Derivation of the Transfer Function and Impulse Response for Fresnel Diffraction

Maxwell showed that the electric field must satisfy the wave equation:

$$\boldsymbol{\nabla}^2\underline{\mathbf{E}} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\underline{\mathbf{E}}}{\partial t^2} = 0$$

which means that the individual components ("polarizations") also must satisfy this equation:

$$\nabla^2 E_j - \frac{n^2}{c^2}\frac{\partial^2 E_j}{\partial t^2} = 0$$

where $j = x, y, z$ specifies the component and $\mathrm{v} \equiv \frac{c}{n}$, where n is the refractive index. The complete solution for the electric field is the sum of the three polarizations over the integral of fields for all values of $\underline{\mathbf{k}}$ and ω weighted by (generally complex-valued) weight function $F[\underline{\mathbf{k}},\omega] = F[k_x,k_y,k_z,\omega]$, which is analogous to the frequency spectrum in the Fourier transform:

$$\underline{\mathbf{E}}[x_1,y_1,z_1,t] = \sum_{j=1}^{3}\hat{\mathbf{a}}_j\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\int_{-\infty}^{+\infty}dk_z\;\left(F[k_x,k_y,k_z,\omega]\exp\left[i\left(k_x x + k_y y + k_z z - \omega t\right)\right]\right)$$

where the summation over j allows us to assign any polarization to the electric field. From this point forward we ignore the summation over polarizations, but the linearity of the equation allows us to add any additional contributions at the end if needed. If we apply the well-known trial solution $\underline{\mathbf{E}}[x,y,z,t] = \underline{\mathbf{E}}_0\exp[i(\underline{\mathbf{k}}\bullet\underline{\mathbf{r}} - \omega t)]$ to the wave equation in vacuum, we obtain two expressions that are related by the wave equation:

$$\begin{aligned}
\boldsymbol{\nabla}^2\underline{\mathbf{E}} &= \boldsymbol{\nabla}^2\left(\underline{\mathbf{E}}_0\exp[i(\underline{\mathbf{k}}\bullet\underline{\mathbf{r}} - \omega t)]\right)\\
&= \left(\underline{\mathbf{E}}_0\exp[-i\omega t]\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\exp[i(k_x x + k_y y + k_z z)]\\
&= \left(\underline{\mathbf{E}}_0\exp[-i\omega t]\right)\cdot\left(k_x^2+k_y^2+k_z^2\right)\exp[i(k_x x + k_y y + k_z z)]\\
&= i^2\left|\underline{\mathbf{k}}\right|^2\underline{\mathbf{E}} = -k^2\underline{\mathbf{E}}
\end{aligned}$$

$$\frac{\partial^2\underline{\mathbf{E}}}{\partial t^2} = (-i\omega)^2\,\underline{\mathbf{E}} = -\omega^2\underline{\mathbf{E}}$$

Thus the wave equation has the form yields the identity:

$$\boldsymbol{\nabla}^2\underline{\mathbf{E}} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\underline{\mathbf{E}}}{\partial t^2} = -k^2\underline{\mathbf{E}} - \frac{\omega^2}{c^2}\underline{\mathbf{E}} = 0 \Longrightarrow k^2 = \left(\frac{\omega}{c}\right)^2 \Longrightarrow \boxed{k = \frac{\omega}{c}}$$

This relation applies a constraint to the general expression that may be expressed as the 1-D Dirac delta function $\delta\left[k - \frac{\omega}{c}\right]$. Thus the constrained solution for the electric field is:

$$\underline{\mathbf{E}}[x_1,y_1,z_1,t] = \int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\int_{-\infty}^{+\infty}dk_z\;\left(F[k_x,k_y,k_z,\omega]\exp[i(k_x x + k_y y + k_z z - \omega t)]\right)\;\delta\left[k - \frac{\omega}{c}\right]$$

