# SIMG-717-20052 Solution Set #6

1. The following signals are applied independently to LSI systems:

$$s_1[x] = e^{-x} \cdot STEP[x]$$
$$s_2[x] = RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])$$

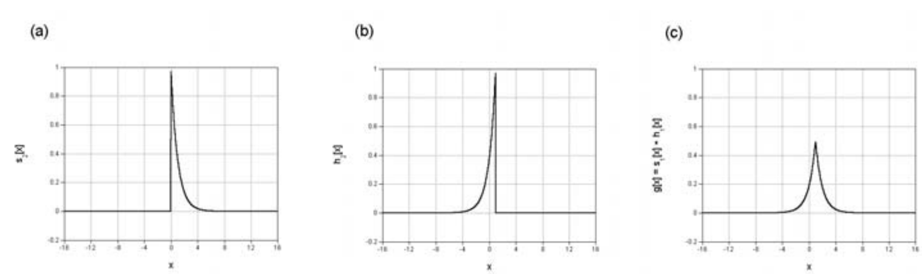
For both cases:

(a) Describe the impulse response $h[x]$ and transfer function $H[\xi]$ of the matched filter that will maximize the output at $x = 2$. (assume that $H[0] = 1$). Sketch the output.

$$a1 : \textit{maximum output at } x = 2$$
$$\Longrightarrow \textit{find a function that is a max at } x = 0 \textit{ and then translate via } \delta[x-2]$$
$$s_1[x] \bigstar s_1[x] = s_1[x] * s_1^*[-x] \ \textit{ is hermitian with maximum at origin}$$
$$h[x] = s_1^*[-x] * \delta[x-2] = \left(e^{-(-x)} \cdot STEP[-x]\right)^* * \delta[x-2]$$
$$= e^{x} \cdot STEP[-x] * \delta[x-2]$$
$$= e^{x-2} \cdot STEP[-(x-2)] = \boxed{e^{x-2} \cdot STEP[-x+2] = h[x]}$$

$$H[\xi] = \mathcal{F}_1\left\{e^{x-2} \cdot STEP[-x+2]\right\} = \left(\left(\mathcal{F}_1\left\{e^{-x} \cdot STEP[x]\right\}\right)\Big|_{\xi \to -\xi}\right) \cdot \mathcal{F}_1\{\delta[x-2]\}$$
$$= \frac{1}{1 + 2\pi i(-\xi)} \cdot \exp[-2\pi i \cdot 2 \cdot \xi] = \boxed{\frac{\exp[-4\pi i \xi]}{1 - 2\pi i \xi} = H[\xi]}$$

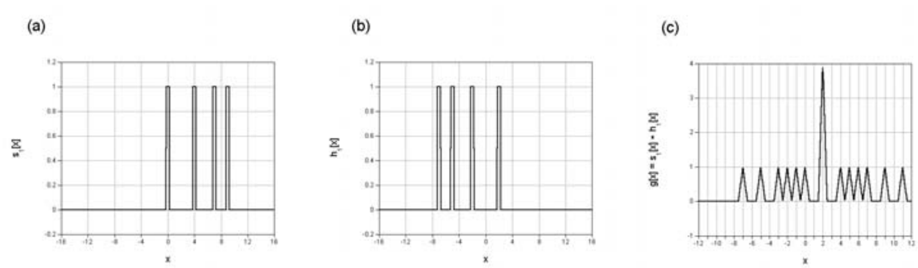
$$(s_1[x] \bigstar s_1[x]) * \delta[x-2] = \mathcal{F}_1^{-1}\left\{|S_1[\xi]|^2\right\} * \delta[x-2]$$
$$= \mathcal{F}_1^{-1}\left\{\left|\frac{1}{1 + 2\pi i \xi}\right|^2\right\} * \delta[x-2]$$
$$= \mathcal{F}_1^{-1}\left\{\frac{1}{1 + (2\pi\xi)^2}\right\} * \delta[x-2] = \frac{1}{2} e^{-|x-2|}$$



