1051-733-20072 **Homework #1 Solutions**

1.

Wavelength is related to velocity and frequency by $c = \lambda \nu$.
The wavelength of the red light can be found as:

$$\lambda = \frac{c}{\nu} = \frac{3\times10^{8}\, m/s}{5\times10^{14}\, s^{-1}} = 0.6\times10^{-6}\, m$$

Similarly, the wavelength of a 60Hz electromagnetic wave can be found as:

$$\lambda = \frac{c}{\nu} = \frac{3\times10^{8}\, m/s}{60 s^{-1}} = 0.5\times10^{7}\, m$$

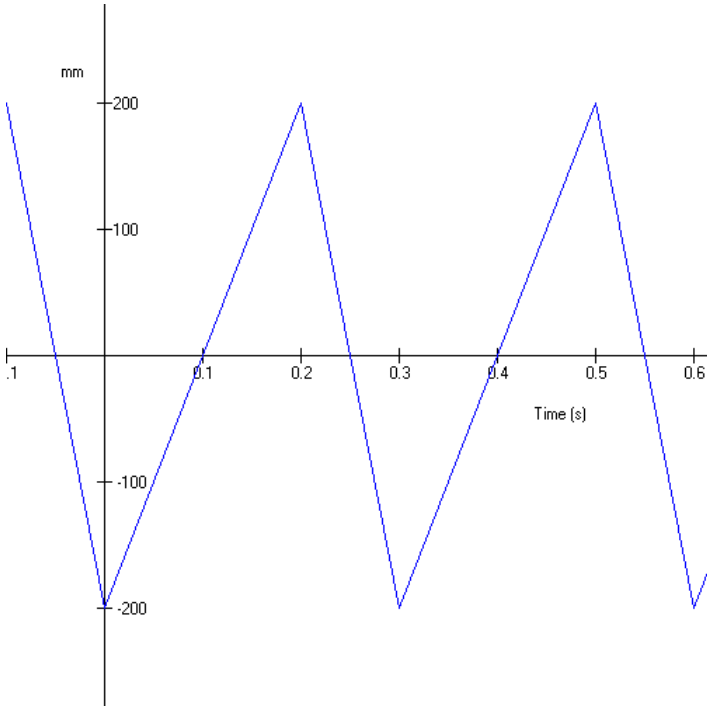
The wavelength of the 60Hz wave is orders of magnitude longer than that of the red light.

2.

(a)

The spatial period is the distance between two points of constant phase (ex. between two peaks). This can be read directly off the figure as 300mm

(b)



(c)

The frequency can by found using $\nu = \frac{1}{T}$, where T is the period of oscillation.

From part b we can see that the period is 0.3s giving us $\nu = \frac{1}{T} = \frac{1}{.3s} = 3.33 Hz$.

3.

(a)

The wave equation written in terms of wavelength and period has the form:

$$\psi(z,t) = A_o \cos\left(\frac{2\pi}{\lambda} z + \frac{2\pi}{T} t + \phi_o\right)$$

Since the wave has a displacement of zero at t=0, z=0, our wave equation needs to equal zero at this point. A shift of the cosine (initial phase) of $\pi/2$ accomplishes this. The remaining unknowns (wavelength, period, and amplitude) are given in the problem. Substitution gives us:

$$\psi(z,t) = A_o \cos\left(\frac{2\pi}{\lambda} z + \frac{2\pi}{T} t + \phi_o\right) = 2\cos\left(2\pi\left(\frac{1}{5m} z + \frac{1}{3s} t\right) \pm \pi/2\right)$$

(b)

The wave equation written in terms of propagation constant and velocity has the form:

$$\psi(z,t) = A_o \cos\left(k_o\left(z + v_\phi t\right) + \phi_o\right) \text{ where } k_o = 2\pi/\lambda \text{ and } v_\phi = \lambda/T$$

Substitution gives us:

$$\psi(z,t) = A_o \cos\left(k_o\left(z + v_\phi t\right) + \phi_o\right) = 2\cos\left(\frac{2\pi}{5m}\left(z + \frac{5m}{3s} t\right) \pm \pi/2\right)$$

(c)

Euler's equation, $e^{ix} = \cos(x) + i\sin(x)$, can be used to write the solution to part a in complex form as:

$$\psi(z,t) = \mathrm{Re}\left\{A_o e^{i\left(2\pi\left(\frac{1}{\lambda} z + \frac{1}{T} t\right) + \phi_o\right)}\right\} = \mathrm{Re}\left\{2e^{i\left(2\pi\left(\frac{1}{5m} z + \frac{1}{3s} t\right) \pm \pi/2\right)}\right\}$$

4.