If we assume that the light is monochromatic with frequency ω_0 (wavelength $\lambda_0 = \frac{2\pi c}{\omega_0}$), then the integral over ω may be evaluated immediately by setting the temporal frequency dependence to a Dirac delta function:

$$F[k_x,k_y,k_z,\omega] = F[k_x,k_y,k_z]\cdot\delta[\omega - \omega_0]$$

which leads to the expression for the electric field::

$$\begin{aligned}
&\underline{\mathbf{E}}[x_1,y_1,z_1,t]\\
&= \int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\int_{-\infty}^{+\infty}dk_z\;\left(\int_{-\infty}^{+\infty}d\omega\;F[k_x,k_y,k_z]\cdot\delta[\omega-\omega_0]\;\delta\left[k-\frac{\omega}{c}\right]\;\exp[i(k_x x + k_y y + k_z z - \omega t)]\right)\\
&= \exp[-i\omega_0 t]\;\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\int_{-\infty}^{+\infty}dk_z\;\left(F[k_x,k_y,k_z]\;\exp[i(k_x x + k_y y + k_z z)]\right)\;\delta\left[k-\frac{\omega_0}{c}\right]
\end{aligned}$$

where the sifting property of the Dirac delta function has been used. This remaining 1-D Dirac delta function reduces the 3-D integral to 2-D. This means that the 3-D representation of $\underline{\mathbf{E}}$ for one wavelength may be evaluated from a 2-D function. To evaluate ththat function, we need to integrate over one of the components of $\underline{\mathbf{k}}$. We'll select the integral of k_z, because we will define the source distribution in the 2-D $x-y$ plane. Therefore, we must put the 1-D Dirac delta function $\delta\left[k-\frac{\omega_0}{c}\right]=\delta\left[\left(k_x^2+k_y^2+k_z^2\right)^{\frac{1}{2}}-\frac{\omega_0}{c}\right]$ in the form of $\delta\left[k_z-(k_z)_0\right]$, where $(k_z)_0$ is the positive number that satisfies:

$$\left(k_x^2+k_y^2+(k_z)_0^2\right)^{\frac{1}{2}}-\frac{\omega_0}{c}=0\Longrightarrow\boxed{(k_z)_0=\left[\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}}$$

In other words, the argument of the 1-D Dirac delta function is a *function* of k_z, and we must recall that expression. In this case, we have:

$$\begin{aligned}\delta\left[k-\frac{\omega_0}{c}\right] &= \delta\left[\left(k_x^2+k_y^2+k_z^2\right)^{\frac{1}{2}}-\frac{\omega_0}{c}\right]=\delta\left[g\left[k_z\right]\right]\\ \text{where } g\left[k_z\right] &\equiv \left(k_x^2+k_y^2+k_z^2\right)^{\frac{1}{2}}-\frac{\omega_0}{c}\end{aligned}$$

Dirac Delta Function of Functional Argument

Recall that if $g\left[x_0=0\right]$, then:

$$\delta\left[g\left[x\right]\right]=\frac{\delta\left[x-x_0\right]}{\left|\frac{dg}{dx}\right|_{x=x_0}}$$

so we need to evaluate the derivative $\frac{dg}{dk_z}$ using the chain rule:

$$\frac{dg}{dk_z}=\frac{1}{2}\left(k_x^2+k_y^2+k_z^2\right)^{-\frac{1}{2}}\cdot 2k_z=\frac{k_z}{\left(k_x^2+k_y^2+k_z^2\right)^{\frac{1}{2}}}$$

At the root $(k_z)_0$, this derivative takes the value:

$$\begin{aligned}\left.\frac{dg}{dk_z}\right|_{k_z=(k_z)_0} &= \frac{(k_z)_0}{\left(k_x^2+k_y^2+(k_z)_0^2\right)^{\frac{1}{2}}}\\ &= \frac{\left[\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}}{\left(k_x^2+k_y^2+\left[\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right]\right)^{\frac{1}{2}}}\\ &= \frac{\left[\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}}{\left(\frac{\omega_0^2}{c^2}\right)^{\frac{1}{2}}}\\ &= \frac{\left[\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}}{\left(\frac{\omega_0}{c}\right)}=\left[1-\frac{c^2}{\omega_0^2}\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}\end{aligned}$$

