SIMG-722, *Imaging Laboratory*, Winter 2002-2003

1 Fourier Transform via Multiplication and Convolution with Quadratic-Phase Functions

In the fall quarter you investigated how to evaluate 1-D and 2-D convolutions with optical systems. This laboratory applies those observations to evaluate the 2-D Fourier transform using 2-D quadratic-phase functions of the form $e^{\pm i\pi\frac{x^2+y^2}{r^2}}$ as modulations and as the impulse responses of filters via the "chirp Fourier transform" algorithm. This algorithm for evaluating the Fourier transform appears often in various imaging applications, and is particularly important in optics because optical propagation may be modeled as convolution with an appropriately scaled quadratic-phase function.

1.1 1-D "*M-C-M*" CHIRP FOURIER TRANSFORM

The 1-D Fourier transform of $f[x]$ is the integral of its product with the complex linear-phase exponential:

$$\mathcal{F}\{f[x]\} \equiv F[\xi] = \int_{-\infty}^{+\infty} f[x]\ e^{-2\pi i\xi x}\ dx \tag{1}$$

The exponent may be rewritten as the sum of three terms:

$$\begin{aligned}(\xi - x)^2 &= x^2 + \xi^2 - 2x\xi \\ \Longrightarrow -2x\xi &= (\xi - x)^2 - x^2 - \xi^2 \end{aligned} \tag{2}$$

Note that the dimensions of x [length] and ξ [cycles/length] do not match, which means that this operation is not legitimate in its basic form. However, we may create an expression that relates these terms and is always valid by applying normalization factors α that have dimensions of length so that the components are dimensionless. The factor α is called the "chirp rate", as it describes "how fast" the quadratic-phase function varies with distance. The new expression is

$$-2\xi x = \left(\alpha\xi - \frac{x}{\alpha}\right)^2 - \left(\frac{x}{\alpha}\right)^2 - (\alpha\xi)^2 \tag{3}$$

The length dimension of the chirp rate α ensures that both $\frac{x}{\alpha}$ and $\alpha\xi$ are dimensionless and may be subtracted without difficulty. This result is substituted into the Fourier transform and the phase factor that does not vary with x is extracted from the integral:

$$\begin{aligned} F[\xi] &= \int_{-\infty}^{+\infty} f[x]\ e^{+i\pi\left(\alpha\xi - \frac{x}{\alpha}\right)^2}\ e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\ e^{-i\pi(\alpha\xi)^2}\ dx \\ &= e^{-i\pi\alpha^2\xi^2} \int_{-\infty}^{+\infty} \left(f[x]\ e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\ e^{+i\pi\left(\alpha\xi - \frac{x}{\alpha}\right)^2}\ dx \end{aligned}$$

After changing variables to $v \equiv \alpha^2\xi$ (which has dimensions of length), we obtain:

$$F[\xi] = e^{-i\pi\left(\frac{v}{\alpha}\right)^2} \int_{-\infty}^{+\infty} \left(f[x]\ e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) e^{+i\pi\left(\frac{v-x}{\alpha}\right)^2}\ dx \tag{4}$$

which has the form of a convolution in the space domain that produces an output function of ν:

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(f[x]\, e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\ e^{+i\pi\left(\frac{v-x}{\alpha}\right)^2}\ dx &= \int_{-\infty}^{+\infty} r[x]\ \ h[v-x]\ \ dx && (5) \\ &= r[x] * h[x]\big|_{x\to\nu} = \left(f[x]\, e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) * \left(e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\Big|_{x\to\nu} && (6) \end{aligned}$$

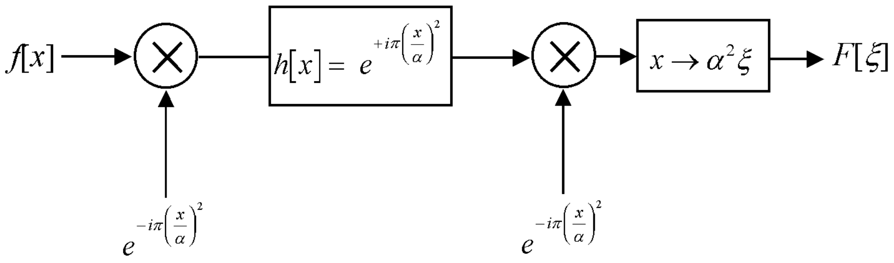


Figure 1: Schematic of 1-D *M-C-M* chirp Fourier transformer

In words, this result shows that the Fourier transform may be written as a cascade of three operations: multiplication by a "downchirp" quadratic-phase function with chirp rate α and a negative sign, convolution with an "upchirp" quadratic-phase with the same chirp rate and evaluated at v, and finally a second multiplication by a "downchirp". The coordinate of the result of the convolution is remapped $\nu \to \alpha^2\xi$ to obtain the Fourier transform:

$$F[\xi] = \left(\left[f[x]\;\; e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\Big|_{x\to v} \right)\;\; e^{-i\pi\left(\frac{v}{\alpha}\right)^2}\Big|_{v\to\alpha^2\xi} \tag{7}$$

Note that the coordinates may be remapped to the frequency domain after the multiplication, which means that all operations with the quadratic-phase signals are performed in the space domain:

$$F[\xi] = \left(\left[f[x]\;\; e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \right)\;\; e^{-i\pi\left(\frac{v}{\alpha}\right)^2}\Big|_{x\to\alpha^2\xi} \tag{8}$$

The sequence of operations suggests an obvious name of "M-C-M" chirp Fourier transformer" for this process. In words, the recipe for the M-C-M chirp Fourier transform is:

1. multiply $f[x]$ by the 1-D "downchirp" $e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$,
2. filter the result by convolving with the "upchirp" impulse response $e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$,
3. multiply the result by a "downchirp" $e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$, and
4. redefine the coordinates by replacing x with $\alpha^2\xi$.

A schematic diagram and an example are shown in the figures. Note that only the phase of the output is affected by the third step in the process, which therefore can be eliminated if only the magnitude (or power) spectrum is required.

If we skip the coordinate "remapping" to the frequency domain, we obtain a relationship between two functions in the space domain where the the "output" has the same functional form as the Fourier transform of the input:

$$\left(\left[f[x]\;\; e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \right)\;\; e^{-i\pi\left(\frac{v}{\alpha}\right)^2} = \mathcal{F}\{f[x]\}\big|_{\xi=\frac{x}{\alpha^2}} = F\left[\frac{x}{\alpha^2}\right] \tag{9}$$

In words, this cascade of the three space-domain operations applied to $f[x]$ yields a function *in the space domain* whose amplitude at coordinate x is equal to the amplitude of the Fourier transform of $f[x]$ evaluated at frequency $\xi = \frac{x}{\alpha^2}$. This is an interesting and very useful result in itself, as we shall see later, but it also provides the starting point for deriving yet another Fourier relationship involving chirp functions in the next section.

