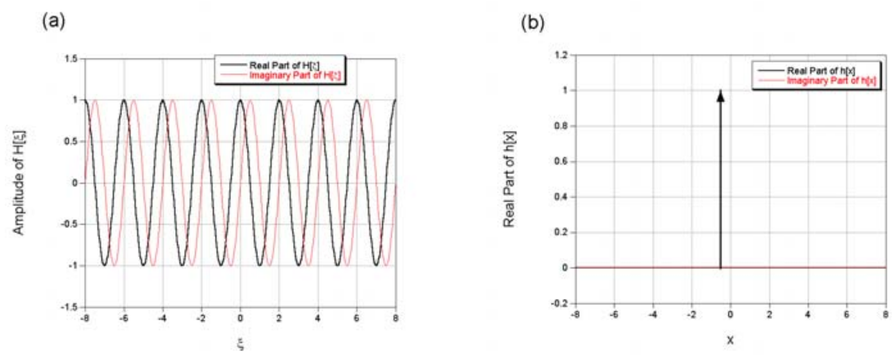
SIMG-717-20052 Solution Set #5

1. For each of the transfer functions, sketch $H_n[\xi]$ and evaluate and sketch the corresponding impulse response $h_n[x]$. Also classify the filters as lowpass, highpass, phase, etc.

   (a) $H_1[\xi] = e^{+i\pi\xi}$

$$h_1[x] = \mathcal{F}_1^{-1}\{\exp[+i\pi\xi]\} = \mathcal{F}_1^{-1}\left\{\exp\left[+2\pi i\frac{\xi}{2}\right]\right\} = \delta\left[x+\frac{1}{2}\right]$$

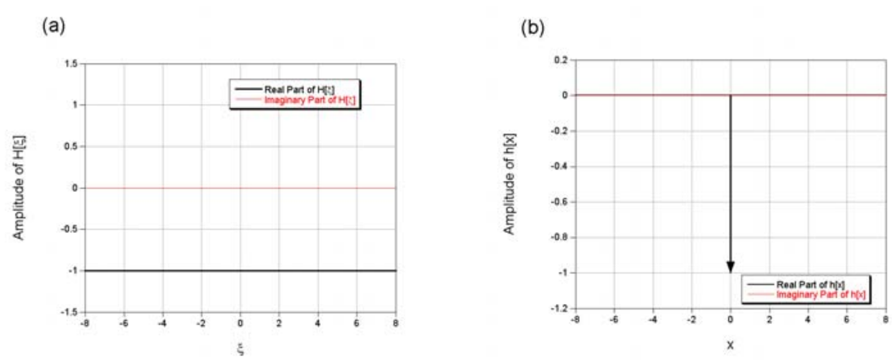
allpass (translates the function)



   (b) $H_2[\xi] = e^{+i\pi} = -1 + 0i$

$$h_2[x] = \mathcal{F}_1^{-1}\left\{e^{+i\pi}\right\} = \mathcal{F}_1^{-1}\left\{e^{+i\pi}\cdot 1[\xi]\right\} = e^{+i\pi}\cdot\delta[x] = -\delta[x]$$

allpass (inverts the function)