$$\begin{aligned} a2 \quad &: \quad s_2[x] = RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\ (s_2[-x])^* &= RECT[2x] * (\delta[x] + \delta[x+4] + \delta[x+7] + \delta[x+9]) \end{aligned}$$

$$\boxed{h_2[x] = (s_2[-x])^* * \delta[x-2] = RECT[2x] + (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7])}$$

$$\begin{aligned} s_2[x] * h_2[x] &= (RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])) \\ &* RECT[2x] + (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7]) \\ &= (RECT[2x] * RECT[2x]) * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\ &* (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7]) \\ &= \frac{1}{2} TRI[2x] \\ &* \delta[x+7] + \delta[x+5] + \delta[x+3] + \delta[x+2] + \delta[x+1] + \delta[x] + 4 \cdot \delta[x-2] \\ &+ \delta[x-4] + \delta[x-5] + \delta[x-6] + \delta[x-7] + \delta[x-9] + \delta[x-11] \end{aligned}$$




(b) Is it possible to construct a transfer function $H[\xi]$ that, when applied to $s[x]$, will produce $g[x] = \delta[x-2]$? Explain your reasoning.

*since the first function has no zeros in its spectrum (except at $\xi = \pm\infty$), an "inverse" matched filter exists, whereas the second function has zeros in its spectrum due to the rectangle function, and thus the "inverse matched" filter may not be constructed.*

2. The transfer functions listed below describe the action of different LSI systems. The goal of this problem is to find the corresponding "inverse filter" in both domains, i.e.,

$$\begin{aligned} W[\xi] &= (H[\xi])^{-1} \\ w[x] &= \mathcal{F}^{-1}\{W[\xi]\} \end{aligned}$$

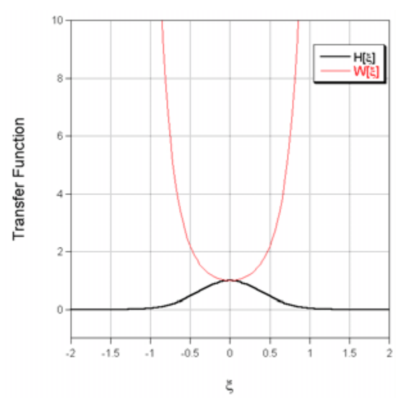
In the situations where the inverse filter does not exist, we will instead evalute the *"pseudoinverse"* filter

$$\begin{aligned} \hat{W}[\xi] &= \left\{ \begin{array}{ccc} (H[\xi])^{-1} & for & H[\xi] \neq 0 \\ 0 & for & H[\xi] = 0 \end{array} \right\} \\ \hat{w}[x] &= \mathcal{F}^{-1}\left\{\hat{W}[\xi]\right\} \end{aligned}$$

In each case, determine which of the inverse filters is appropriate and sketch its transfer function $W_n[\xi]$ or $\hat{W}_n[\xi]$. Classify the action of the appropriate filter as lowpass, highpass, etc. ALSO in those cases where it is possible, evaluate and sketch the appropriate impulse response ($w_n[x]$ or $\hat{w}_n[x]$) of the appropriate inverse filter. You may use reasonable approximations where appropriate – the sketches may be helpful here.