An equivalent expression in the frequency domain is obtained by remapping the coordinates of the input function and the chirps to the frequency domain first, via $x = \alpha^2\xi$, to obtain a relation

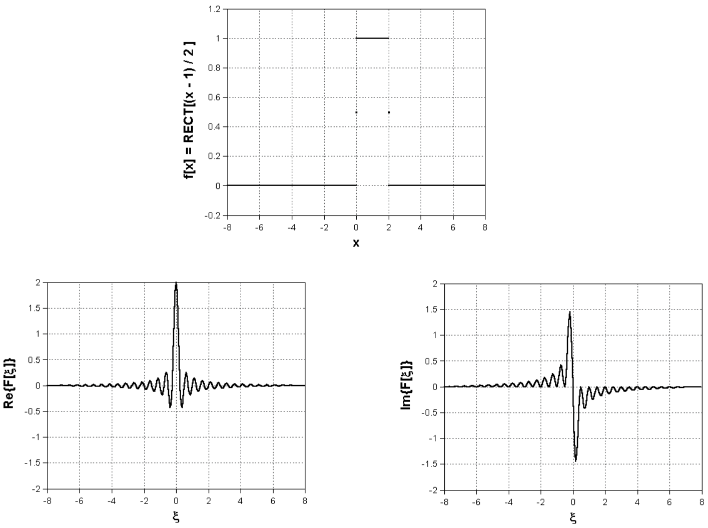


Figure 2: Function $f[x]$ used to demonstrate *M-C-M* and *C-M-C* chirp Fourier transforms and its spectrum $F[\xi]$ as real and imaginary parts.

between two frequency-domain functions $f\left[\alpha^2\xi\right]$ and $F[\xi]$:

$$\begin{aligned} F[\xi] &= e^{-i\pi\alpha^2\xi^2}\int_{-\infty}^{+\infty}\left(f\left[\alpha^2\xi\right]\ e^{-i\pi\alpha^2\xi^2}\right)\ e^{+i\pi\alpha^2\left(\xi-\frac{x}{\alpha^2}\right)^2}\ d\left(\alpha^2\xi\right) \\ &= |\alpha|^2\left(\left(f\left[\alpha^2\xi\right]\ e^{-i\pi\alpha^2\xi^2}\right)*e^{+i\pi\alpha^2\xi^2}\right)\ e^{-i\pi\alpha^2\xi^2} \end{aligned} \tag{10}$$

1.2 1-D "*C-M-C*" CHIRP FOURIER TRANSFORM

The "duality" of convolution and multiplication of functions in the two domains leads to an alternate algorithm for the chirp transform. The derivation uses the "transform-of-a-transform" theorem and the reversal corollary in the "space-domain" *M-C-M* chirp Fourier transform. There are some subtle features of the derivation that are easy to miss. Consider the *M-C-M* transform of the particular input function $F\left[-\frac{x}{a^2}\right]$:

$$\begin{aligned} \left(\left[F\left[-\frac{x}{\alpha^2}\right]\ e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right]*e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\ e^{-i\pi\left(\frac{v}{\alpha}\right)^2} &= \mathcal{F}\left\{F\left[-\frac{x}{\alpha^2}\right]\right\}\Big|_{\xi=\frac{x}{\alpha^2}} \\ &= |\alpha|^2\ \mathcal{F}\left\{f\left[\alpha^2\xi\right]\right\}\big|_{\xi=\frac{x}{\alpha^2}} = \alpha^2 f[x] \end{aligned} \tag{11}$$

The 1-D Fourier transform of both sides of this equation may be evaluated via the known transforms of the chirp functions and the filter and modulation theorems:

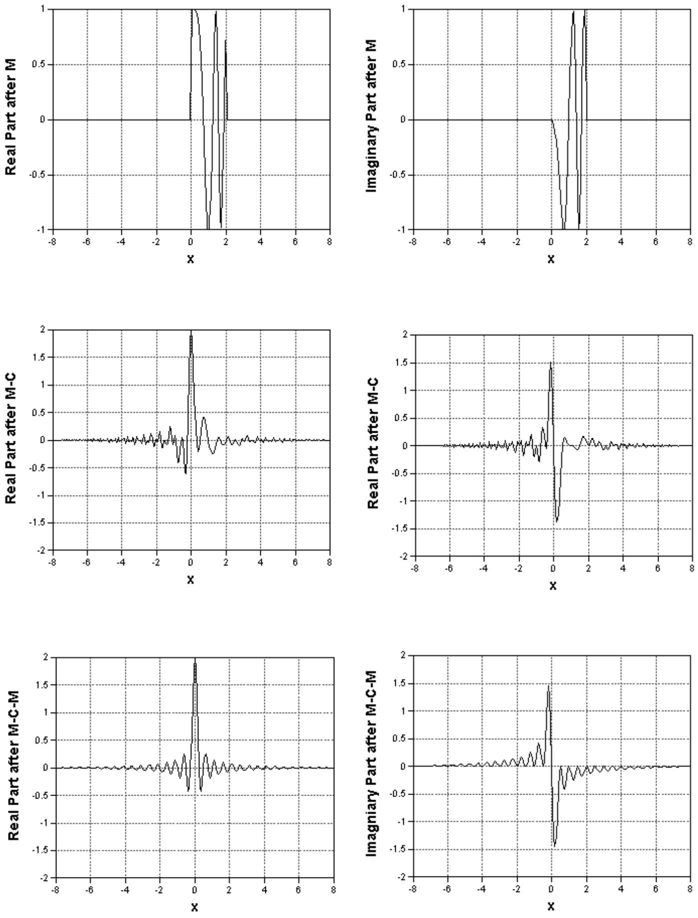


Figure 3: Steps in evaluation of M-C-M chirp Fourier transform with $e^{\pm i\pi x^2}$ (i.e., $\alpha = 1$).

$$
\begin{aligned}
\alpha^2 \mathcal{F}\{f[x]\} &= \mathcal{F}\left\{\left(\left[F\left[-\frac{x}{\alpha^2}\right]\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} \\
\alpha^2 F[\xi] &= \left(\left(|\alpha|^2 f\left[+\alpha^2\xi\right] * \mathcal{F}\left\{e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\}\right)\cdot \mathcal{F}\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\}\right) * \mathcal{F}\left\{e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} \\
&= \alpha^2\left(\left(f\left[\alpha^2\xi\right] * \left[|\alpha|\ e^{-i\frac{\pi}{4}}\ e^{+i\pi\alpha^2\xi^2}\right]\right)\cdot\left[|\alpha|\ e^{+i\frac{\pi}{4}}\ e^{-i\pi\alpha^2\xi^2}\right]\right) * \left[|\alpha|\ e^{-i\frac{\pi}{4}}\ e^{+i\pi\alpha^2\xi^2}\right] \\
&\Longrightarrow \alpha^2 F[\xi] = |\alpha|^5\ e^{-i\frac{\pi}{4}}\left[\left(\left[f\left[\alpha^2\xi\right] * e^{+i\pi\alpha^2\xi^2}\right]\cdot e^{-i\pi\alpha^2\xi^2}\right) * e^{+i\pi\alpha^2\xi^2}\right] \\
&\Longrightarrow F[\xi] = |\alpha|^3\ e^{-i\frac{\pi}{4}}\left[\left(\left[f\left[\alpha^2\xi\right] * e^{+i\pi\frac{\left(\alpha^2\xi\right)^2}{\alpha^2}}\right]\cdot e^{-i\pi\alpha^2\xi^2}\right) * e^{+i\pi\alpha^2\xi^2}\right] \qquad (12)
\end{aligned}
$$

Because the arguments of the component functions in the first convolution on the right side are scaled by α^2, the expression for a scaled convolution applies:

$$
\begin{aligned}
\text{Given that } f[x] * h[x] &= g[x] \\
\text{Then } f\left[\frac{x}{b}\right] * h\left[\frac{x}{b}\right] &= |b|\ g\left[\frac{x}{b}\right]
\end{aligned}
$$