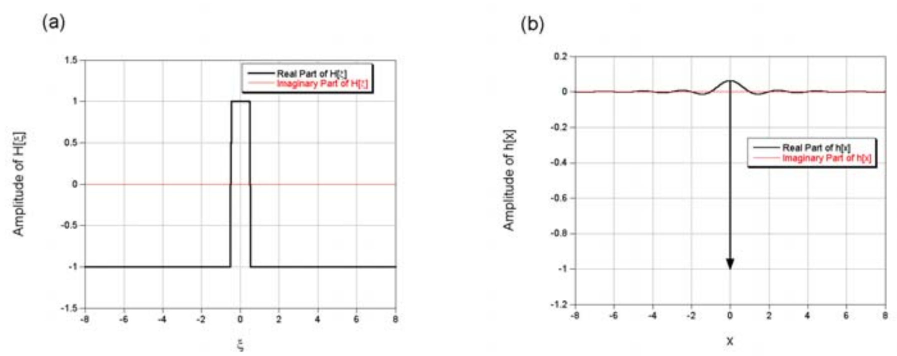


(c) $H_3[\xi] = e^{+i\pi(1-RECT[2\pi])}$

$$\begin{aligned}
h_1[x] &= \mathcal{F}_1^{-1}\left\{e^{+i\pi(1-RECT[2\pi])}\right\} \\
|2\xi| &< \frac{1}{2} \Longrightarrow |\xi| < \frac{1}{4} \Longrightarrow RECT[2\xi] = 1 \Longrightarrow H_3[\xi] = \exp[+i\pi(1-1)] = 1 \\
|2\xi| &> \frac{1}{2} \Longrightarrow |\xi| > \frac{1}{4} \Longrightarrow RECT[2\xi] = 0 \Longrightarrow H_3[\xi] = e^{+i\pi(1-0)} = -1 \\
|2\xi| &= \frac{1}{2} \Longrightarrow \xi = \pm\frac{1}{4} \Longrightarrow RECT[2\xi] = \frac{1}{2} \Longrightarrow H_3[\xi] = e^{+i\pi\left(1-\frac{1}{2}\right)} = +i
\end{aligned}$$

$$\begin{aligned}
H_3[\xi] &= 2 \cdot RECT[2\xi] - 1[\xi] + \left(\textit{two isolated points with amplitude } +i \textit{ at } \xi = \pm\frac{1}{4}\right) \\
h_3[x] &= \mathcal{F}_1^{-1}\{2 \cdot RECT[2\xi] - 1[\xi]\} = 2 \cdot \frac{1}{2} SINC\left[\frac{x}{2}\right] - \delta[x] = SINC\left[\frac{x}{2}\right] - \delta[x]
\end{aligned}$$

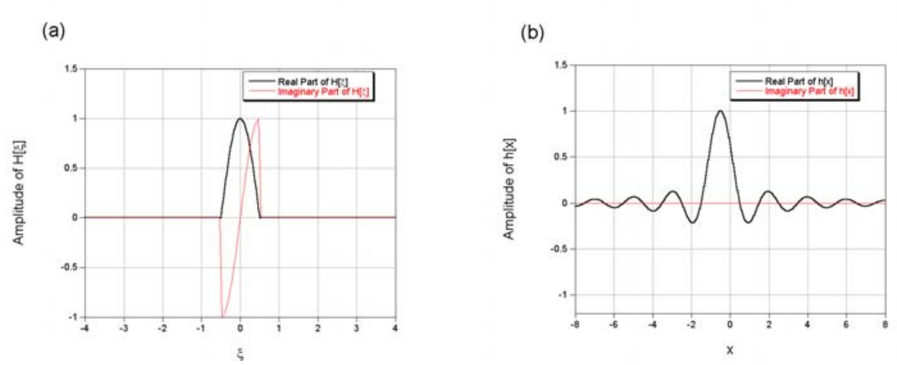
allpass (phase filter)



(d) $H_4[\xi] = RECT[\xi] \cdot H_1[\xi] = RECT[\xi] \cdot e^{+i\pi\xi}$

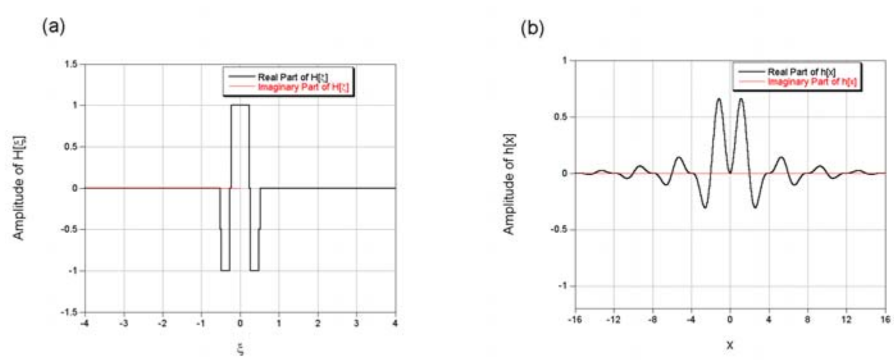
$$\begin{aligned}
H_4[\xi] &= RECT[\xi] \cdot e^{+i\pi\xi} = RECT[\xi] \cdot e^{+i\pi\xi} \\
h_4[x] &= SINC[x] * \delta\left[x + \frac{1}{2}\right] = SINC\left[x + \frac{1}{2}\right]
\end{aligned}$$

