SIMG-717-20052 Solution Set #2

1. Use the results derived in class to evaluate the 1-D Fourier transforms of:

$$f_1[x] = J_0[2\pi\xi_0 x]$$
$$f_2[x] = \frac{J_1(2\pi\xi_0 x)}{2\pi\xi_0 x}$$

Sketch the functions and their Fourier transforms.

(a) $f_1[x] = J_0[2\pi\xi_0 x]$

we know from the Hankel transform that the 2-D transform of J_0 *is a ring delta function*

$$\mathcal{F}_2\{J_0[2\pi\xi_0 r]\} = \frac{1}{2\pi\xi_0}\delta(\rho-\xi_0)$$

we also know from the Radon transform that the inverse Fourier transform of the convolution of the ring delta with a line delta is the "central slice" of the J_0

$$\mathcal{F}_2^{-1}\left\{\frac{1}{2\pi\xi_0}\delta(\rho-\xi_0) * (\delta[\xi]\cdot 1[\eta])\right\} = J_0[2\pi\xi_0 r]\cdot(1[x]\cdot\delta[y]) = J_0[2\pi\xi_0 x]\cdot(1[x]\cdot\delta[y])$$

$$\begin{aligned}
\textit{so, } \mathcal{F}_1\{J_0[2\pi\xi_0 x]\} &= \left.\frac{1}{2\pi\xi_0}\delta(\rho-\xi_0) * (\delta[\xi]\cdot 1[\eta])\right|_{\eta=0} \\
&= \mathcal{F}_2\{J_0[2\pi\xi_0 r]\cdot(1[x]\cdot\delta[y])\} \\
&= \mathcal{F}_2\{J_0[2\pi\xi_0 x]\cdot(1[x]\cdot\delta[y])\} \\
&= \frac{1}{2\pi\xi_0}\delta(\rho-\xi_0) * (\delta[\xi]\cdot 1[\eta]) \\
&= \frac{1}{2\pi\xi_0}\int_{\eta=-\infty}^{+\infty}\delta\left(\sqrt{\xi^2+\eta^2}-\xi_0\right)d\eta
\end{aligned}$$

which we evaluated in class

$$\begin{aligned}
\mathcal{F}_2\{J_0[2\pi\xi_0 r]\cdot(1[x]\cdot\delta[y])\} &= \frac{1}{2\pi\xi_0}\frac{2}{\sqrt{1-4\left(\frac{\xi}{2\xi_0}\right)^2}}RECT\left[\frac{\xi}{2\xi_0}\right] \\
&= \frac{1}{\pi\xi_0}\frac{1}{\sqrt{1-\left(\frac{\xi}{\xi_0}\right)^2}}RECT\left[\frac{\xi}{2\xi_0}\right] \\
&= \frac{1}{\pi}\frac{1}{\sqrt{\xi_0^2-\xi^2}}RECT\left[\frac{\xi}{2\xi_0}\right]
\end{aligned}$$

(b) $f_2[x] = \frac{J_1(2\pi\xi_0 x)}{2\pi\xi_0 x} = \frac{1}{2}\frac{2J_1(2\pi\xi_0 x)}{(2\pi\xi_0 x)} = \frac{1}{2} SOMB[2\xi_0 x]$