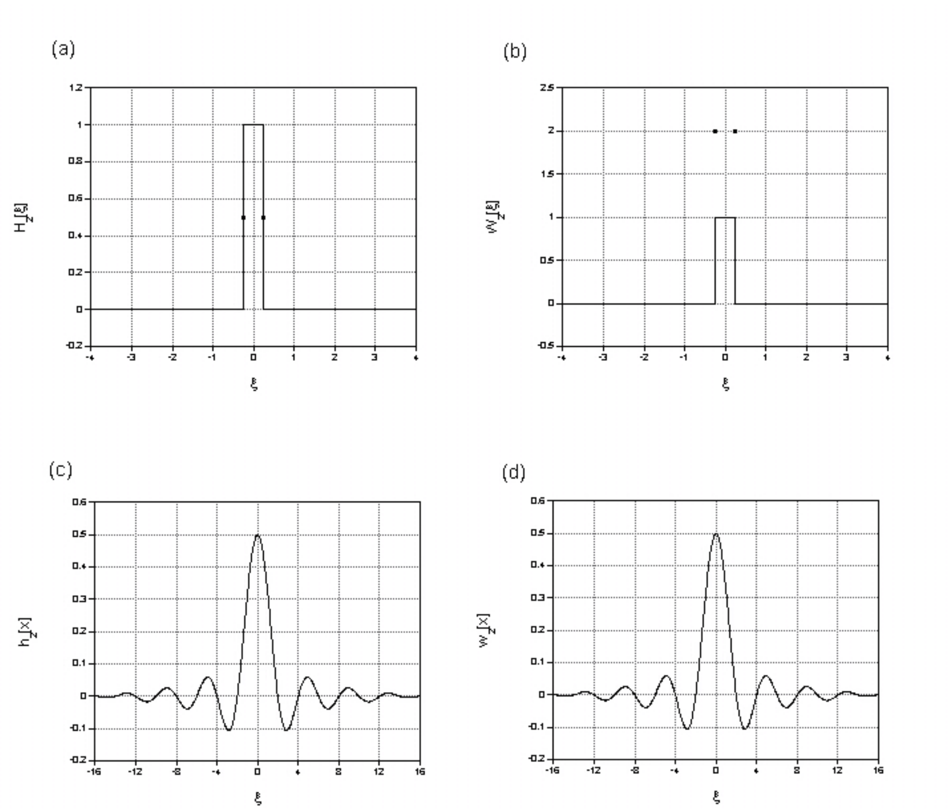
(a) $H_1[\xi] = GAUS[\xi]$

$$\begin{aligned} h_1[x] &= GAUS[x]\,,\ \textit{a lowpass filter (averager)} \\ W_1[\xi] &= (GAUS[\xi])^{-1} = \left(e^{-\pi\xi^2}\right)^{-1} = \exp\left[+\pi\xi^2\right] \\ w_1[x] &= \mathcal{F}_1^{-1}\left\{\exp\left[+\pi\xi^2\right]\right\}\ \textit{for which we have no analytic expression,} \\ & \quad \textit{but it clearly is a highboost filter} \end{aligned}$$



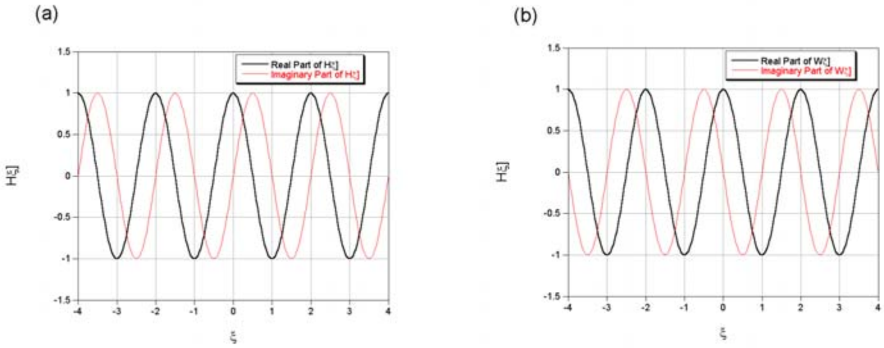
(b) $H_2[\xi] = RECT[2\xi]$

$$\begin{aligned}
H_2[x] &= RECT[2\xi]\,, \textit{ the ideal lowpass filter} \\
\hat{W}_2[\xi] &= (RECT[2\xi])^{-1} = \begin{cases} 1 & if \quad |\xi| < \frac{1}{4} \\ 0 & if \quad |\xi| > \frac{1}{4} \\ 2 & if \quad \xi = \pm\frac{1}{4} \end{cases} = RECT[2\xi] \\
&\qquad \textit{the isolated values at the edges have no area and thus no effect on an integral} \\
\hat{w}_2[x] &= \mathcal{F}_1^{-1}\{RECT[2\xi]\} = \frac{1}{2} SINC\left[\frac{x}{2}\right]
\end{aligned}$$



