior is not modelled by Eq.(46) because it is valid only for large $|\xi|$. Note that the rate of oscillation of the spectrum with ξ decreases with the order n.