(a) Starting with the wave equation and substituting the given values gives us:

$$\psi(x,t) = A_o \cos\left(\frac{2\pi}{\lambda}\left(x + v_\phi t\right) + \phi_o\right) = 5m\cos\left(\frac{2\pi}{50m}\left(x + v_\phi t\right) + \phi_o\right)$$

The wave at t=0 can be represented by:

$$\psi(x,0) = 5m\cos\left(\frac{2\pi}{50m}\left(x + v_\phi(0)\right) + \phi_o\right) = 5m\cos\left(\frac{x\pi}{25m} + \phi_o\right)$$

(b)

Going back to our original description of the wave and substituting in the velocity gives us:

$$\psi(x,t) = 5[m]\cos\left(\frac{2\pi}{50[m]}\left(x + v_\phi t\right) + \phi_o\right) = 5[m]\cos\left(\frac{2\pi}{50[m]}\left(x + 2[m/s]t\right) + \phi_o\right)$$

Note the positive sign indicates propagation along the negative x-direction. The wave at t=4s is given as:

$$\psi(x,t) = 5[m]\cos\left(\frac{2\pi}{50[m]}\left(x + 2[m/s](4[s])\right) + \phi_o\right) = 5[m]\cos\left(\frac{\pi}{25[m]}\left(x + 8[m]\right)\right) + \phi_o$$

5.

The proof utilizes Euler’s Equation and the sum of a an exponential and its complex conjugate (found once again using Euler’s) as:

$$e^{ix}+e^{-ix}=(\cos x+i\sin x)+(\cos x-i\sin x)=2\cos x$$

The proof follows:

$$\psi(z,t)=A\sin\left(k_o z+w_o t\right)+A\sin\left(k_o z+w_o t+\phi_o\right)$$

Using Euler's Equation

$$=\operatorname{Im}\left\{Ae^{i(k_o z+w_o t)}+Ae^{i(k_o z+w_o t+\varphi_o)}\right\}$$

$$=A\operatorname{Im}\left\{e^{ik_o z}e^{iw_o t}e^{\frac{i\varphi_o}{2}}e^{\frac{-i\varphi_o}{2}}+e^{ik_o z}e^{-iw_o t}e^{\frac{i\varphi_o}{2}}e^{\frac{i\varphi_o}{2}}\right\}$$

$$=A\operatorname{Im}\left\{e^{i\left(k_o z+\frac{\varphi_o}{2}\right)}\left(e^{i\left(w_o t-\frac{\varphi_o}{2}\right)}+e^{-i\left(w_o t-\frac{\varphi_o}{2}\right)}\right)\right\}$$

Using the equation for the sum of two complex exponentials and Euler's Equation

$$=A\operatorname{Im}\left\{\left(\cos\left(k_o z+\frac{\varphi_o}{2}\right)+i\sin\left(k_o z+\frac{\varphi_o}{2}\right)\right)\left(2\cos\left(w_o t-\frac{\varphi_o}{2}\right)\right)\right\}$$

$$=2A\sin\left(k_o z+\frac{\varphi_o}{2}\right)\cos\left(w_o t-\frac{\varphi_o}{2}\right)$$

6.

(a)

By a simple manipulation we can see that we have the equation of a traveling wave.

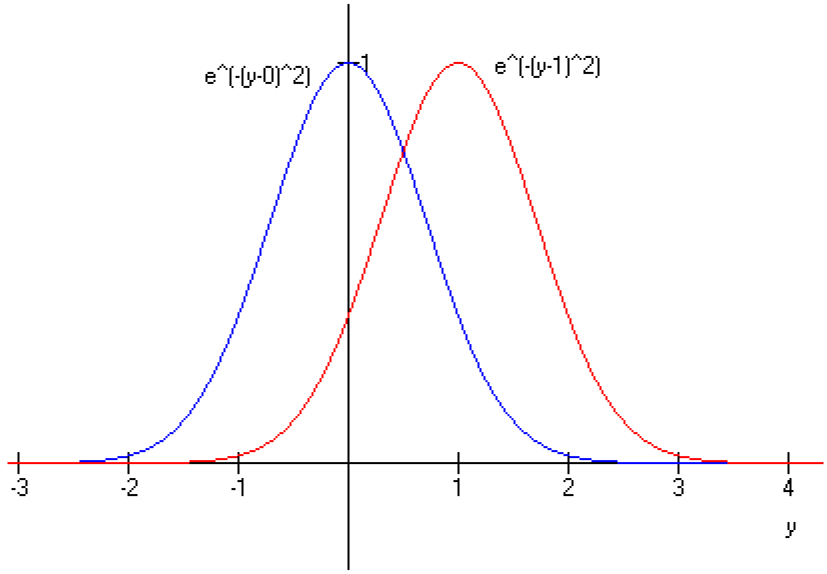
$$\psi(y,t)=e^{-\left(a^2y^2+b^2t^2-2abty\right)}=e^{-\left(a\left(y-\frac{b}{a}t\right)\right)^2}$$