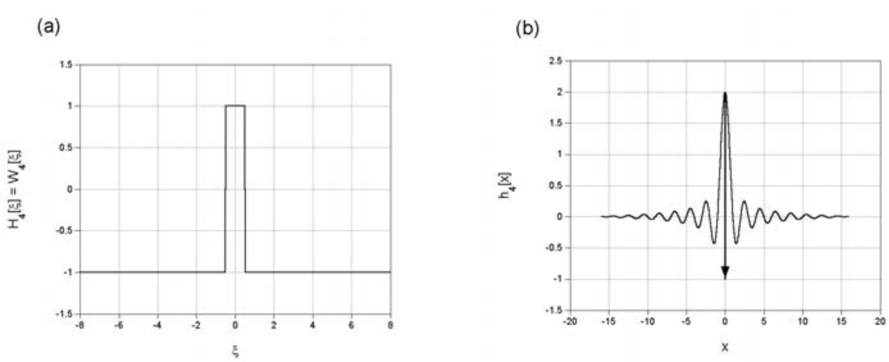

(c) $H_3[\xi] = e^{+i\pi\xi}$

$$\begin{aligned}
H_3[\xi] &= e^{+i\pi\xi} \textit{ (an allpass linear-phase filter)} \\
H_3[\xi] &= e^{+2\pi i\xi\cdot\frac{1}{2}} \implies h_3[x] = \delta\left[x + \frac{1}{2}\right] \\
W_3[\xi] &= \left(e^{+i\pi\xi}\right)^{-1} \implies w_3[x] = \delta\left[x - \frac{1}{2}\right]
\end{aligned}$$




(d) $H_4[\xi] = e^{+i\pi(1-RECT[\xi])}$

$$
\begin{aligned}
H_4[\xi] &= \begin{cases} e^{+i\pi} = -1 & if \quad RECT[\xi] = 0 \implies |\xi| > \frac{1}{2} \\ e^{0} = +1 & if \quad RECT[\xi] = +1 \implies |\xi| < \frac{1}{2} \\ e^{+i\frac{\pi}{2}} = +i & if \quad RECT[\xi] = +\frac{1}{2} \implies \xi = \pm\frac{1}{2} \end{cases} \\
&= (2 \cdot RECT[\xi] - 1[\xi]) + \left(i \cdot \frac{1}{2} \text{ } if \text{ } \xi = \pm\frac{1}{2}\right) \\
&\quad \textit{The isolated imaginary values have no impact on an integral} \\
&\quad \textit{This is an allpass filter with “nonlinear” phase.} \\
h_4[x] &= \mathcal{F}_1^{-1}\{2 \cdot RECT[\xi] - 1[\xi]\} = 2 \cdot SINC[x] - \delta[x] \\
W_4[\xi] &= (H_4[\xi])^{-1} = \begin{cases} \frac{1}{-1} = -1 & if \quad RECT[\xi] = 0 \implies |\xi| > \frac{1}{2} \\ \frac{1}{+1} = +1 & if \quad RECT[\xi] = +1 \implies |\xi| < \frac{1}{2} \\ \frac{1}{+i} = -i & if \quad RECT[\xi] = +\frac{1}{2} \implies \xi = \pm\frac{1}{2} \end{cases} \\
w_4[x] &= \mathcal{F}_1^{-1}\{2 \cdot RECT[\xi] - 1[\xi]\} = 2 \cdot SINC[x] - \delta[x] = h_4[x]
\end{aligned}
$$



3. Design the Wiener or Wiener-Helstrom filter (whichever is appropriate) for the following input signals, impulse responses, and noise power spectra.