When applied to the inner expression in the last equation, we obtain:

$$
f\left[\alpha^2\xi\right] * e^{+i\pi\frac{\left(\alpha^2\xi\right)^2}{\alpha^2}} = \frac{1}{\alpha^2}\left[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\right]\Big|_{u\to\alpha^2\xi} \qquad (13)
$$

where u has dimensions of length. The Fourier transform of the M-C-M sequence of operations becomes:

$$
\begin{aligned}
F[\xi] &= |\alpha|^3\ e^{-i\frac{\pi}{4}}\left[\left(\frac{1}{\alpha^2}\left[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\right]\Big|_{u\to\alpha^2\xi}\cdot e^{-i\pi\alpha^2\xi^2}\right) * e^{+i\pi\alpha^2\xi^2}\right] \\
&= |\alpha|\ e^{-i\frac{\pi}{4}}\left[\left(\left[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\right]\Big|_{u\to\alpha^2\xi}\cdot e^{-i\pi\alpha^2\xi^2}\right) * e^{+i\pi\alpha^2\xi^2}\right] \qquad (14)
\end{aligned}
$$

As in the derivation of the *M-C-M* transform, the output coordinates of the first convolution may be "remapped" after the multiplication by the downchirp:

$$
\begin{aligned}
F[\xi] &= |\alpha|\ e^{-i\frac{\pi}{4}}\Bigg(\bigg[\bigg[\bigg(\bigg(\bigg[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\cdot e^{-i\pi\left(\frac{u}{\alpha}\right)^2}\bigg)\bigg)\bigg]\bigg|_{u\to\alpha^2\xi} * e^{+i\pi\alpha^2\xi^2}\Bigg) \\
&= |\alpha|\ e^{-i\frac{\pi}{4}}\Bigg(\bigg[\bigg[\bigg(\bigg(\bigg[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\cdot e^{-i\pi\left(\frac{u}{\alpha}\right)^2}\bigg)\bigg)\bigg]\bigg|_{u\to\alpha^2\xi} * e^{+i\pi\frac{\left(\alpha^2\xi\right)^2}{\alpha^2}}\Bigg) \qquad (15)
\end{aligned}
$$

thus leaving another scaled convolution that also may be rewritten:

$$
\begin{aligned}
F[\xi] &= |\alpha|\ e^{-i\frac{\pi}{4}}\Bigg(\frac{1}{\alpha^2}\bigg[\bigg[\bigg(\bigg[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\cdot e^{-i\pi\left(\frac{u}{\alpha}\right)^2}\bigg) * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\bigg]\bigg|_{u\to\alpha^2\xi}\Bigg) \\
&= \frac{1}{|\alpha|}e^{-i\frac{\pi}{4}}\Bigg(\bigg[\bigg[\bigg(\bigg[f[u] * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\cdot e^{-i\pi\left(\frac{u}{\alpha}\right)^2}\bigg) * e^{+i\pi\left(\frac{u}{\alpha}\right)^2}\bigg]\bigg]\bigg|_{u\to\alpha^2\xi}\Bigg) \qquad (16)
\end{aligned}
$$

We can use any desired symbol for the dummy space-domain variable of integration u. The obvious choice is x, which produces an expression for the Fourier transform of $f[x]$ as a cascade of three operations with appropriate chirp functions: a convolution with an "upchirp", multiplication by a "downchirp", and convolution with an "upchirp".

$$
F[\xi] = \frac{1}{|\alpha|}e^{-i\frac{\pi}{4}}\Bigg(\bigg[\bigg[\bigg(\bigg[f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\bigg]\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\bigg) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\bigg]\bigg]\bigg|_{x\to\alpha^2\xi}\Bigg) \qquad (17)
$$

This sequence of operations for this second route to the Fourier transform via quadratic-phase operations is:

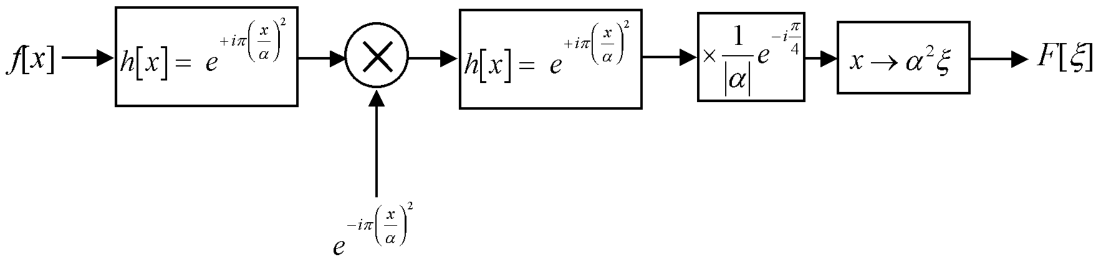


Figure 4: Schematic of 1-D *C-M-C* chirp Fourier trnasformer.

1. filter the input with the upchirp impulse response $e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$,
2. multiply the filtered signal by a downchirp $e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$,
3. filter the result with the upchirp $e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$,
4. multiply the result by the constant factor $|\alpha|^{-1}\ e^{-i\frac{\pi}{4}}$ and redefine the coordinates by replacing x with $\alpha^2\xi$.

The obvious shorthand name for this sequence of operations is the "*C-M-C*" chirp Fourier transform", and will prove very useful for describing the action of imaging systems in monochromatic (coherent) light.

As before, we may skip the coordinate remapping to produce an operation that is wholly in the space domain:

$$\boxed{F\left[\frac{x}{\alpha^2}\right] = \frac{1}{|\alpha|} e^{-i\frac{\pi}{4}} \left[\left(\left[f\left[x\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right]} \tag{18}$$

Note that all multiplications are by upchirps and all convolution filters have downchirp impulse responses in both the *M-C-M* and *C-M-C* algorithms. If appropriate filters can be constructed, either algorithm may be implemented with three appropriate filters; the multiplicative upchirps may be generated by applying impulses to filters with upchirp responses. The appropriate multiplication upchirps may be generated by applying an impulse to the appropriate filter. Block diagrams and illustrative example are shown in the figures.

