SIMG-721, *Imaging Laboratory*, Fall 2002

1 Convolution Concepts

1.1 2-D convolution

By direct analogy with the 1-D case, a 2-D convolution denoted by the linear operator $\mathcal{O}$ and that is specified by a 2-D function $h[x,y]$. This 2-D function may again be called the *impulse response*, but also is commonly called the *point spread function*. The mathematical expression for a 2-D convolution is is:

$$\begin{aligned}\mathcal{O}\{f[x,y]\} &\equiv f[x,y] * h[x,y] = g[x,y] \\ &= \iint_{-\infty}^{+\infty} f[u,v]\ \ h[x-u,y-v]\ \ du\ dv \end{aligned} \tag{1}$$

This operation is implemented via the same set of four operations as in the 1-D case:

1. reversal of the 2-D impulse respone $h[x,y]$ (equivalent to rotation about the origin of coordinates by $180°$)
2. translation of the impulse response relative to the origin of coordinates
3. multiplication by the "input" function f
4. integration to calculate the "volume" of the product function

The sequence of operations is repeated for all possible translations.

This lab explores an optical method for computing the 2-D convolution that is based on the (approximate) behavior of light as "rays", i.e., it assumes that there is no diffraction (bending) of light. The system actually computes the optical *crosscorrelation* of two functions (that must be real valued in this system. The general expression for 2-D crosscorrelation is similar to 2-D convolution and is denoted by a five-pointed star ("pentagram):

$$f[x,y] \bigstar h[x,y] \equiv \iint_{-\infty}^{+\infty} f[u,v]\ \ h^*[u-x,v-y]\ \ du\ dv \tag{2}$$

Note the differences: crosscorrelation includes a complex conjugation operation (which has no effect for real-valued functions) and the order of the arguments in the impulse response is reversed. The integration variables can be changed to demonstrate that :

$$f[x,y] \bigstar h[x,y] = f[x,y] * h^*[-x,-y] \tag{3}$$

And if $h[x,y]$ is real valued, this reduces to:

$$f[x,y] \bigstar h[x,y] = f[x,y] * h[-x,-y]\ \ if\ \ h \in \Re \tag{4}$$

In short, a system that can evaluate the crosscorrelation may also be used to evaluate the convolution by "reversing" the impulse response.

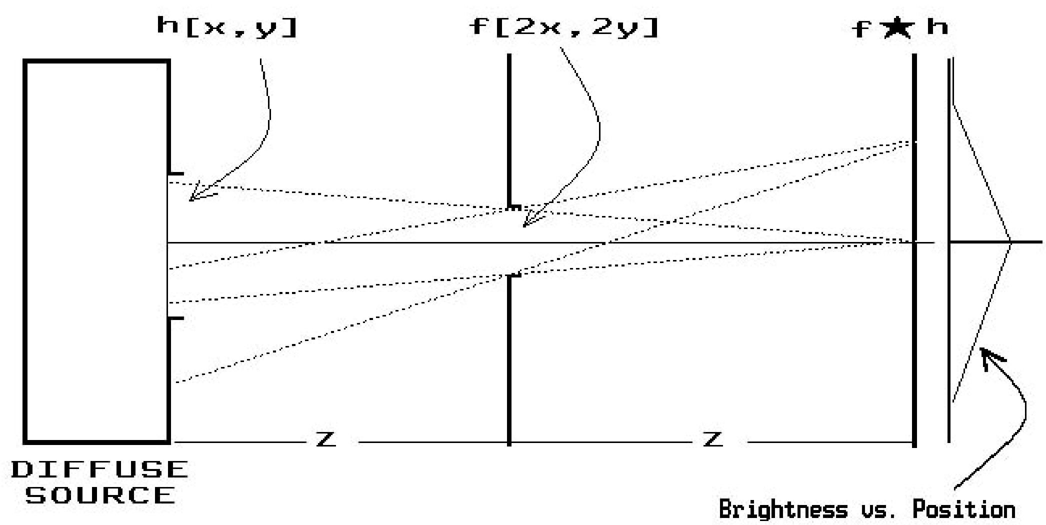
1.2 Optical System

The optical "correlator" system is quite simple, but will be the basis for a system to evaluate the 2-D Fourier transform in the winter quarter.