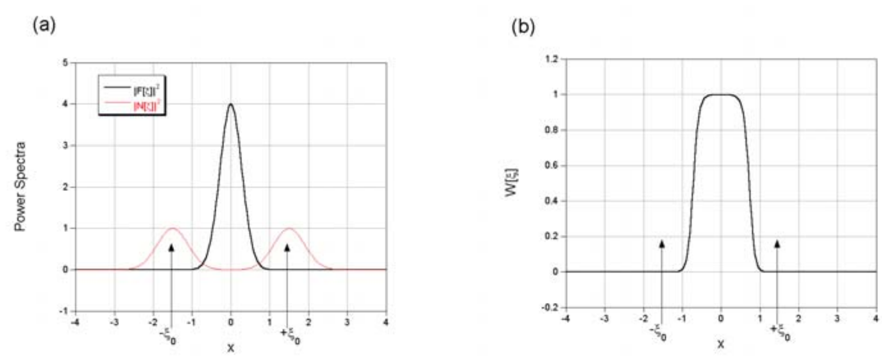
   (a) $f[x] = 2 \cdot GAUS[x]$, $h[x] = \delta[x]$, $|N[\xi]|^2 = GAUS[\xi + \xi_0] + GAUS[\xi - \xi_0]$

$$\begin{aligned} h[x] &= \delta[x] \implies \textit{Wiener Filter} \\ W[\xi] &= \frac{1}{1 + \frac{|N[\xi]|^2}{|F[\xi]|^2}} \end{aligned}$$

*I plotted an example with* $\xi_0 = 1$

$$\begin{aligned} F[\xi] &= 2 \cdot GAUS[\xi] = 2 \cdot \exp\left[-\pi\xi^2\right] \\ |F[\xi]|^2 &= 4 \cdot \left(\exp\left[-\pi\xi^2\right]\right)^2 = 4 \cdot \exp\left[-\pi\left(\sqrt{2}\cdot\xi\right)^2\right] \\ |N[\xi]|^2 &= GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0] = \exp\left[-\pi(\xi-\xi_0)^2\right] + \exp\left[-\pi(\xi+\xi_0)^2\right] \\ &= \exp\left[-\pi\xi^2\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\exp\left[+2\pi\xi\xi_0\right] + \exp\left[-\pi\xi^2\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\exp\left[-2\pi\xi\xi_0\right] \\ &= \exp\left[-\pi\xi^2\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\left(\exp\left[+2\pi\xi\xi_0\right] + \exp\left[-2\pi\xi\xi_0\right]\right) \end{aligned}$$

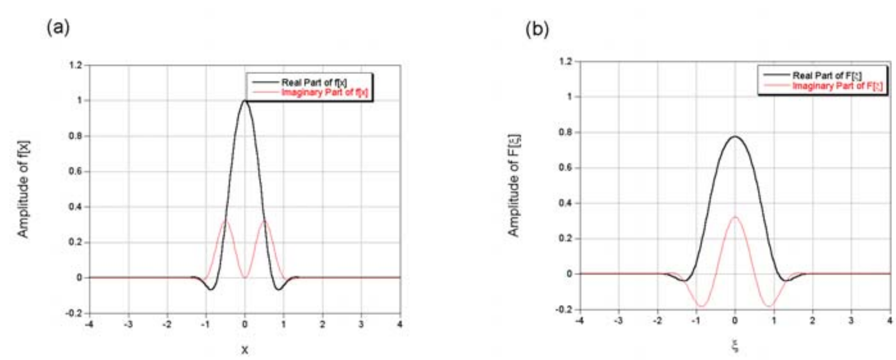
$$\begin{aligned} \frac{|N[\xi]|^2}{|F[\xi]|^2} &= \frac{\exp\left[-\pi(\xi-\xi_0)^2\right] + \exp\left[-\pi(\xi+\xi_0)^2\right]}{4\cdot\exp\left[-\pi\left(\sqrt{2}\cdot\xi\right)^2\right]} \\ &= \frac{1}{4}\cdot\left(\exp\left[-\pi\xi^2\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\left(\exp\left[+2\pi\xi\xi_0\right]+\exp\left[-2\pi\xi\xi_0\right]\right)\right)\cdot\exp\left[+4\pi\xi^2\right] \\ &= \frac{1}{4}\cdot\exp\left[+3\pi\xi^2\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\left(\exp\left[+2\pi\xi\xi_0\right]+\exp\left[-2\pi\xi\xi_0\right]\right) \end{aligned}$$