It can be seen that the phase term has the form of a traveling wave.

$$\Phi(y,t)=a\left(y-\frac{b}{a}t\right)=k_o\left(y-v_\phi t\right)$$

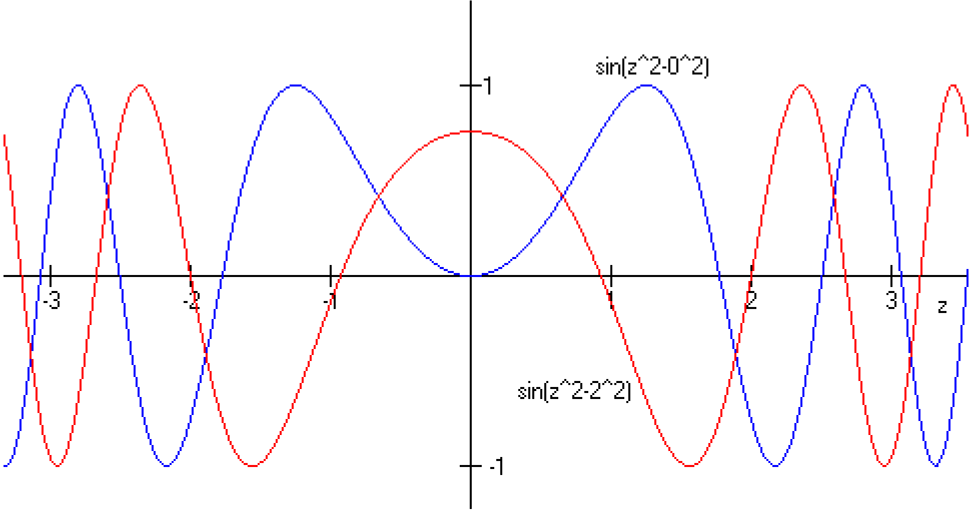
The negative sign indicates the wave is traveling along the positive y-direction. The velocity is $v_\phi = b/a$.

The plot below shows the wave at t=0 and t=1. All constants were assigned a value of one.



(b)

This is not a traveling wave. The powers of t and z prevent a linear relationship between the two. This is easily verified by plotting the function at multiple times. All constants were assigned a value of one.



(c)

By a simple manipulation we can see that we have the equation of a traveling wave.

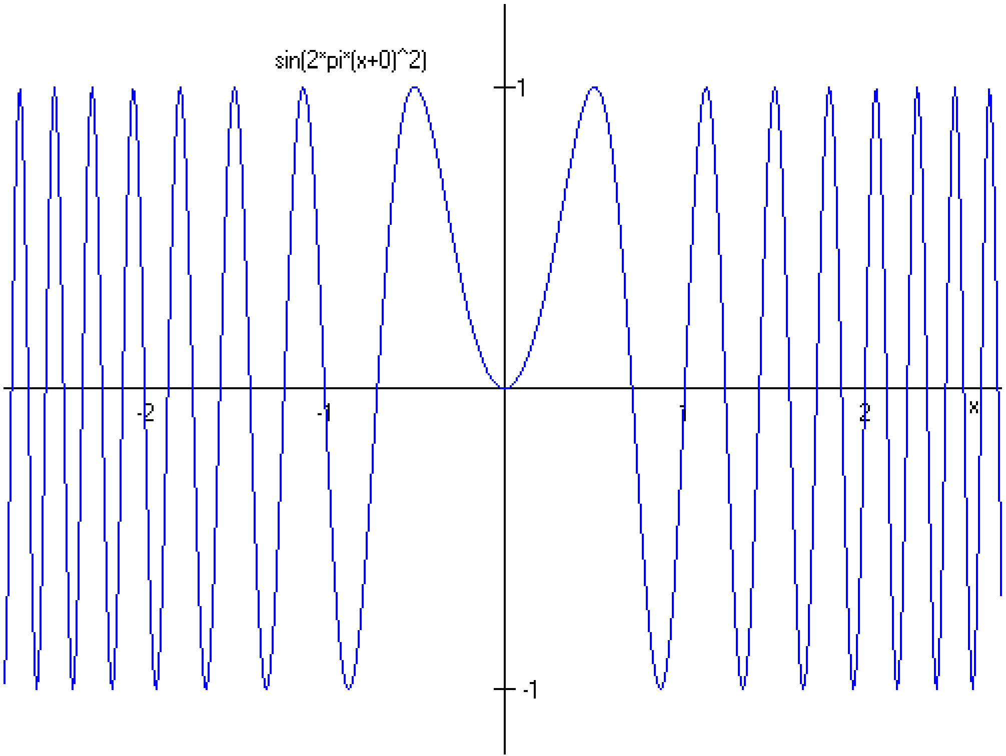
$$\psi(x,t) = A\sin\left(2\pi\left(\frac{x}{a}+\frac{t}{b}\right)^2\right) = A\sin\left(2\pi\left(\frac{1}{a}\left(x+\frac{a}{b}t\right)\right)^2\right)$$