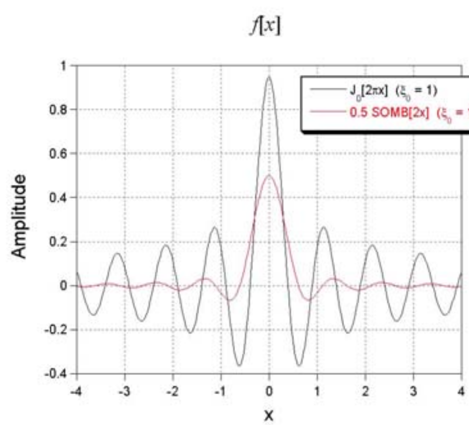
we know from the Hankel transform that

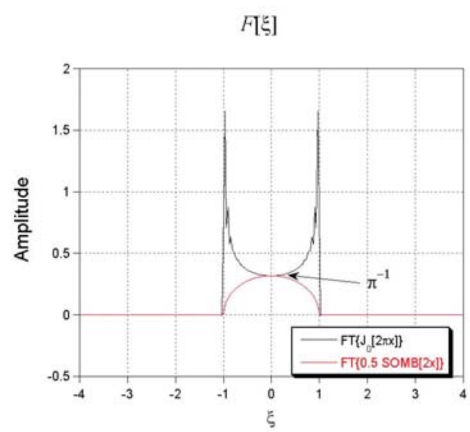
$$\mathcal{F}_2\left\{\frac{1}{2}SOMB[2\xi_0 r]\right\} = \frac{1}{2}\left(\frac{4}{\pi}\left[\frac{1}{(2\xi_0)^2}CYL\left(\frac{\rho}{2\xi_0}\right)\right]\right) = \frac{1}{2\pi\xi_0^2}CYL\left(\frac{\rho}{2\xi_0}\right)$$

we also know from the Radon transform that:

$$\mathcal{F}_2^{-1}\{F[\xi,\eta] * \delta[\xi]\,1[\eta]\} = f[x,y]\cdot 1[x]\,\delta[y] = f[x,0]\cdot 1[x]\,\delta[y]$$

$$\begin{aligned}
so,\ \mathcal{F}_2\left\{\frac{1}{2}SOMB[2\xi_0 x]\cdot 1[x]\,\delta[y]\right\} &= \frac{1}{2\pi\xi_0^2}CYL\left(\frac{\rho}{2\xi_0}\right) * \delta[\xi]\,1[\eta] \\
&= \frac{1}{2\pi\xi_0^2}\int_{\eta=-\infty}^{+\infty} CYL\left(\frac{\sqrt{\xi^2+\eta^2}}{2\xi_0}\right) d\eta \\
&= \frac{1}{2\pi\xi_0^2}\int_{\eta=-\sqrt{\xi_0^2-\xi^2}}^{+\sqrt{\xi_0^2-\xi^2}} d\eta \\
&= \frac{1}{2\pi\xi_0^2}\left(2\sqrt{\xi_0^2-\xi^2}\right) RECT\left[\frac{\xi}{2\xi_0}\right] \\
&= \frac{1}{\pi\xi_0^2}\sqrt{\xi_0^2-\xi^2}\,RECT\left[\frac{\xi}{2\xi_0}\right] \\
&= \frac{1}{\pi\xi_0}\sqrt{1-\left(\frac{\xi}{\xi_0}\right)^2}\,RECT\left[\frac{\xi}{2\xi_0}\right]
\end{aligned}$$





2. Find expressions for the moments of the following functions and use them to evaluate the areas, mean values, and variances.

   (a) $f[x] = SINC[x]$
   We know that the spectrum is

$$F[\xi] = RECT[\xi]$$

which we would expect to have an "unusual" power series. Evaluate the moments of $f[x]$:

$$\mathrm{m}_\ell = \int_{-\infty}^{+\infty} SINC[x]\ x^\ell\ dx = \frac{1}{(-2\pi i)^\ell} \frac{d^\ell}{d\xi^\ell} RECT[\xi]\bigg|_{\xi=0} = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathrm{m}_0 &= \int_{-\infty}^{+\infty} SINC[x]\ dx = RECT[\xi]|_{\xi=0} = 1 \\ \mathrm{m}_1 &= \mu_{(SINC[x])} = \frac{1}{-2\pi i} \frac{d}{d\xi}(RECT[\xi])\bigg|_{\xi=0} = 0 \end{aligned}$$

from moment theorem : $\mathrm{m}_2 = \dfrac{1}{(-2\pi i)^2} \dfrac{d2}{d\xi^2}(RECT[\xi])\bigg|_{\xi=0}$

which looks like it is zero. However, by direct integration:

$$\mathrm{m}_2 = \int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x}\right) \cdot x^2 dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} (x \cdot \sin \pi x)\ dx,$$

which is the area of the even function that grows with increasing $|x|$. The area is not well defined, which means that this is not a useful series for the spectrum.