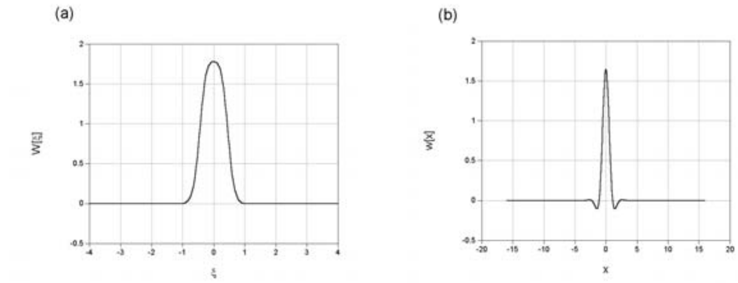
(b) $f[x] = GAUS\left[\frac{x}{b}\right] \cdot \exp\left[+i\pi x^2\right]$, $h[x] = RECT[x]$, $|N[\xi]|^2 = GAUS[\xi + \xi_0] + GAUS[\xi - \xi_0]$

$$\begin{aligned}
h[x] &= RECT[x] \implies \textit{Wiener-Helstrom Filter} \\
W[\xi] &= \frac{H^*[\xi]}{|H[\xi]|^2 + \frac{|N[\xi]|^2}{|F[\xi]|^2}} \\
\textit{I plotted an example with } b &= 1 \textit{ and } \xi_0 = 1
\end{aligned}$$

$$\begin{aligned}
H[\xi] &= SINC[\xi] \implies H^*[\xi] = SINC[\xi] \implies |H[\xi]|^2 = SINC^2[\xi] \\
F[\xi] &= |b| \cdot GAUS[b\xi] * \left(\exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\xi^2\right]\right) \\
&= |b| \cdot \exp\left[-\pi (b\xi)^2\right] * \left(\exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\xi^2\right]\right) \\
&\quad \textit{which is complicated, but which approximates to} \\
F[\xi] &\cong |b| \cdot \exp\left[-\pi (b\xi)^2\right] \cdot \exp\left[+i\pi\xi^2\right] \\
|F[\xi]|^2 &= |b|^2 \cdot \exp\left[-2\pi (b\xi)^2\right] = |b|^2 \cdot \exp\left[-\pi\left(\left(\sqrt{2}\right) b\xi\right)^2\right] \\
&= |b|^2 \, GAUS\left[\sqrt{2} b\xi\right]
\end{aligned}$$




$$\begin{aligned}
W[\xi] &= \frac{H^*[\xi]}{|H[\xi]|^2 + \frac{|N[\xi]|^2}{|F[\xi]|^2}} = \frac{SINC[\xi]}{SINC^2[\xi] + \frac{GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0]}{|b|^2 GAUS\left[\sqrt{2} b\xi\right]}} \\
\textit{I plotted an example with } \xi_0 &= 1 \textit{ and } b = 1
\end{aligned}$$