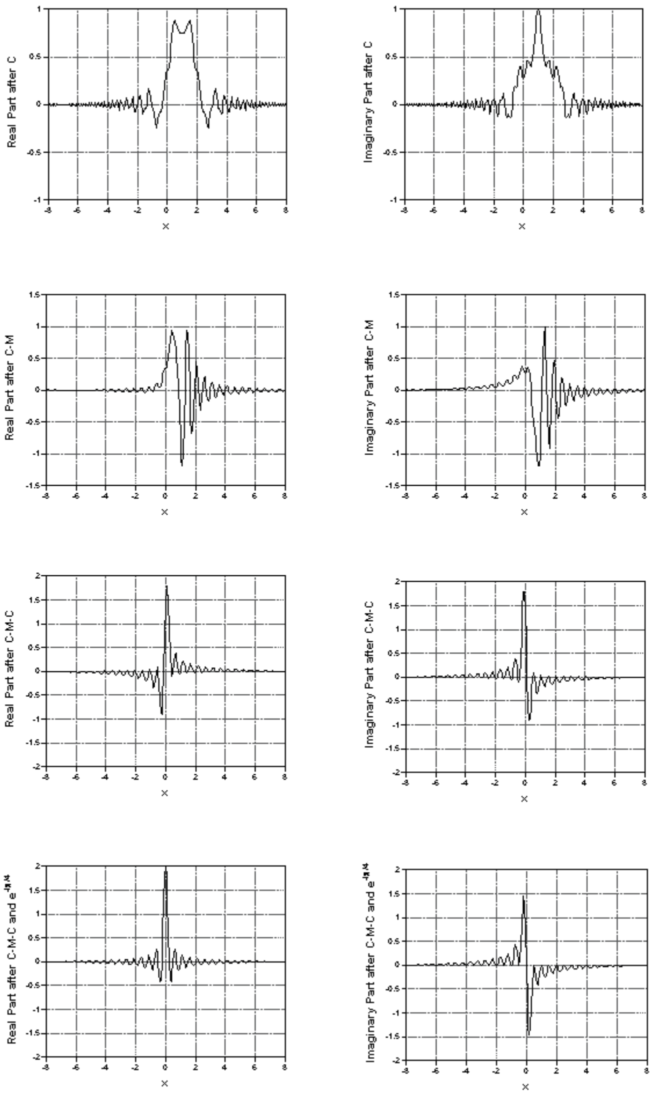
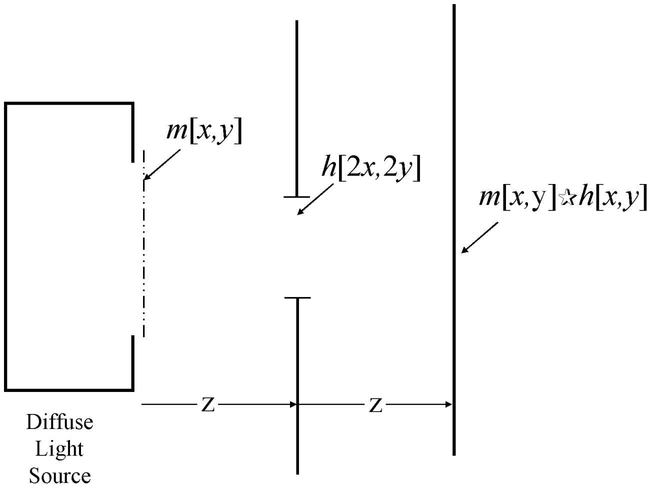


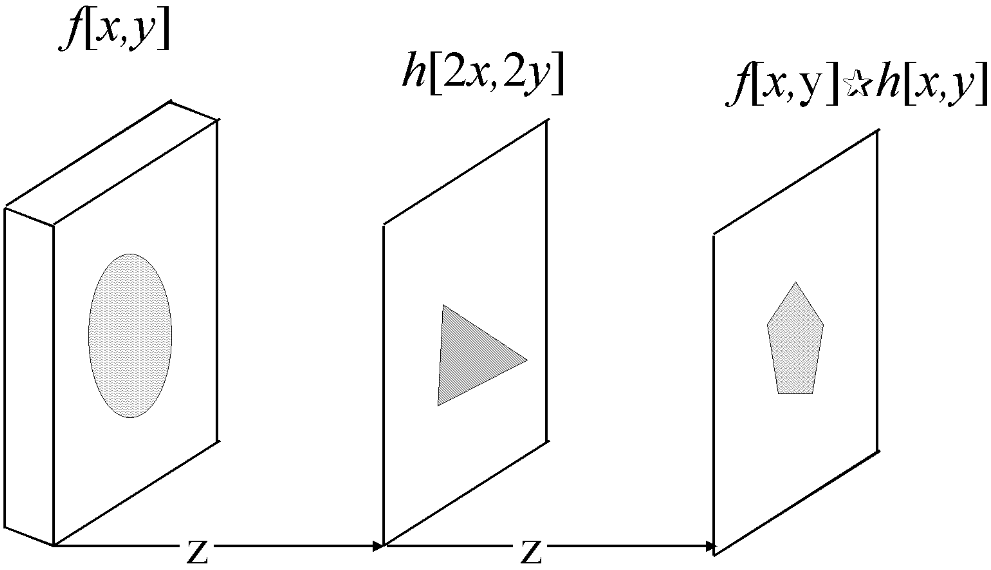
Figure 5: Steps in *C*-*M*-*C* chirp Fourier transformer

1.3 Chirp Fourier Transform System Based on Ray Optics

The 2-D optical "correlator" system that you investigated in the fall quarter is the basis for a 2-D chirp Fourier transformer. The light source of the optical system is a "light box" that is assumed to emit light "equally" in all directions, so that the intensity of the emitted light is assumed to be a uniform two-dimensional function proportional to $1[x,y]$. The light is modulated by a 2-D transparency with transmittance $m[x,y]$ (which may be bitonal or gray scale). Since we assume that the light box emits rays uniformly in all directions, the "pattern" of light after the transparency $m[x,y]$ is proportional to $m[x,y]$ .The light then propagates in straight lines until it encounters a second transparency with transmittance $h[x,y]$ located at $z = z$. Since the light from the box as assumed to travel in all directions and in straight lines (as "rays"), each point on the second transparency "sees" light from all points in $m[x,y]$; the rays from the various points on m travel along straight-line paths at different angles. This is the source of the "translation" operation. The rays continue to an observation plane located a z_2 from the second transparency. Each location at the output plane "sees" the intensity at a coordinate in the distant transparency (m) through a point in the near transparency (h), and hence the intensity in the distant transparency is multiplied by the transmittance at a single position coordinate in the nearer transparency. Different locations of the detector will "see" a shifted $m[x,y]$ relative to $h[x,y]$ due to parallax.The light arriving at that single output location is the summation (integral) of the point-by-point products of the intensity of the distant (input-plane) aperture and the transmittance of the midplane aperture. The coordinates of the function f at the midplane must be minified to ensure that it has the same apparent scale as m when seen by a sensor in the output plane. If the two distances z_1 and z_2 are equal, then $h[x,y]$ must be half as large as $m[x,y]$, i.e., the scaled midplane function is $h[2x,2y]$. Such a system is shown in the figures as "side" and "perspective" views:



"Side" view of the optical correlator, showing "input plane" where transparency $m[x,y]$ is placed and "impulse response" $h[x,y]$. The output is a scaled replica of $m[x,y] \bigstar h[x,y]$.



"Perspective" view of the optical correlator.

The intensity of the light at the output plane is the crosscorrelation rather than the convolution because neither function has been "reversed". Since the impulse response $h[x,y]$ is a parameter of the "system", "reverse" the first transparency by "rotating" it about its center by 180° to produce $h[-x,-y]$. The light pattern at the output plane is measured either by direct exposure on a camera sensor (without a lens) or by imaging the pattern cast on an optical diffuser (a ground glass or even a simple piece of plain white paper) with a camera and lens. If you use direct exposure, then the transparencies will have to be small and the propagation distances will have to be short to ensure that the pattern "fits" on the sensor.