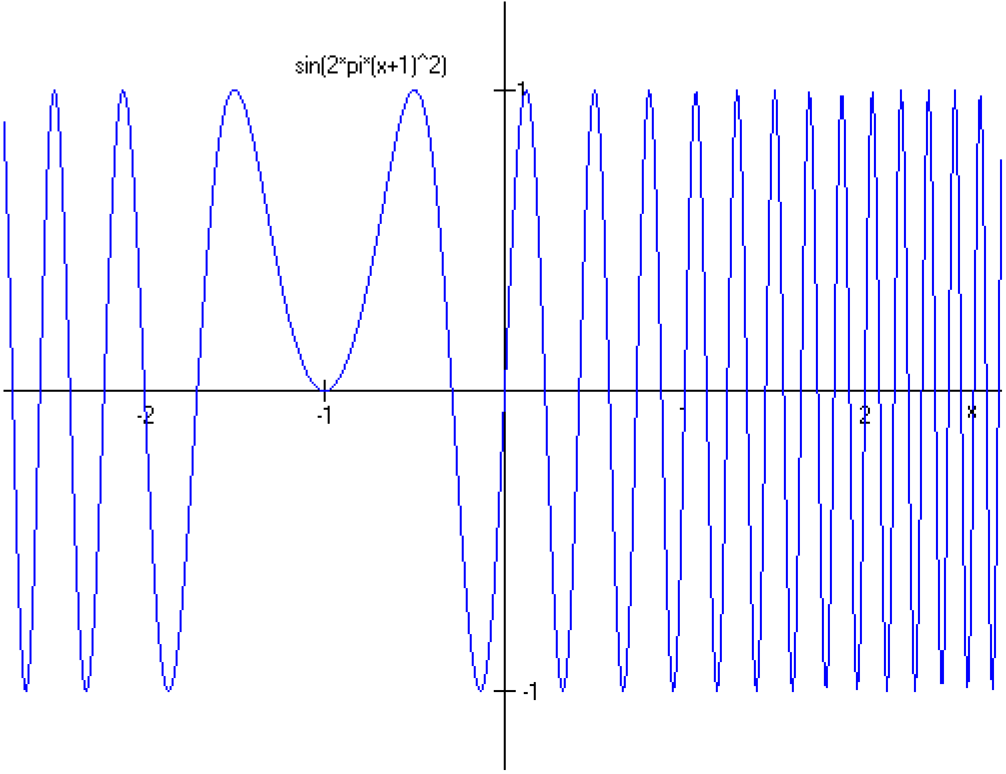
It can be seen that the phase term has the form of a traveling wave.

$$\Phi(x,t) = \frac{1}{a}\left(x+\frac{a}{b}t\right) = k_o\left(x - v_\phi t\right)$$

The positive sign indicates the wave is traveling along the negative x-direction. The velocity is $v_\phi = a/b$.

The plots below shows the wave at t=0 and t=1. All constants were assigned a value of one.


sin(2*pi*(x+0)^2)
1
-1
-2
-1
1
2
x


sin(2*pi*(x+1)^2)
1
-1
-2
-1
1
2
x

(d)

By a simple manipulation we can see that we have the equation of a traveling wave.