Thus the expression for the 1-D Dirac delta function in terms of $(k_z)_0$ is:

$$\begin{aligned}\delta\left[k-\frac{\omega_0}{c}\right] &= \frac{\delta\left[k_z-(k_z)_0\right]}{\left|\left.\frac{dg}{dk_z}\right|_{k_z=(k_z)_0}\right|}\\ &= \frac{1}{\left[1-\frac{c^2}{\omega_0^2}\left(k_x^2+k_y^2\right)\right]^{\frac{1}{2}}}\delta\left[k_z-\left(\frac{\omega_0^2}{c^2}-\left(k_x^2+k_y^2\right)\right)^{\frac{1}{2}}\right]\end{aligned}$$

Evaluate the Integral over k_z

When evaluating the 1-D integral over k_z, the 1-D Dirac delta function allows us to substitute $(k_z)_0 = \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}}$ at every appearance of k_z:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dk_z \ \ \left(F\left[k_x, k_y, k_z\right] \exp\left[i\left(k_x x + k_y y + k_z z\right)\right]\right) \ \delta\left[k - \frac{\omega_0}{c}\right] \\
= &\int_{-\infty}^{+\infty} dk_z \ \ \left(F\left[k_x, k_y, k_z\right] \exp\left[i\left(k_x x + k_y y + k_z z\right)\right]\right) \cdot \left(\frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}} \delta\left[k_z - \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}}\right]\right) \\
= &\left(F\left[k_x, k_y, (k_z)_0\right] \ \frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}}\right) \ \exp\left[i\left(k_x x + k_y y + \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}} z\right)\right] \\
= &\left(F\left[k_x, k_y, (k_z)_0\right] \ \frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}}\right) \ \exp\left[i\left(k_x x + k_y y\right)\right] \ \exp\left[i\left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}} z\right]
\end{aligned}$$

Note that the only dependence on z appears in the last term. To simplify (and shorten!) this expression, we rename part of the integrand:

$$\begin{aligned}
A\left[k_x, k_y; z\right] &\equiv F\left[k_x, k_y, \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}}\right] \ \frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}} \ \exp\left[i\,(k_z)_0\, z\right] \\
&= F\left[k_x, k_y, \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}}\right] \ \frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}} \ \exp\left[i\left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}} z\right]
\end{aligned}$$

The same function evaluated at $z = 0$ is:

$$A\left[k_x, k_y; z = 0\right] = F\left[k_x, k_y, \left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}}\right] \ \frac{1}{\left[1 - \frac{c^2}{\omega_0^2}\left(k_x^2 + k_y^2\right)\right]^{\frac{1}{2}}}$$

The entire integral now may be written as:

$$\begin{aligned}
\underline{\mathbf{E}}\left[x_1, y_1, z_1, t\right] &= \exp\left[-i\omega_0 t\right] \cdot \left(\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \ A\left[k_x, k_y; z\right] \ \exp\left[i\left(k_x x + k_y y\right)\right]\right) \\
&= \exp\left[-i\omega_0 t\right] \cdot \left(\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \ \left(A\left[k_x, k_y; 0\right] \ \exp\left[i\left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}} z\right]\right) \ \exp\left[i\left(k_x x + k_y y\right)\right]\right)
\end{aligned}$$

which has the form of an inverse 2-D Fourier transform if we identify $\frac{\omega_0}{c} = \frac{2\pi}{\lambda_0}$, $k = |\underline{\mathbf{k}}| = \frac{2\pi}{\lambda}$, $k_x = 2\pi\xi$, and $k_y = 2\pi\eta$. This leads to:

$$\begin{aligned}
\exp\left[i\,(k_z)_0\, z\right] &= \exp\left[i\left(\frac{\omega_0^2}{c^2} - \left(k_x^2 + k_y^2\right)\right)^{\frac{1}{2}} z\right] \\
&= \exp\left[i\frac{2\pi}{\lambda_0}\left(1 - \lambda^2\left(\left(\xi^2 + \eta^2\right)\right)\right)^{\frac{1}{2}} z\right]
\end{aligned}$$

and:

$$\exp\left[i\left(k_x x + k_y y\right)\right] = \exp\left[+2\pi i\left(\xi x + \eta y\right)\right]$$