1.4 Experiments with Ray Optics Fourier transformer:

1.4.1 Materials and Equipment:

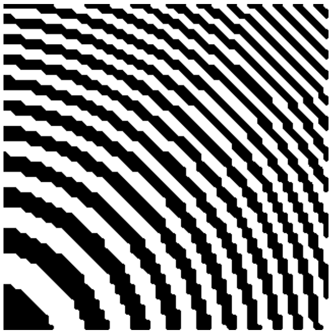
- Light Box
- Transparencies of "chirped gratings" (Fresnel zone plates) of two sizes, one approximately half the size of the other
- Cardboard to mount gratings
- Ground glass (tissue or plain white paper may be used)
- Method to record patterns (photographically or by hand drawing)

1.4.2 Experiments:

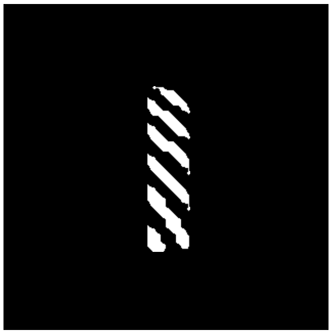
1. To implement the chirp Fourier transform algorithm in this system, we use two replicas of bitonal functions of the form

$$h[x,y] = STEP\left[\cos\left[\pi\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{\alpha^2}}\right]\right] \tag{19}$$

where $[x_0,y_0]$ is the center of the pattern and α is the "chirp rate". These are versions of the "Fresnel zone plate". One copy (full-sized) is located at the light box and a the second smaller replica is at the "midplane" z_1. If these two transparencies are used in the system, the observed pattern located at $z=2z_1$ will be the crosscorrelation of these two patterns and will approximate a 2-D Dirac delta function. This is the 2-D Fourier transform of the unit constant and serves as a baseline for the other transforms. Observe and record the results.



"Off-axis Fresnel Zone Plate", which is a thresholded off-axis circularly symmetric chirp function



Fresnel zone plate multiplied by rectangle function; this would be placed at the "input" plane of the system (labeled $h[x,y]$ in previous figure).

2. To evaluate the Fourier transform of a 2-D object pattern, we need to multiply the first zone plate by the object transparency. Construct some SIMPLE objects, e.g., a rectangle, pairs of rectangles, multiple rectangles at equal spacings (approximation of $RECT\left[\frac{x}{b}\right] * COMB[x]$), one or more circles, etc. Place them over the first zone plate, observe the resulting pattern and record it. You will have implemented an approximation of the following operation:

$$g[x,y] = (f[x,y] \cdot h[x,y]) \bigstar h[2x,2y]$$

   where $h[x,y]$ is defined above.

3. You might want to construct a computer model of this system in IDL^{TM} or other software and compare computed and observed results. One advantage of the computer model is that you can digitally increase the contrast of the observed pattern to make the various features easier to see.

4. Note that the third chirp (the last M) is missing from this calculation. Explain how this absence affects the result.

1.5 Chirp Fourier Transform System Based on Wave Optics

We can observe that the "points of constant phase" of light emitted by a point source have a spherical shape. If allowed to propagate a "sufficient distance" (to the Fresnel diffraction region), the spherical wave may be modeled as a parabolic wave. In other words, light propagation from a point source generates a two-dimensional chirp function. If the observation plane is located at a large distance z_1 from an aperture ("object") whose size in the plane at z_0 is constrained so that $z_1 >> \sqrt{x_0^2 + y_0^2}$, then the distance R between any point on the object and any observation point is approximately z_1. Therefore the reciprocal distance R^{-1} is may be approximated by z_1^{-1}. In other words, the amplitude of the impulse response does not vary the observation plane located at the coordinate z_1.

The large values of the temporal frequency of optical radiation and the rapid velocity of propagation mean that small variations in the distance R between source and observation points will produce significant variations in the optical phase. This means that the variation in R across the observation plane must be considered in the phase computation; it is not sufficient to approximate the factor of R in the phase by a numerical constant. In this calculation, the distance may be written as a power series:

$$
\begin{aligned}
\left|\underline{\mathbf{R}}_1 - \underline{\mathbf{R}}_0\right| &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + z_1^2} = \sqrt{z_1^2 \left(1 + \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2}\right)} \\
&= z_1 \sqrt{1 + \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2}} \\
&= z_1 \left(1 + \frac{1}{2}\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2} - \frac{1}{8}\frac{\left((x_1 - x_0)^2 + (y_1 - y_0)^2\right)^2}{z_1^4} + \cdots\right) \qquad (20)
\end{aligned}
$$

If z_1 is sufficiently large, terms of second and larger order may be assumed to be sufficiently close to zero that they may be ignored, leaving the approximation:

$$
R \simeq z_1 \left(1 + \frac{1}{2}\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2}\right) = z_1 + \frac{1}{2}\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{2z_1} \qquad (21)
$$

Because the spherical wavelet in the resulting integral is a function of the difference of the input and output coordinates ($x_1 - x_0$ and $y_1 - y_0$), the process is a convolution.with a quadratic-phase factor:

$$
\begin{aligned}
\underline{\mathbf{E}}\left[x_1, y_1; z_1\right] &\simeq \mathcal{E}_0 \; \frac{1}{i\lambda\left|z_1\right|} \; e^{+2\pi i \frac{z_1}{\lambda}} \; \iint_{-\infty}^{+\infty} a\left[x_0, y_0\right] \; e^{+i\pi \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{\lambda z_1}} \; dx_0 \; dy_0 \\
&= \mathcal{E}_0 \; \frac{1}{i\lambda\left|z_1\right|} \; e^{+2\pi i \frac{z_1}{\lambda}} \; \left(a\left[x, y\right] * e^{+i\pi \frac{x^2 + y^2}{\lambda z_1}} \bigg|_{x \to x_1, y \to y_1} \right) \qquad (22)
\end{aligned}
$$

This is the *Fresnel approximation* of the electric-field amplitude at the observation plane due to the source distribution $a\left[x_0, y_0\right]$ located at the origin. In this case, the spherical-wave impulse response in this convolution integral is approximated by a waveform that has constant magnitude and a *parabolic* shape; it is the *quadratic-phase factor* or *chirp.* The convolution of the source distribution and the quadratic-phase impulse response is sometimes called the *Fresnel transform.*

In words, we clearly can compute the Fresnel diffraction pattern from an arbitrary source distribution at the observation plane specified by z_1 by calculating the appropriate quadratic-phase impulse response $h\left[x, y; z_1\right]$ and convolving with the source distribution a $\left[x_0, y_0\right]$. The observed intensity is the time average of the squared magnitude of the convolution. For small values of z_1 (near the threshold distance for Fresnel diffraction), the image will "resemble" the shape of the source distribution but with oscillating intensity near edges due to the chirp factor. As z_1 is increased, the chirp rate of the quadratic-phase factor and the spatial frequency of the oscillations will decrease.

Measured at a distance z_1 down the optic axis, the impulse response is a quadratic-phase factor with chirp rate $\alpha = \sqrt{\frac{1}{\lambda z_1}}$, which decreases with increasing z_1. In other words, the "chirp rate" of the quadratic-phase factor is:

$$h\left[x; z_1\right] = \frac{1}{i\lambda z_1} e^{+\frac{2\pi i}{z_1}}\ e^{+i\pi\frac{x^2}{\lambda z_1}} = \frac{1}{\lambda z_1} e^{+i\left(\frac{2\pi}{z_1} - \frac{\pi}{2}\right)}\ e^{+i\pi\frac{x^2}{\lambda z_1}} \tag{23}$$