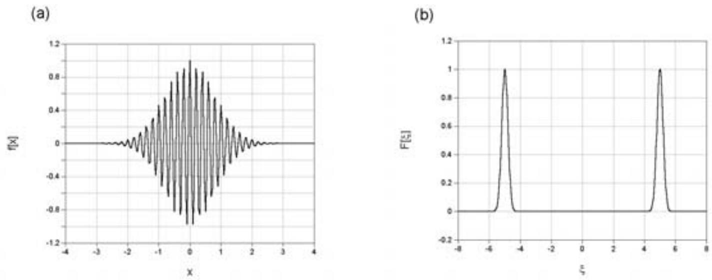


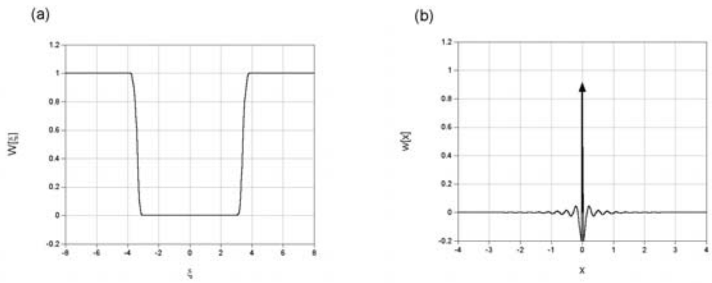

(c) $f[x] = GAUS\left[\frac{x}{2}\right] \cdot \cos[10\pi x]$, $h[x] = \delta[x]$, $|N[\xi]|^2 = GAUS[\xi]$

$$\begin{aligned} h[x] &= \delta[x] \implies \textit{Wiener Filter} \\ W[\xi] &= \frac{1}{1 + \frac{|N[\xi]|^2}{|F[\xi]|^2}} \end{aligned}$$

$$\begin{aligned} F[\xi] &= 2 \cdot GAUS[2\xi] * \frac{1}{2}(\delta[\xi+5] + \delta[\xi-5]) = GAUS[2(\xi+5)] + GAUS[2(\xi-5)] \\ &= GAUS\left[\frac{\xi+5}{\left(\frac{1}{2}\right)}\right] + GAUS\left[\frac{\xi-5}{\left(\frac{1}{2}\right)}\right] = \exp\left[-\pi\left(\frac{\xi+5}{\left(\frac{1}{2}\right)}\right)^2\right] + \exp\left[-\pi\left(\frac{\xi-5}{\left(\frac{1}{2}\right)}\right)^2\right] \end{aligned}$$

$$\begin{aligned} |F[\xi]|^2 &= \left|\exp\left[-\pi\left(\frac{\xi+5}{\left(\frac{1}{2}\right)}\right)^2\right] + \exp\left[-\pi\left(\frac{\xi-5}{\left(\frac{1}{2}\right)}\right)^2\right]\right|^2 \\ &= \exp\left[-2\pi\left(\frac{\xi+5}{\left(\frac{1}{2}\right)}\right)^2\right] + \exp\left[-2\pi\left(\frac{\xi-5}{\left(\frac{1}{2}\right)}\right)^2\right] \\ &\quad +2\exp\left[-\pi\left(\frac{\xi+5}{\left(\frac{1}{2}\right)}\right)^2\right] \cdot \exp\left[-\pi\left(\frac{\xi-5}{\left(\frac{1}{2}\right)}\right)^2\right] \\ &\cong \exp\left[-\pi\left(\frac{\xi+5}{\left(\frac{1}{4}\right)}\right)^2\right] + \exp\left[-\pi\left(\frac{\xi-5}{\left(\frac{1}{4}\right)}\right)^2\right] \\ &= GAUS\left[\frac{\xi+5}{\left(\frac{1}{4}\right)}\right] + GAUS\left[\frac{\xi-5}{\left(\frac{1}{4}\right)}\right] \\ |N[\xi]|^2 &= GAUS[\xi] = \exp\left[-\pi\xi^2\right] \end{aligned}$$


(a)
f[x]
x
(b)
F[ξ]
ξ


(a)
W[ξ]
ξ
(b)
w[x]
x

4. Consider a system with the input functions $f_n[x]$ listed below. The functions are translated by an arbitrary and unknown distance $x_0$, so that the actual input function is $f_n[x - x_0]$. The goal of this problem is to construct the "matched filter" for these inputs, i.e., determine the impulse response $m_n[x]$ and/or transfer function $M_n[\xi]$ that will help determine $x_0$. In the ideal case, we can construct a filter such that:

$$f_n[x - x_0] * m_n[x] = \delta[x - x_0]$$

For each of the input functions listed below, determine the transfer functions $M_n[\xi]$ that produce the ideal input (if possible). ALSO, in those cases where it is possible, evaluate and sketch the impulse response $m_n[x]$ of the matched filter. Again, you may use reasonable approximations where appropriate.