lowpass filter with linear phase



(e) $H_5[\xi] = RECT[\xi] \cdot H_3[\xi] = RECT[\xi] \cdot e^{+i\pi(1-RECT[2\xi])}$

$$\begin{aligned}
H_5[\xi] &= RECT[\xi] \cdot e^{+i\pi(1-RECT[2\xi])} = RECT[\xi] \cdot (2 \cdot RECT[2\xi] - 1[\xi]) \\
h_5[x] &= SINC[x] * \left(SINC\left[\frac{x}{2}\right] - \delta[x]\right) \\
&= SINC[x] * SINC\left[\frac{x}{2}\right] - SINC[x] * \delta[x] \\
&= \mathcal{F}_1^{-1}[RECT[\xi] \cdot 2RECT[2\xi]] - SINC[x] \\
&= SINC\left[\frac{x}{2}\right] - SINC[x] \\
&\quad \textit{lowpass filter}
\end{aligned}$$

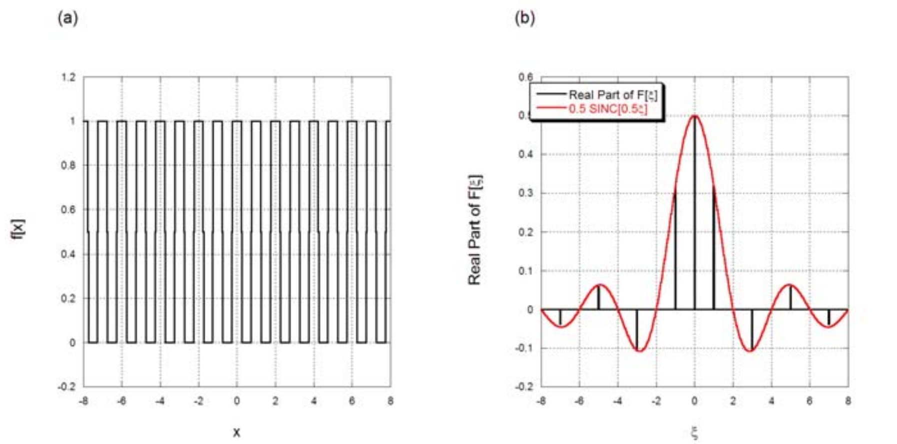


2. For the function:

$$f[x] = COMB[x] * RECT[2x]$$

(a) Sketch $F[\xi]$

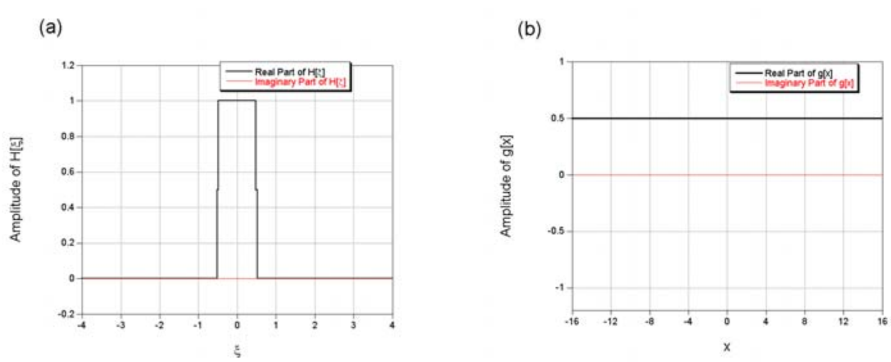
$$\begin{aligned} & \textit{50\% duty-cycle square wave} \\ F[\xi] &= COMB[\xi] \cdot \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \end{aligned}$$



For each of the transfer functions listed, sketch $H[\xi]$, the corresponding impulse response $h[x]$, and the output that results if $f[x]$ is applied to the input. Also classify the filters as lowpass, highpass, phase, etc.