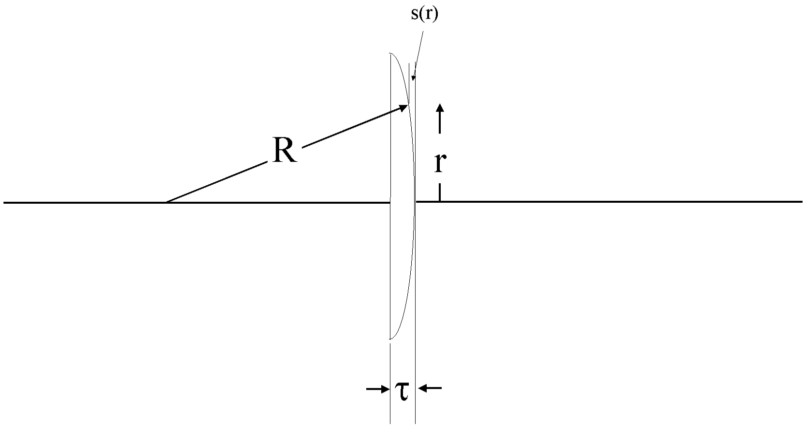
Since the amplitude evaluated at z_1 is the convolution of the input amplitude with an impulse response, then the action of the system also may be expressed in the frequency domain as a transfer function. In this case, we have:

$$\begin{aligned} H\left[\xi, \eta; z_1\right] &= \mathcal{F}_2\left\{\frac{1}{i\lambda\left|z_1\right|}\ e^{+2\pi i\frac{z_1}{\lambda}} e^{+i\pi\left(\frac{x^2+y^2}{\lambda z_1}\right)}\right\} \\ &= \frac{1}{i\lambda\left|z_1\right|}\ e^{+2\pi i\frac{z_1}{\lambda}} \mathcal{F}_2\left\{e^{+i\pi\left(\frac{x^2+y^2}{\lambda z_1}\right)}\right\} \\ &= \frac{1}{i\lambda\left|z_1\right|}\ e^{+2\pi i\frac{z_1}{\lambda}} \left(\sqrt{\lambda z_1} e^{+i\frac{\pi}{4}}\right)^2 e^{-i\pi\lambda z_1\left(\xi^2+\eta^2\right)} = e^{+2\pi i\frac{z_1}{\lambda}}\ e^{-i\pi\lambda z_1\left(\xi^2+\eta^2\right)} \end{aligned} \tag{24}$$

This form demonstrates that the propagation of light in the Fresnel approximation acts as an allpass quadratic-phase filter with an additional constant phase from propagation down the optical axis. The allpass character of the filter makes intuitive sense because this is a passive system that adds no energy to the signal. In other words, the integrated energy in the signal is conserved by propagation in the Fresnel diffraction region.

1.6 Action of Lens

Optical systems typically are used to form images of the source distribution by redirecting the radiation from point sources to point images. For example, consider a slab of glass with maximum thickness τ and with a convex spherical face with radius of curvature R (i.e., a plano-convex lens). The thickness function of the glass is found by using the Pythagorean formula in the following picture:



Schematic of plano-convex lens, showing center of curvature of surface with radius R and the "sag" $s(r)$, where r is the radial distance from the optical axis.

$$\left(R - s\left[r\right]\right)^2 + r^2 = R^2 \rightarrow r^2 - 2R\ s\left[r\right] + s^2\left[r\right] = 0 \tag{25}$$

If both R and r are much larger than $s\left[r\right]$, then factors of size $\left(s\left[r\right]\right)^2$ may be ignored. The result is a simple formula, the *sag formula* for a spherical lens:

$$s\left[r\right] \simeq \frac{r^2}{2R} \tag{26}$$

In words, the sag formula approximates the spherical surface by a paraboloid. This is exactly analogous to the approximation for a spherical wavefront as a paraboloid in the Fresnel diffraction region.

The approximate thickness of glass as a function of radial distance from the optical axis is the difference between the maximum thickness and the parabolic sag:

$$\tau[r] = \tau - s[r] = \tau - \frac{r^2}{2R} \tag{27}$$

which measures the thickness of glass as a function of radial distance r from the optical axis. The phase delay as a function of radial distance from the optic axis is the sum of the phase delay through the glass and the phase delay through air in the sag region. The delay due to the glass is the number of wavelengths multiplied by the refractive index

$$\begin{aligned}\phi[r] &= \left(\frac{2\pi}{\left(\frac{\lambda}{n}\right)} \times \textit{distance traveled in glass}\right) + \left(\frac{2\pi}{\lambda} \times \textit{distance traveled in vacuum}\right) \\ &= n\frac{2\pi}{\lambda}\left(\tau - \frac{r^2}{2R}\right) + \frac{2\pi}{\lambda}\left(\frac{r^2}{2R}\right) = n\frac{2\pi}{\lambda}\tau - n\frac{\pi r^2}{\lambda R} + \frac{\pi r^2}{\lambda R} \\ &= n\frac{2\pi}{\lambda}\tau - \frac{\pi r^2}{\lambda R}(n-1)\end{aligned} \tag{28}$$

The first term is the fixed phase delay of light traveling the fixed distance τ through glass (which applies only to "rays" through the center of the lens), while the second term decreases the phase angle away from the optical axis due to the more rapid velocity in the vacuum of the sag region. Since the sag is a function of the radial distance r from the optical axis, so clearly is the phase delay.

The electric field exiting the lens as a function of radial position from the optical axis is the product of the incident field and the phase delays:

$$\underline{\mathbf{E}}[r,t] = \underline{\mathbf{E_0}}[r]\;\; e^{+2\pi i\frac{n\ell_0}{\lambda}}\;\; e^{-i\pi(n-1)\frac{r^2}{\lambda R}} \tag{29}$$

i.e., it is the sum of constant-phase and quadratic-phase signals.

In the common case of a lens has two positive spherical surfaces with radii R_1 and $-R_2$, the phase transmittance of the lens has two contributions due to the two sags:

$$\begin{aligned}t[r] &= e^{+2\pi i\frac{n\ell_0}{\lambda}}\;\; e^{-i\pi(n-1)\frac{r^2}{\lambda R_1}}\;\; e^{-i\pi(n-1)\frac{r^2}{\lambda(-R_2)}} \\ &= e^{+2\pi i\frac{n\ell_0}{\lambda}}\;\; e^{-i\pi(n-1)\frac{r^2}{\lambda}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}\end{aligned} \tag{30}$$

The expression may be simplified by defining a factor f that satisfies the condition:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{31}$$

This may be a familiar equation; it is the *lens-maker's formula* that relates the *focal length* f of the lens to the index of refraction n and the radii of curvature R_1 and R_2 of the two surfaces. The effect of the lens is a change in the phase of the transmitted light by the optical transmission $t[r]$:

$$t[r] = e^{+2\pi i\frac{n\ell_0}{\lambda}}\;\; e^{-i\pi\frac{r^2}{\lambda f}}$$

Note that if f is negative, the optical phase of the lens is:

$$t(r) = e^{+2\pi i\frac{n\ell_0}{\lambda}}\;\; e^{-i\pi\frac{r^2}{\lambda|f|}} \tag{32}$$

Note that this expression is a pure phase function. Lenses have finite diameters that may be introduced into the optical transmission as a real-valued function $p(r)$ (or $p[x,y]$) that defines the *pupil* of the optical system. The pupil function will often be a binary function, but we can (and will) use pupil functions that vary slowly and continuously over the radial distance r. The resulting transmittance function of the general spherical lens is:

$$t\left(r\right)=p\left(r\right)\ e^{+2\pi i\frac{n\ell_0}{\lambda}}\ e^{-i\pi\frac{r^2}{\lambda f}} \tag{33}$$

The negative sign in the chirp exponent indicates that the lens has positive power.

A lens changes the radius of curvature of the incident wave by adding or subtracting a phase factor that is a quadratic function of the distance from the optical axis.