$$\sigma_f^2 = \frac{\mathrm{m}_2}{\mathrm{m}_0} - \left(\frac{\mathrm{m}_1}{\mathrm{m}_0}\right)^2 = \frac{\infty}{1} - \left(\frac{0}{1}\right)^2 = \infty$$

   (b) $g[x] = SINC^2[x]$

$$\begin{aligned} \mathrm{m}_\ell &= \int_{-\infty}^{+\infty} SINC^2[x]\ x^\ell\ dx = \frac{1}{(-2\pi i)^\ell} \frac{d^\ell}{d\xi^\ell} TRI[\xi]\bigg|_{\xi=0} \\ \mathrm{m}_0 &= \frac{1}{(-2\pi i)^0} TRI[0] = 1 \\ \mathrm{m}_1 &= \frac{1}{(-2\pi i)} \frac{d}{d\xi} TRI[\xi]\bigg|_{\xi=0} = 0 \text{ because } TRI[\xi] \text{ is even} \\ \mathrm{m}_2 &= \frac{1}{(-2\pi i)^2} \frac{d^2}{d\xi^2} TRI[\xi]\bigg|_{\xi=0} = -\frac{1}{4\pi^2} \frac{d^2}{d\xi^2} TRI[\xi]\bigg|_{\xi=0} = -\frac{1}{4\pi^2}(-\infty) = +\infty \end{aligned}$$

because the second derivative of $TRI[\xi]$ is not defined

by direct integration, we have

which is well defined and infinite!

$$\sigma^2_{(SINC^2[x])} = \frac{m_2}{m_0} - \left(\frac{m_1}{m_0}\right)^2 = \frac{\infty}{1} - \left(\frac{0}{1}\right)^2 = +\infty$$

The variance of $SINC^2[x]$ *is infinite!!*

(c) $h[x] = \frac{1}{b} e^{-\left(\frac{x}{b}\right)} STEP[x]$

We know the spectrum:

$$\begin{aligned}
h[x] &= \frac{1}{b} e^{-\left(\frac{x}{b}\right)} STEP[x] = \frac{1}{b} e^{-\left(\frac{x}{b}\right)} STEP\left[\frac{x}{b}\right] \\
H[\xi] &= \frac{1}{1+2\pi i(b\xi)} = \frac{1-2\pi i(b\xi)}{1+(2\pi b\xi)^2} \\
\Re\{H[\xi]\} &= \frac{1}{1+(2\pi b\xi)^2} \\
\Im\{H[\xi]\} &= \frac{-2\pi(b\xi)}{1+(2\pi b\xi)^2}
\end{aligned}$$

$$\begin{aligned}
H[0] &= 1 \Longrightarrow m_0 = 1 \\
\frac{d}{d\xi}\left(\frac{1}{1+2\pi i b\xi}\right) &= -\left(\frac{1}{1+2\pi i b\xi}\right)^2 \cdot (-2\pi i b) \\
\frac{d}{d\xi}\left(\frac{1}{1+2\pi i b\xi}\right)\bigg|_{\xi=0} &= -2\pi i b \Longrightarrow m_1 = \left(\frac{1}{-2\pi i}\right)\left(\frac{d^\ell F[\xi]}{d\xi^\ell}\right)\bigg|_{\xi=0} = b \\
&\Longrightarrow \mu_h = \frac{m_1}{m_0} = b \\
\frac{d^2}{d\xi^2}\left(\frac{1}{1+2\pi i b\xi}\right) &= 2\left(\frac{1}{1+2\pi i b\xi}\right)^3 \cdot (-2\pi i b)^2 \\
\frac{d^2}{d\xi^2}\left(\frac{1}{1+2\pi i b\xi}\right)\bigg|_{\xi=0} &= -8\pi^2 b^2 \Longrightarrow m_2 = \left(\frac{1}{-2\pi i}\right)^2\left(\frac{d^\ell F[\xi]}{d\xi^\ell}\right)\bigg|_{\xi=0} = 2b^2 \\
\sigma_h^2 &= \frac{m_2}{m_0} - \left(\frac{m_1}{m_0}\right)^2 = \frac{2b^2}{1} - \left(\frac{b}{1}\right)^2 = b^2
\end{aligned}$$