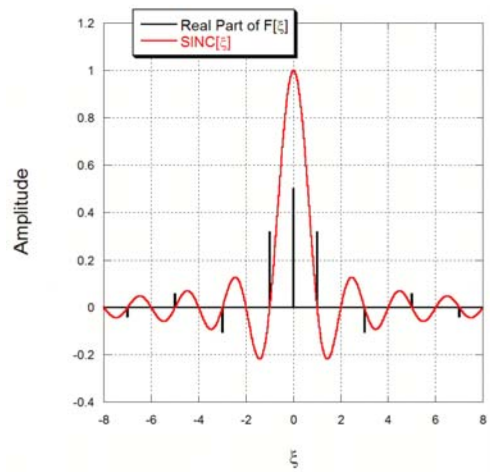
(b) $H[\xi] = RECT[\xi]$

$$G[\xi] = F[\xi] \cdot H[\xi] = \frac{1}{2}\delta[\xi] \implies g[x] = \frac{1}{2} \cdot 1[x]$$



(c) $h[x] = RECT[x]$

$$H[\xi] = SINC[\xi] \implies G[\xi] = COMB[\xi] \cdot \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \cdot SINC[\xi]$$

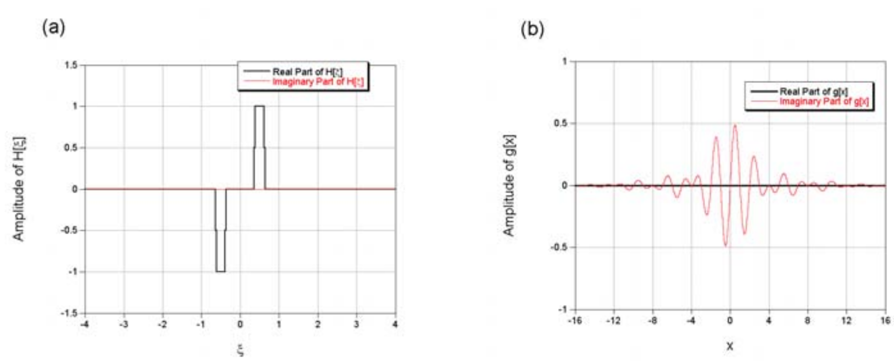


$$G[\xi] = \frac{1}{2} \cdot \delta[\xi] \implies g[x] = \frac{1}{2} \cdot 1[x] \quad \textit{(same output as \#2b)}$$

(d) $H[\xi] = -RECT[4\xi + 2] + RECT[4\xi - 2]$

$$\begin{aligned} H[\xi] &= -RECT\left[4\left(\xi + \frac{1}{2}\right)\right] + RECT\left[4\left(\xi - \frac{1}{2}\right)\right] \\ &= -RECT\left[\frac{\left(\xi + \frac{1}{2}\right)}{\left(\frac{1}{4}\right)}\right] + RECT\left[\frac{\left(\xi - \frac{1}{2}\right)}{\left(\frac{1}{4}\right)}\right] \\ &= RECT\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right] * \left(-\delta\left[\xi + \frac{1}{2}\right] + \delta\left[\xi - \frac{1}{2}\right]\right) \end{aligned}$$

$$\begin{aligned} h[x] &= \frac{1}{4} SINC\left[\frac{x}{4}\right] \cdot -2 \cdot \sin\left[2\pi \cdot \frac{1}{2} \cdot x\right] \\ &= -\frac{1}{2} SINC\left[\frac{x}{4}\right] \cdot \sin[\pi x] \end{aligned}$$



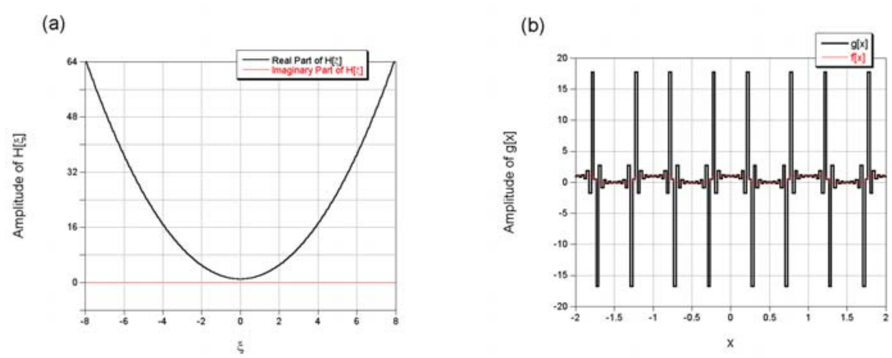
note that the rectangles in $H[\xi]$ do not enclose any of the Dirac delta functions in $F[\xi]$, which means that:

$$G[\xi] = F[\xi] \cdot H[\xi] = 0[\xi] \implies g[x] = 0[x]$$

(e) $H[\xi] = 1 + \xi^2$

$$\begin{aligned} h[x] &= \mathcal{F}_1^{-1}\left\{1 - \frac{1}{4\pi^2}(2\pi i\xi)^2\right\} \\ &= \mathcal{F}_1^{-1}\{1\} - \frac{1}{4\pi^2}\mathcal{F}_1^{-1}\left\{(2\pi i\xi)^2\right\} \\ &= \delta[x] - \frac{1}{4\pi^2}\delta''[x] \end{aligned}$$