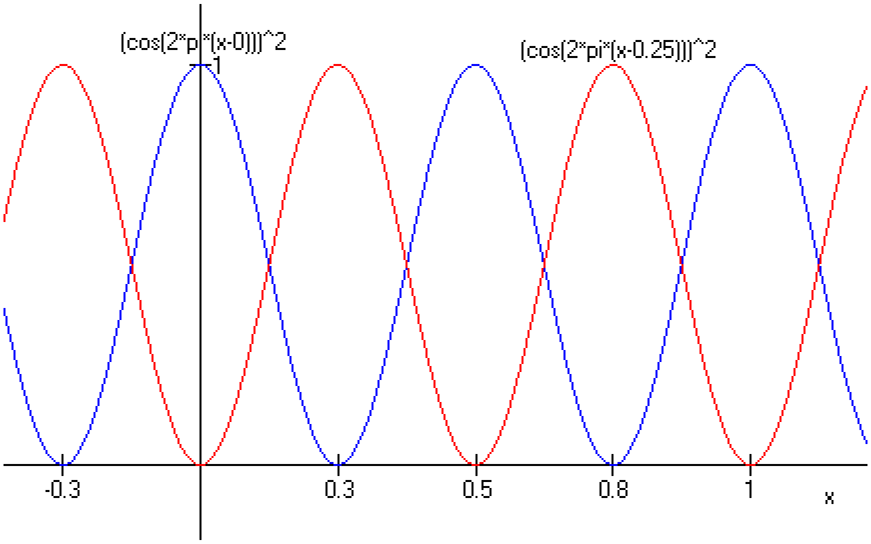
$\psi(x,t) = A\cos^2(2\pi(t-x)) = A\cos^2(2\pi(x-t))$

It can be seen that the phase term has the form of a traveling wave.

$\Phi(x,t) = x - t = k_o(x - v_\phi t)$

The negative sign indicates the wave is traveling along the positive x-direction. The velocity is $v_\phi = 1$.

The plot below shows the wave at t=0 and t=0.25. All constants were assigned a value of one.



7.

(a)

$\sin^2(\Phi(t))$ oscillates from zero to one, centered about one-half. If $T = \tau$ or any other multiple of the period, it is obvious that the upper half will cancel with the bottom half and the average will be 0.5. For $T >> \tau$ we use the following relation: $\sin^2\Theta = \frac{1}{2} - \frac{1}{2}\cos 2\Theta$.

Taking the time average we have:

$$\left\langle \sin^2(\Phi(t)) \right\rangle = \frac{1}{T}\int_t^{t+T} \sin^2(\Phi(t'))dt'$$

$$= \frac{1}{2T}\int_t^{t+T} dt' - \frac{1}{2T}\int_t^{t+T} \cos(\Phi(t'))dt'$$

$$= \frac{1}{2} - \frac{1}{2T}\int_t^{t+T} \cos(\Phi(t'))dt'$$

taking the limit

$$= \frac{1}{2} - \lim_{T\to\infty} \frac{1}{2T}\int_t^{t+T} \cos(\Phi(t'))dt'$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

(b)

Using the results of part a and the relationship $\sin^2\Theta + \cos^2\Theta = 1$ we can confirm the time average of $\cos^2(\Phi(t))$ as 0.5.

$$\left\langle \cos^2(\Phi(t)) \right\rangle = \left\langle 1 - \sin^2(\Phi(t)) \right\rangle = \left\langle 1 \right\rangle - \left\langle \sin^2(\Phi(t)) \right\rangle = 1 - \frac{1}{2} = \frac{1}{2}$$

(c)

The following relationship simplifies the problem: $\sin\Theta\cos\Theta = \dfrac{\sin 2\Theta}{2}$

As in part a, the time average of $\dfrac{\sin(2\Phi(t))}{2}$ for $T = \tau$ or any other multiple of the period is obvious. The function oscillations around zero from positive to negative one-half. If an integer number of oscillations are considered the top and bottom half will cancel out giving a time average of 0. For $T >> \tau$ we have:

$$\left\langle \frac{\sin(2\Phi(t))}{2} \right\rangle = \frac{1}{T}\int_t^{t+T} \frac{\sin(2\Phi(t'))}{2}dt'$$

taking the limit

$$= \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} \frac{\sin(2\Phi(t'))}{2}dt'$$

$$= 0$$