*The transfer function of the ideal matched filter is:*

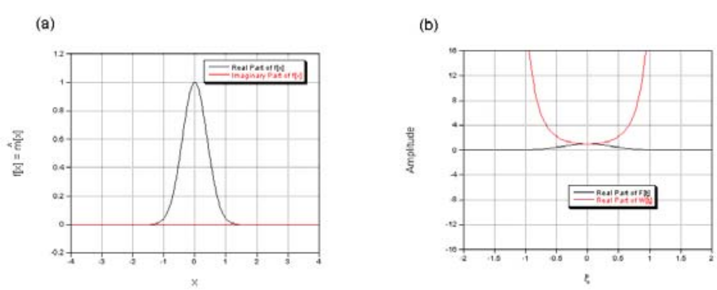
$$M[\xi] = \frac{1}{F[\xi]}$$

$$\Longrightarrow m[x] = \mathcal{F}_1^{-1}\left\{\frac{1}{F[\xi]}\right\}$$

$$= \mathcal{F}_1^{-1}\left\{\frac{F^*[\xi]}{|F[\xi]|^2}\right\} = f^*[-x] * \mathcal{F}_1^{-1}\left\{\frac{1}{|F[\xi]|^2}\right\}$$

(a) $f_1[x] = GAUS[x]$

$$\begin{aligned}
F[\xi] &= \exp\left[-\pi\xi^2\right] \\
M[\xi] &= \exp\left[+\pi\xi^2\right] \\
m[x] &= \mathcal{F}_1^{-1}\left\{\exp\left[+\pi\xi^2\right]\right\} \\
&\quad \textit{for which we have no analytic expression,} \\
&\quad \textit{but which is a differencing operator} \\
\hat{m}[x] &= f^*[-x] = f_1[x] = GAUS[x], \\
&\quad \textit{which will average out more of any noise,} \\
&\quad \textit{but will also produce a "wider" function}
\end{aligned}$$



(b) $f_2[x] = RECT[2x]$

$$\begin{aligned}
F[\xi] &= \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \\
M[\xi] &= \frac{2}{SINC\left[\frac{\xi}{2}\right]} \text{ where } SINC\left[\frac{\xi}{2}\right] \neq 0 \\
\hat{M}[\xi] &= M^*[\xi] = M[\xi] \implies \hat{m}[x] = f_2[x] = RECT[2x] \\
f_2[x] * \hat{m}[x] &= RECT[2x] * RECT[2x] = \frac{1}{2} TRI[2x]
\end{aligned}$$

(c) $f_3[x] = e^{+i\pi x}$

$$\begin{aligned}
F_3[\xi] &= \delta\left[\xi + \frac{1}{2}\right] \\
M_3[\xi] &= \frac{1}{F_3[\xi]} \text{ which doesn't exist} \\
\hat{M}_3[\xi] &= F_3^*[\xi] \implies \hat{m}_3[x] = f_3^*[-x] = \left(e^{+i\pi(-x)}\right)^* = e^{-i\pi(-x)} = e^{+i\pi x} = f_3[x] \\
f_3[x] * f_3^*[-x] &= e^{+i\pi x} * e^{+i\pi x} \text{ which is not well defined}
\end{aligned}$$

(d) $f_4[x] = e^{+i\pi(1-RECT[x])}$

$$\begin{aligned}
f_4[x] &= 2 \cdot RECT[x] - 1[x] \\
F_4[\xi] &= 2 \cdot SINC[\xi] - \delta[\xi] \\
M_4[\xi] &= \frac{1}{F_4[\xi]} = ? \\
\hat{m}_4[x] &= f_4^*[-x] = f_4[x] \\
& f_4[x] * \hat{m}_4[x] \text{ is not well defined; it is infinite everywhere.}
\end{aligned}$$