The exact expression for the integral becomes:

$$\underline{\mathbf{E}}[x_1, y_1, z_1, t] = (2\pi)^2 \exp[-i\omega_0 t] \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \left(A[\xi, \eta; 0] \cdot \exp\left[i\frac{2\pi}{\lambda_0} \left(1 - \lambda_0^2 \left(\xi^2 + \eta^2\right)\right)^{\frac{1}{2}} z \right] \right) \exp[+2\pi i (\xi x + \eta y)]$$

We will ignore the temporal phase term $\exp[-i\omega_0 t]$ from this point forward, because it contributes no spatial dependence. The corresponding expression in the frequency domain is:

$$\mathcal{F}_2 \{\underline{\mathbf{E}}[x_1, y_1, z_1, t]\} = \left(A[\xi, \eta; 0] \cdot \exp\left[i\frac{2\pi}{\lambda_0} \left(1 - \lambda_0^2 \left(\xi^2 + \eta^2\right)\right)^{\frac{1}{2}} z \right] \right)$$

Approximating the Dependence on z

The task now is to make approximations that allow the integral to be evaluated more readily. The obvious candidate for this treatment is to remove the square root from $\exp\left[i\frac{2\pi}{\lambda_0} \left(1 - \lambda_0^2 \left(\xi^2 + \eta^2\right)\right)^{\frac{1}{2}} z \right]$. Recall the binomial theorem:

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + \cdots$$

If $|\varepsilon| << 1$, then this simplifies to:

$$(1+\varepsilon)^n \cong 1 + n\varepsilon$$

If the radial spatial frequency $\rho = \left[\xi^2 + \eta^2\right]^{\frac{1}{2}}$ is small compared to the reciprocal of the wavelength ($\rho << \lambda_0^{-1}$), then we can substitute $n = \frac{1}{2}$ and $\varepsilon = -\lambda^2 \left(\xi^2 + \eta^2\right)$:

$$\begin{aligned} \exp\left[\frac{2\pi i}{\lambda_0} \left(1 - \lambda_0^2 \left(\xi + \eta^2\right)\right)^{\frac{1}{2}} z \right] &\cong \exp\left[\frac{2\pi i}{\lambda_0} \left(1 - \frac{1}{2}\lambda_0^2 \left(\xi^2 + \eta^2\right)\right) z \right] \\ &= \exp\left[i\frac{2\pi z}{\lambda_0} \right] \cdot \exp\left[-i\pi\lambda_0 z \left(\xi^2 + \eta^2\right)\right] \end{aligned}$$

Thus we have the expression:

$$\mathcal{F}_2 \{\underline{\mathbf{E}}[x_1, y_1, z_1, t]\} \cong A[2\pi\xi, 2\pi\eta; 0] \cdot \left(\exp\left[i\frac{2\pi z}{\lambda_0} \right] \cdot \exp\left[-i\pi\lambda_0 z \left(\xi^2 + \eta^2\right)\right] \right) \ \ if \ \ \xi^2 + \eta^2 << \lambda_0^{-2}$$

Transfer Function of Light Propagation

This expression has the form of a modulation in the frequency domain, so we can identify the multiplicative term as a transfer function:

$$\boxed{H[\xi, \eta; z_1] = \exp\left[i\frac{2\pi z_1}{\lambda_0} \right] \cdot \exp\left[-i\pi\lambda_0 z_1 \left(\xi^2 + \eta^2\right)\right]}$$

which describes the effect of light propagation on the field evaluated at $z = 0$ for $\rho << \lambda_0^{-1}$, which defines in the Fresnel diffraction region.