The transfer function increases faster than the SINC function decays, so this transfer function will significantly amplify the amplitude at large spatial frequencies, and thus it "sharpens" the function. This gives significant "overshoots" at the edges, as shown in the illustration made using the DFT.



3. Show that the autocorrelation of the impulse response of the constant-phase and linear-phase allpass filters are on-axis Dirac delta functions.

$$\begin{aligned}
\textit{transfer function of constant-phase allpass filter is} &: \\
H[\xi] &= 1[\xi]\cdot\exp[+i\phi] \\
\textit{impulse response of constant-phase allpass filter is} &: \\
h[x] &= \delta[x]\cdot\exp[+i\phi]
\end{aligned}$$

$$\begin{aligned}
h[x]\bigstar h[x] &= h[x]*h^*[-x] = (\delta[x]\cdot\exp[+i\phi])*(\delta[x]\cdot\exp[+i\phi])^* \\
&= (\delta[x]\cdot\exp[+i\phi])*(\delta[x]\cdot\exp[-i\phi]) \\
&= (\delta[x]*\delta[x])\cdot\exp[+i\phi]\cdot\exp[-i\phi] \\
&= \delta[x]*\delta[x] = \boxed{\delta[x] = h[x]\bigstar h[x]}
\end{aligned}$$

$$\begin{aligned}
\textit{transfer function of linear-phase allpass filter is} &: \\
H[\xi] &= 1[\xi]\cdot\exp[+i2\pi\xi x_0] \\
\textit{impulse response of constant-phase allpass filter is} &: \\
h[x] &= \delta[x-x_0]
\end{aligned}$$

$$\begin{aligned}
h[x]\bigstar h[x] &= h[x]*h^*[-x] = (\delta[(+x)-x_0])*(\delta[(-x)-x_0])^* \\
&= \delta[x-x_0]*\delta[-x-x_0] \\
&= \delta[x-x_0]*\delta[-(x+x_0)] \\
&= \delta[x-x_0]*\delta[x+x_0] \\
&= \delta[x-(x_0-x_0)] = \boxed{\delta[x] = h[x]\bigstar h[x]}
\end{aligned}$$

4. Evaluate the *M-C-M* or *C-M-C* chirp Fourier transforms (whichever is more convenient)

$$\begin{aligned} F\left[\frac{x}{\alpha^2}\right] &= \left(\left[f[x]\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \\ &= \frac{1}{|\alpha|}\cdot e^{-i\frac{\pi}{4}}\cdot\left(\left\{\left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \end{aligned}$$

for the following functions:

(a) $f[x] = \delta[x]$
use the M-C-M

$$\begin{aligned} &\left(\left(\delta[x]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right]\right) * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \left([\delta[x]\cdot 1[x]] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \left(\delta[x] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \boxed{1[x]} \end{aligned}$$

(b) $f[x] = \delta[x - x_0]$
use the M-C-M

$$\begin{aligned} &\left(\delta[x-x_0]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \left(\left[\delta[x-x_0]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right]\right] * \exp\left[+-i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \exp\left[-i\pi\left(\frac{x_0}{\alpha}\right)^2\right]\left(\delta[x-x_0] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \exp\left[-i\pi\left(\frac{x_0}{\alpha}\right)^2\right]\left(\exp\left[+i\pi\left(\frac{x-x_0}{\alpha}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \exp\left[-i\pi\left(\frac{x_0}{\alpha}\right)^2\right]\left(e^{+i\pi\left(\frac{x}{\alpha}\right)^2}e^{+i\pi\left(\frac{x_0}{\alpha}\right)^2}e^{+2\pi i\frac{xx_0}{\alpha^2}}\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\ &= \left(e^{-i\pi\left(\frac{x_0}{\alpha}\right)^2}\cdot e^{+i\pi\left(\frac{x_0}{\alpha}\right)^2}\right)\cdot\left(e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{+2\pi i\frac{xx_0}{\alpha^2}} \\ &= \boxed{\exp\left[+2\pi i x_0\left(\frac{x_0}{\alpha^2}\right)\right]}\ \textit{a linear-phase exponential} \end{aligned}$$