1.7 Optical Chirp Fourier Transformer

We have derived the image $g\left[x\right]$ at the observation plane due to a coherent source distribution $f\left[x\right]$ located at the origin and generated by a lens of focal length f located at $z=z_1$. Implicit in the derivation was the assumption that light emitted from all points in the source had the same phase. The output is the convolution of the input with the impulse response $h\left[x;z_1,f,z_2\right]$:

$$\begin{aligned} g\left[x,y\right] &= f\left[x,y\right]*h\left[x,y;z_1,f,z_2\right] \\ &= f\left[x,y\right]*\left(\left[\frac{1}{i\lambda z_1}\ e^{+2\pi i\frac{z_1}{\lambda}}\ e^{+i\pi\frac{x^2+y^2}{\lambda z_1}}\right]\ p\left[x,y\right]\ e^{-i\pi\frac{x^2+y^2}{\lambda f}}\ *\ \frac{1}{i\lambda z_2}e^{+i\pi\frac{x^2+y^2}{\lambda z_2}}\right) \end{aligned} \tag{34}$$

The system is *usually* shift invariant. However, there are distances z_1 and z_2 for which the derivation is invalid and the system is shift variant. If the input is located in the front focal plane of the lens ($z_1=f$), then the product of the leading phase terms is unity and the imaging equation is satisfied only for $z_2=\infty$ with magnification$\frac{z_2}{z_1}=\infty$.

$$\begin{aligned} h\left[x,y;z_1=f,f,z_2\right] &= \left(\left[\frac{1}{i\lambda f}e^{+i\pi\frac{x^2}{\lambda f}}\right]\ p\left[x,y\right]\ e^{-i\pi\frac{x^2+y2}{\lambda f}}\ *\ \frac{1}{i\lambda z_2}\ e^{+i\pi\frac{x^2+y^2}{\lambda z_2}}\right) \\ &= \left(\frac{1}{-\lambda^2 f z_2}\right)\ \left(p\left[x,y\right]*e^{+i\pi\frac{x^2+y^2}{\lambda z_2}}\right). \end{aligned} \tag{35}$$

If $p\left[x,y\right]=1\left[x,y\right]$, i.e., the lens has infinite diameter, the impulse response becomes:

$$h\left[x,y;f,f,z_2\right]=\left(\frac{1}{-\lambda^2 f z_2}\right)\ \left(1\left[x,y\right]*e^{+i\pi\frac{x^2+y^2}{\lambda z_2}}\right)$$

The impulse response is the convolution of two functions with unit magnitude and infinite support. The convolution integral is:

$$1\left[x\right]*e^{+i\pi\frac{x^2}{\lambda z_2}}=\int_{-\infty}^{+\infty}e^{+i\pi\frac{u^2}{\lambda z_2}}\ 1\left[x-u\right]\ du=\int_{-\infty}^{+\infty}e^{+i\pi\frac{u^2}{\lambda z_2}}\ du=\sqrt{\lambda z_2}\ e^{+\frac{i\pi}{4}}$$

$$1\left[x,y\right]*e^{+i\pi\frac{x^2+y^2}{\lambda z_2}}=\int_{-\infty}^{+\infty}e^{+i\pi\frac{u^2+v^2}{\lambda z_2}}\ 1\left[x-u,y-v\right]\ du\ dv=\int_{-\infty}^{+\infty}e^{+i\pi\frac{u^2+v^2}{\lambda z_2}}\ du=i\lambda z_2$$

where the central-ordinate theorem was used to evaluate the integral. This is easy to understand by recognizing that the lens transforms a spherical wave emitted by a source in the front focal plane to a plane wave. The "tilt" (i.e., phase) of the plane wave depends on the source position. Because the impulse response varies with source location, *this imaging system is not space invariant.* If $p\left[x\right]$ has compact support, the impulse response is a plane wave of limited extent, but the impulse response is still space variant. Under these conditions, we cannot define an overall system impulse response $h\left[x;z_1,f,z_2\right]$, but rather we must sequentially convolve, multiply, and convolve with quadratic phase factors.

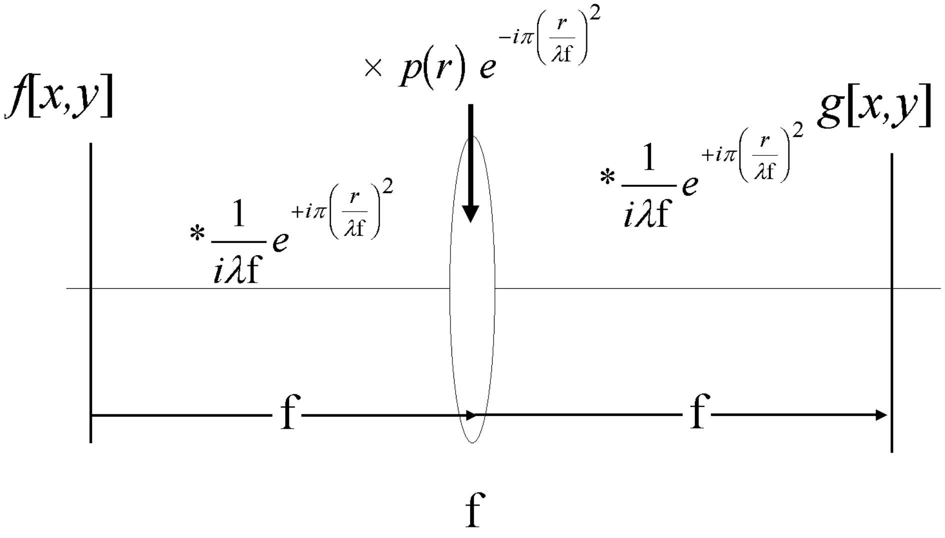
In similar fashion, consider the coherent image of a uniphase source observed at the back focal plane of the lens, i.e., $z_2=f$. If the object-to-lens distance is infinite, the distribution at z_2 will be an image with zero magnification. Thus the image position will be fixed regardless of object location; this system is also shift variant.

The combination of the two cases has $z_1 = z_2 = f$; no geometric image is formed. From our discussion of Fraunhofer diffraction, it is perhaps already clear what this system is doing; the "image" of a point source located on axis is a plane wave with wavefronts perpendicular to the axis. In other words, the output wave has constant magnitude and phase over any plane orthogonal to the optical axis. The lens has "moved" the region where spherical wavefronts may be considered to be planar from ∞ to any point to the right of the lens. The distribution of light due to an on-axis point source at any location behind the lens is a constant magnitude and phase. As the source point moves off axis, the magnitude remains constant and the phase becomes a linear function of x. This system is calculating the (space-variant) Fourier transform of the source distribution via an algorithm with three steps:

1. convolve $f[x]$ with $e^{+i\pi\frac{x^2}{\lambda f}}$ by propagating a distance f,
2. multiply by $e^{-i\pi\frac{x^2}{\lambda f}}$ with a positive lens of focal length f, and
3. convolve with $e^{+i\pi\frac{x^2}{\lambda f}}$ by propagating through f.

The "image" in the back focal plane of the lens will be a scaled copy of the Fourier transform with $\xi = \frac{x}{\lambda f}$. We have seen this sequence of operations before – this is *chirp Fourier transform*. A similar algorithm may be derived with these steps:

If a coherent object distribution (all object points in phase) is "imaged" with a lens, the system is space variant if the object is in the front focal plane and/or if the output is observed in the back focal plane. The "image" distribution is then a scaled Fourier transform of the object.



Schematic of optical *C-M-C* chirp Fourier transformer: the lens with focal length f and pupil function $p(r)$ multiplies the first convolution by a quadratic phase with a negative sign.