Impulse Response of Light Propagation

We can then use the known Fourier transform to evaluate the corresponding impulse response in the Fresnel diffraction region.

$$\begin{aligned}
h[x,y;z_1] &= \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\mathcal{F}_2\left\{\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right]\right\} \\
&= \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\mathcal{F}_1\left\{\exp\left[-i\pi\lambda_0 z_1\xi^2\right]\right\}\cdot\mathcal{F}_1\left\{\exp\left[-i\pi\lambda_0 z_1\eta^2\right]\right\} \\
&= \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\left(\frac{1}{\sqrt{\lambda_0 z_1}}\exp\left[-i\frac{\pi}{4}\right]\exp\left[+i\pi\frac{x^2}{\lambda_0 z_1}\right]\right)\cdot\left(\frac{1}{\sqrt{\lambda_0 z_1}}\exp\left[-i\frac{\pi}{4}\right]\exp\left[+i\pi\frac{y^2}{\lambda_0 z_1}\right]\right) \\
&= \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\left(\frac{1}{\sqrt{\lambda_0 z_1}}\right)^2\left(\exp\left[-i\frac{\pi}{4}\right]\right)^2\exp\left[+i\pi\frac{\left(x^2+y^2\right)}{\lambda_0 z_1}\right] \\
&= \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\frac{1}{\lambda_0 z_1}(-i)\exp\left[+i\pi\frac{\left(x^2+y^2\right)}{\lambda_0 z_1}\right]
\end{aligned}$$

This the impulse response of light propagation in the Fresnel diffraction region:

$$\boxed{h[x,y;z_1] = \exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\cdot\frac{1}{i\lambda_0 z_1}\exp\left[+i\pi\frac{\left(x^2+y^2\right)}{\lambda_0 z_1}\right]}$$

In other words, the electric field at the plane $z = z_1$, which we will call $g[x,y;z_1]$, can be derived from the field at the plane $z = 0$, which we will call $f[x,y;0]$ via the 2-D convolution:

$$g[x,y;z_1] \cong \left(\frac{1}{i\lambda_0 z_1}\exp\left[i\frac{2\pi z_1}{\lambda_0}\right]\right)\left(f[x,y;0]*\exp\left[+i\pi\frac{\left(x^2+y^2\right)}{\lambda_0 z_1}\right]\right)$$

$$\boxed{g[x,y;z_1] \cong \left(\frac{1}{\lambda_0 z_1}\exp\left[i\frac{2\pi z_1}{\lambda_0}-i\frac{\pi}{2}\right]\right)\left(f[x,y;0]*\exp\left[+i\pi\frac{\left(x^2+y^2\right)}{\lambda_0 z_1}\right]\right)}$$

Interpretation of Fresnel Diffraction

Since we can evaluate the equation for light propagation from a 2-D distribution of monochromatic light in the plane $z = 0$ as a convolution, then the process of light propagation in this approximation must be linear and shift invariant. The impulse response is proportional to a quadratic-phase factor, and thus is an "allpass" filter. The leading factor of z_1^{-1} is due to the inverse square law for light propagating "straight down" the z-axis from $z = 0$ to $z = z_1$. The constant-phase factor $\exp\left[i\frac{2\pi z_1}{\lambda_0}\right]$ represents the change in phase of light along this same path. The approximation assumes that the effect of the inverse square law for light that travels at any angle relative to the z-axis is too small to be considered. The spherical waves emitted by the source(s) in the plane $[x,y;0]$ propagate sufficiently far so that the variation in amplitude with transverse distance $[x,y]$ is ignored (i.e., the inverse square law only and the spherical wavefront is approximated by a paraboloidal wavefront.

The Fourier transform of the quadratic-phase factor demonstrates that the transver function is a "downchirp," a quadratic-phase factor with a negative sign.