(c) $f[x] = 1[x]$
use the M-C-M

$$\begin{aligned}
\left(\left[1[x] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} &= \left(e^{-i\pi\left(\frac{x}{\alpha}\right)^2} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \\
\left(e^{-i\pi\left(\frac{x}{\alpha}\right)^2} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) &= \mathcal{F}_1^{-1}\left\{\left(|\alpha|\, e^{-i\frac{\pi}{4}} e^{+i\pi\alpha^2\xi^2}\right) \cdot \left(|\alpha|\, e^{+i\frac{\pi}{4}} e^{-i\pi\alpha^2\xi^2}\right)\right\} \\
\mathcal{F}_1^{-1}\left\{|\alpha|^2\right\} &= |\alpha|^2 \cdot \delta[x]
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \left(e^{-i\pi\left(\frac{x}{\alpha}\right)^2} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = |\alpha|^2 \cdot \delta[x] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} \\
= \quad & |\alpha|^2 \cdot \left(\delta[x] \cdot e^{-i\pi\left(\frac{0}{\alpha}\right)^2}\right) = \boxed{|\alpha|^2 \cdot \delta[x]}
\end{aligned}$$

(d) $f[x] = \exp[+2\pi i\xi_0 x]$
use the M-C-M

$$F\left[\frac{x}{\alpha^2}\right] = \left(\left[f[x] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$$

$$\begin{aligned}
f[x] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} &= \exp[+2\pi i\xi_0 x] \cdot \exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] \\
&= \exp\left[-i\pi\left(\frac{x}{\alpha} - \alpha\xi_0\right)^2\right] \cdot \exp\left[+i\pi(\alpha\xi_0)^2\right]
\end{aligned}$$

Now do the convolution:

$$\begin{aligned}
& \left[f[x] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \\
= \; & \left(\exp\left[-i\pi\left(\frac{x}{\alpha} - \alpha\xi_0\right)^2\right] \cdot \exp\left[+i\pi(\alpha\xi_0)^2\right]\right) * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right] \\
= \; & \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \left(\exp\left[-i\pi\left(\frac{x}{\alpha} - \alpha\xi_0\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right) \\
= \; & \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \left(\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] * \delta\left[\frac{x}{\alpha} - \alpha\xi\right] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]\right) \\
= \; & \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \left(\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right] * \delta\left[\frac{x}{\alpha} - \alpha\xi\right]\right) \\
= \; & \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \left(|\alpha|^2\, \delta[x] * \delta\left[\frac{x}{\alpha} - \alpha\xi_0\right]\right) \\
= \; & |\alpha|^2 \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \delta\left[\frac{x}{\alpha} - \alpha\xi_0\right] \\
= \; & |\alpha|^2 \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \delta\left[\frac{x - \alpha^2\xi_0}{\alpha}\right] \\
= \; & |\alpha|^3 \exp\left[+i\pi(\alpha\xi_0)^2\right] \cdot \delta\left[x - \alpha^2\xi_0\right]
\end{aligned}$$

Now do the postmultiplication:

$$\begin{aligned}
&\left(\left[f[x]\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right]*e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\\
=\;& |\alpha|^3\exp\left[+i\pi\left(\alpha\xi_0\right)^2\right]\cdot\delta\left[x-\alpha^2\xi_0\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha}\right)^2\right]\\
=\;& |\alpha|^3\exp\left[+i\pi\left(\alpha\xi_0\right)^2\right]\cdot\delta\left[x-\alpha^2\xi_0\right]\cdot\exp\left[-i\pi\left(\frac{\alpha^2\xi_0}{\alpha}\right)^2\right]\\
=\;& |\alpha|^3\cdot\delta\left[x-\alpha^2\xi_0\right]\cdot\left(\exp\left[+i\pi\left(\alpha\xi_0\right)^2\right]\cdot\exp\left[-i\pi\left(\alpha\xi_0\right)^2\right]\right)\\
=\;& |\alpha|^3\cdot\delta\left[x-\alpha^2\xi_0\right]\\
=\;& |\alpha|\cdot\delta\left[\frac{x-\alpha^2\xi_0}{\alpha^2}\right]\\
=\;& |\alpha|\cdot\delta\left[\frac{x}{\alpha^2}-\xi_0\right]\\
&\boxed{F\left[\frac{x}{\alpha^2}\right]=|\alpha|\cdot\delta\left[\left(\frac{x}{\alpha^2}\right)-\xi_0\right]}
\end{aligned}$$