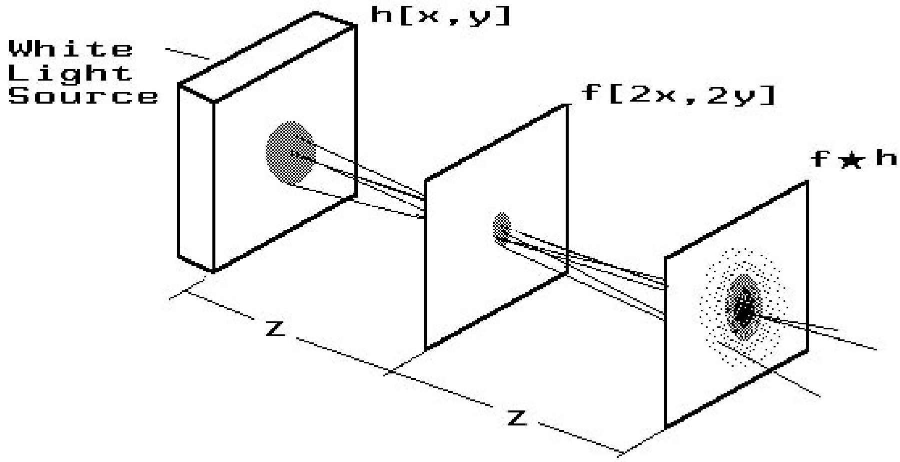
The light source of the optical system is a "light box" that is assumed to emit light "equally" in all directions, so that the intensity of the emitted light is assumed to be a uniform two-dimensional function proportional to $1[x,y]$. The light is modulated by a 2-D transparency with transmittance $m[x,y]$ (which may be bitonal or gray scale). The "pattern" of light after the transparency is proportional to:

$$1[x,y] \times m[x,y] = m[x,y] \tag{5}$$

The light then propagates down the optical axis by some distance z_1, where it encounters a second transparency with transmittance $f[x,y]$. Since the light from the box as assumed to travel in all directions and in straight lines (as "rays"), each point on the second transparency "sees" light from all points in $m[x,y]$; the rays from the various points on m travel along straight-line paths at different angles. This is the source of the "translation" operation. The rays continue to an observation plane located a z_2 from the second transparency. Each location at the output plane "sees" the intensity at a coordinate in the distant transparency (m) through a point in the near transparency (f), and hence the intensity in the distant transparency is multiplied by the transmittance at a single position coordinate in the nearer transparency. Different locations of the detector will "see" a shifted $h[x]$ relative to $f[x,y]$ due to parallax.The light arriving at that single output location is the summation (integral) of the point-by-point products of the intensity of the distant (input-plane) aperture and the transmittance of the midplane aperture. The coordinates of the function f at the midplane must be minified to ensure that it has the same apparent scale as m when seen by a sensor in the output plane. If the two distances z_1 and z_2 are equal, then $f[x,y]$ must be half as large as $m[x,y]$, i.e., the scaled midplane function is $f[2x,2y]$. Such a system is shown in the figures as "side" and "perspective" views:



"Side" view of the optical correlator



"Perspective" view of the optical correlator.

The intensity of the light at the output plane is the crosscorrelation rather than the convolution because neither function has been "reversed". Since the impulse response $h[x,y]$ is a parameter of the "system", I suggest "reversing" the first transparency (actually "rotating" it about its center by 180°) to produce $h[-x,-y]$. The light pattern at the output plane is measured either by direct exposure on a camera sensor (without a lens) or by imaging the pattern cast on an optical diffuser (a ground glass or even a simple piece of plain white paper) with a camera and lens. If you use direct exposure, then the transparencies will have to be small and the propagation distances will have to be short to ensure that the pattern "fits" on the sensor. If your results exhibit noise (and you can BET they will), you may average several images taken under identical conditions.

1.3 Experiments:

This setup will be used to convolve different instances of $h[x,y]$ and $f[2x,2y]$. In each case, "grab" an image of the convolution with the digital camera. Since the "gray-scale" values are important, you want to collect the images using a linear sensor and you should avoid any subsequent nonlinear image processing (linear contrast enhancement is okay).

You can make your own choices of objects, but I suggest trying the following bitonal transparencies:

1. $f[x,y] = h[x,y] = RECT[x,y]$
2. $f[x,y] = RECT[x,y]\,,\ \ h[x,y] = RECT\left[\frac{x}{2},y\right]$
3. $f[x,y] = h[x,y] = CYL\left(\sqrt{x^2+y^2}\right) = CYL(r)$
4. $f[x,y] = CYL(r)\,,\ \ h[x,y] = CYL\left(\frac{r}{2}\right)$
5. $f[x,y] = RECT[x+x_0,y] + RECT[x-x_0,y]\,,\ \ h[x,y] = RECT[x,y]$

You might also try generating "gray-scale" transparencies by making transparencies of some simple halftones. For example, you could try evaluating:

1. $RECT[x,y] * TRI[x,y]$
2. $TRI[x,y] * TRI[x,y]$

A more complicated example that will be the basis for the 2-D optical Fourier transformer is the convolution of thresholded circularly symmetric quadratic-phase sinusoids:

1. $STEP\left[\cos\left[\pi r^2\right]\right] * STEP\left[\cos\left[\pi r^2\right]\right]$ (the individual functions are "Fresnel zone plates"). If you want to try this, use an "off-axis" zone plates as shown:

"Off-axis Fresnel Zone Plate", which is a thresholded off-axis circularly symmetric chirp function

1.4 Questions:

1. Derive the relation between crosscorrelation and convolution, i.e., prove eq. (3).
2. Describe the fundamental constraints on the fidelity of the output of the convolution. In particular, determine what happens if the object and impulse response are small compared to the propagation distances.
3. Describe how to modify the system to compute the crosscorrelation of two real-valued functions.
4. Describe the effect of changing the distances z_1 and z_2 without changing the scales of the two component functions.