1.8 Experiments with 2-D Optical Fourier Transformer:

1.8.1 Materials and Equipment:

- Low power He:Ne Laser (DO NOT look directly at the transmitted beam!)
- Beam expander (negative lens or microscope objective in optical "spatial filter")
- Positive lens to use as collimator or to focus the expanded beam
- Aluminum foil, needles, and single-edged razor blades, used to make input objects (pinhole, multiple pinholes, slits, etc.)
- Film transparencies, including Fresnel zone plate, to serve as input objects

1.8.2 Experiments:

1. Construct an optical Fourier transformer with the laser, beam expander, and lens provided. Two configurations are possible.

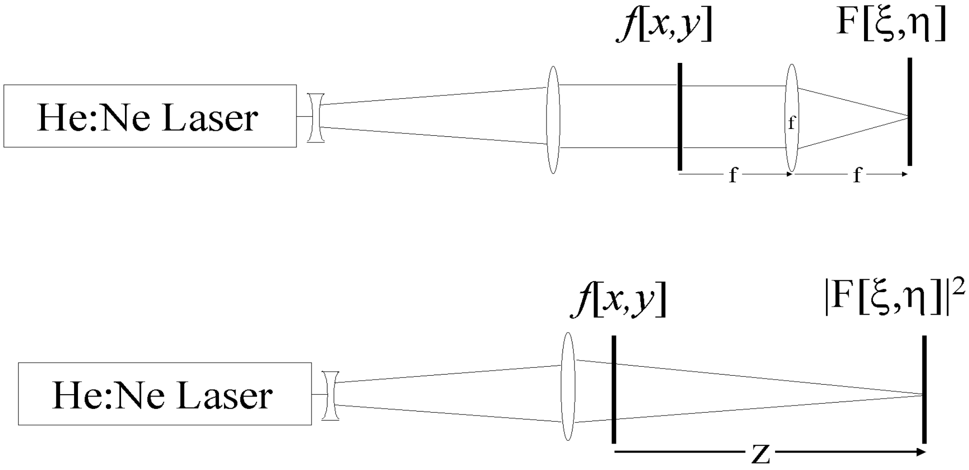
   (a) The first setup uses a lens with negative power to expand the He:Ne laser beam by creating a diverging spherical wave (phase proportional to $e^{+i\pi\frac{r^2}{\lambda \mathrm{f}}}$). A second positive lens is then used to "collimate" the expanding beam, so that the wavefronts are planar (beam is "focused" at $z = +\infty$, so that there is no phase variation with the radial coordinate r). The object transparency $f[x, y]$ is placed anywhere in this beam. The plane wave interacts with $f[x, y]$ and the light is "diffracted". The propagation of light over the distance z in the Fresnel diffraction region is (approximately) modeled by convolution of the original pattern with the impulse response $h(r) = e^{+i\pi\frac{r^2}{\lambda z}}$, where r is the radial distance from the optical axis (center of symmetry). This impulse response is another diverging spherical wave. A second positive lens, with focal length **f** is placed at a distance **f** from the transparency; this multiplies the incident optical field by a "converging" spherical wave $p(r)\, e^{-i\pi\frac{r^2}{\lambda \mathrm{f}}}$, where $p(r)$ is the "pupil function" of the lens. The light propagates another increment of distance **f**, thus convolving with a second diverging spherical wave, to produce a "space-domain" pattern that has the same functional form as the Fourier transform of $f[x, y]$ as $F\left(\frac{r}{\lambda \mathrm{f}}\right)$. Since the eye and other optical detectors actually measure the squared magnitude of the complex amplitude, we actually see $\left|F\left(\frac{r}{\lambda \mathrm{f}}\right)\right|^2$, so the phase of the Fourier transform is not "visible". If the lenses are "perfect", this system exac/tly implements the "chirp Fourier transform" and thus both the magnitude and phase of the Fourier transform are correctly evaluated.

   Note that a second lens with focal length **f** could be placed at a distance **f** "beyond" the Fourier-transform plane and the result observed at yet another distance **f** beyond that lens. This second "system" evaluates the Fourier transform of the first Fourier transform, i.e., it implments the "transform-of-a-transform" theorem to evaluate:

   $$\mathcal{F}\left\{F\left[\frac{x}{\lambda \mathrm{f}}, \frac{y}{\lambda \mathrm{f}}\right]\right\} \propto f[-x, -y]$$

   The first transform could be "modified" by inserting multiplicative "masks" to filter the signal. For example, an aperture centered at the Fourier plane would truncate the "high-frequency" terms of the transmitted Fourier transform to implement a lowpass filter.

   (b) The second setup is based on Fraunhofer diffraction, and thus works in a different way. Though is easier to implement, it only correctly evaluates the squared magnitude of the Fourier transform and is not as easily used to filter the input. That said, this is the setup that I suggest you use. The first positive lens is placed to focus the laser light at some distance z from the lens. Because the collimation lens may be considered as focusing light at $z = +\infty$, this system effectively places $z = +\infty$ at the finite distance. In other words, this has moved the Fraunhofer diffraction region to a short distance away. If a transparency $f[x,y]$ is placed in the converging beam, the pattern formed at the focal spot is proportional to the squared magnitude of the Fourier transform $\left|F\left[\frac{x}{\lambda z},\frac{y}{\lambda z}\right]\right|^2$



Optical Fourier transformers: (1) Setup using collimating lens (to "focus" beam at ∞) and lens of focal length **f** to produce Fourier transform with correct optical phase (note that the pattern observed by eye is actually $|F[\xi,\eta]|^2$, where $\xi=\frac{x}{\lambda \mathrm{f}},\eta=\frac{y}{\lambda \mathrm{f}}$). This apparatus may be used to perform optical filtering by adding a second lens with focal length **f** to compute a second Fourier transform. (2) Simpler setup based on "Fraunhofer diffraction" and using a single lens. The pattern observed at the focus of the lens ("image" of the laser point source) is proportional to the squared magnitude of the Fourier transform of $f[x,y]$, where $\xi=\frac{x}{\lambda z},\eta=\frac{y}{\lambda z}$.

2. Construct some targets by pricking holes of various sizes in aluminum foil. I find that a good technique is to use a finger on the back side of the foil as an anvil and to rotate the needle or straight pin while pricking the foil. This gives a nice round hole and the hole can be enlarged by increasing the pressure on the needle. Also try arrays of holes (e.g. pairs of the same size at different separations, a pair of holes with larger diameter at different spacings, etc.). I also suggest trying to make single slits of different widths and pairs of slits.

3. Evaluate the Fourier transforms of the aluminum-foil objects and compare the observed results to what you would expect from the shape of the function and theorems of the Fourier transform.

4. Evaluate the Fourier transforms of several of the available transparencies, including square apertures, gratings, etc.

5. The sets of slides include a halftoned transparency of a picture of Albert Einstein. This is a bitonal image (only black and white) created from black dots at regular spacings. The shades of gray are approximated by varying the diameters of the black dots. In short, this object includes a *COMB* function. Evaluate the Fourier transform of this image and describe the features you see.

6. Construct "sandwiches" of transparences (e.g., a rectangle and a grating) to demonstrate the modulation theorem in this system.

7. Construct "sandwiches" of a bitonal transparency (e.g., a rectangle) and a Fresnel zone plate. Evaluate the Fourier transform and explain the results – the method of stationary phase might be helpful in this explanation.