the transform of a linear-phase exponential is a translated Dirac delta function

(e) $f[x]=\exp\left[+i\pi x^2\right]\implies F\left[\frac{x}{\alpha^2}\right]=\exp\left[+i\frac{\pi}{4}\right]\exp\left[-i\pi\left(\frac{x}{\alpha^2}\right)^2\right]$
use the M-C-M with $\alpha=1$. *The premultiplication gives:*

$$\exp\left[+i\pi x^2\right]\cdot\exp\left[-i\pi\left(x\right)^2\right]=1[x]$$

Now do the convolution:

$$\begin{aligned}
1[x]*\exp\left[+i\pi x^2\right] &= \mathcal{F}_1^{-1}\left\{\delta[\xi]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\xi^2\right]\right\}\\
&= \mathcal{F}_1^{-1}\left\{\delta[\xi]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot 1\right\}\\
&= 1[x]\cdot\exp\left[+i\frac{\pi}{4}\right]
\end{aligned}$$

Now do the postmultiplication:

$$1[x]\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(x\right)^2\right]=F\left[\frac{x}{1}\right]\implies F[\xi]=\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\xi\right)^2\right]$$

5. Use the C-M-C chirp Fourier transform to find an expression for $f[x] * exp\left[+i\pi\left(\frac{x}{\alpha}\right)^2\right]$ in terms of $F\left[\frac{x}{\alpha^2}\right]$

$$F\left[\frac{x}{\alpha^2}\right] = \frac{1}{|\alpha|}\cdot e^{-i\frac{\pi}{4}}\cdot\left(\left\{\left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)$$

the first convolution has the same form as the desired function

$$\textit{cross-multiply} \;:\; |\alpha|\cdot e^{+i\frac{\pi}{4}}\cdot F\left[\frac{x}{\alpha^2}\right] = \left\{\left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$$

now want to convolve from the right to generate a Dirac delta function that "cancels" the last term

$$\begin{aligned} e^{+i\pi\left(\frac{x}{\alpha}\right)^2} * e^{-i\pi\left(\frac{x}{\alpha}\right)^2} &= \mathcal{F}_1^{-1}\left\{\left(|\alpha|\, e^{+i\frac{\pi}{4}} e^{-i\pi\alpha^2\xi^2}\right)\left(|\alpha|\, e^{-i\frac{\pi}{4}} e^{+i\pi\alpha^2\xi^2}\right)\right\} \\ &= |\alpha|^2\cdot 1[\xi] \implies e^{+i\pi\left(\frac{x}{\alpha}\right)^2} * e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = |\alpha|^2\cdot\delta[x] \\ &\implies \frac{1}{|\alpha|^2} e^{+i\pi\left(\frac{x}{\alpha}\right)^2} * e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = \delta[x] \end{aligned}$$

$$\left\{\left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right\} * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} * \frac{1}{|\alpha|^2} e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = \left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$$

$$|\alpha|\cdot e^{+i\frac{\pi}{4}}\cdot F\left[\frac{x}{\alpha^2}\right] * \frac{1}{|\alpha|^2} e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = \left(f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}$$

now multiply both sides from the right by $e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$

$$\begin{aligned} f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} &= \left(|\alpha|\cdot e^{+i\frac{\pi}{4}}\cdot F\left[\frac{x}{\alpha^2}\right] * \frac{1}{|\alpha|^2} e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \\ &= \frac{1}{|\alpha|} e^{+i\frac{\pi}{4}}\cdot\left\{\left(F\left[\frac{x}{\alpha^2}\right] * e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\} \end{aligned}$$

$$\boxed{f[x] * e^{+i\pi\left(\frac{x}{\alpha}\right)^2} = \frac{1}{|\alpha|} e^{+i\frac{\pi}{4}}\cdot\left\{\left(F\left[\frac{x}{\alpha^2}\right] * e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\}}$$