(d) $GAUS[x]$

$$
\begin{aligned}
GAUS[x] &= e^{-\pi x^2} \implies F[\xi] = GAUS[\xi] = e^{-\pi\xi^2} \\
F[0] &= 1 \Longrightarrow \mathrm{m}_0 = 1 \\
\frac{d}{d\xi}\left(e^{-\pi\xi^2}\right) &= (-2\pi\xi)\, e^{-\pi\xi^2} \\
\frac{d}{d\xi}\left(e^{-\pi\xi^2}\right)\bigg|_{\xi=0} &= 0 \implies \mathrm{m}_1 = 0 \text{ \textit{(as expected because GAUS is even)}} \\
\frac{d^2}{d\xi^2}\left(e^{-\pi\xi^2}\right) &= \left(4\pi^2\xi^2 - 2\pi\right)\cdot e^{-\pi\xi^2} \\
\frac{d^2}{d\xi^2}\left(\left(4\pi^2\xi^2 - 2\pi\right)\cdot e^{-\pi\xi^2}\right)\bigg|_{\xi=0} &= -2\pi \Longrightarrow \mathrm{m}_2 = \left(\frac{1}{-2\pi i}\right)^2 \left(\frac{d^\ell F[\xi]}{d\xi^\ell}\right)\bigg|_{\xi=0} \\
&= \frac{1}{-4\pi^2}\cdot -2\pi = +\frac{1}{2\pi} \\
\sigma^2 &= \frac{\mathrm{m}_2}{\mathrm{m}_0} - \left(\frac{\mathrm{m}_1}{\mathrm{m}_0}\right)^2 = \frac{\left(\frac{1}{2\pi}\right)}{1} - \left(\frac{0}{1}\right)^2 = \frac{1}{2\pi} \cong 0.159
\end{aligned}
$$

Note that all odd moments vanish because GAUS is even.

3. Evaluate the first four moments of the functions in the previous problem and use them to construct an equation that approximates each spectrum; graph the approximate spectra.

$$F[\xi] = \sum_{\ell=0}^{+\infty} \left(\frac{1}{\ell!} \frac{d^\ell F}{d\xi^\ell}\bigg|_{\xi=0} \right) (\xi - 0)^\ell = \sum_{\ell=0}^{+\infty} \left(\frac{(-2\pi i)^\ell \ \mathrm{m}_\ell}{\ell!} \right) \xi^\ell$$

(a) $f[x] = SINC[x]$

$$\begin{aligned} \mathrm{m}_0 &= 1 \\ \mathrm{m}_1 &= 0 \\ \mathrm{m}_2 &= \infty \\ \mathrm{m}_3 &= \infty \end{aligned}$$

Series solution for $RECT[\xi]$ *is not useful*

(b) $g[x] = SINC^2[x]$

$$\begin{aligned} \mathrm{m}_0 &= 1 \\ \mathrm{m}_1 &= 0 \\ \mathrm{m}_2 &= \infty \\ \mathrm{m}_3 &= \infty \end{aligned}$$

Series solution for $TRI[\xi]$ *is not useful*

(c) $h[x] = \frac{1}{|b|} e^{-\left(\frac{x}{b}\right)} STEP[x] = \frac{1}{|b|} STEP\left[\frac{x}{b}\right] \cdot e^{-\left(\frac{x}{b}\right)} \implies H[\xi] = \frac{1}{1+2\pi i(b\xi)}$

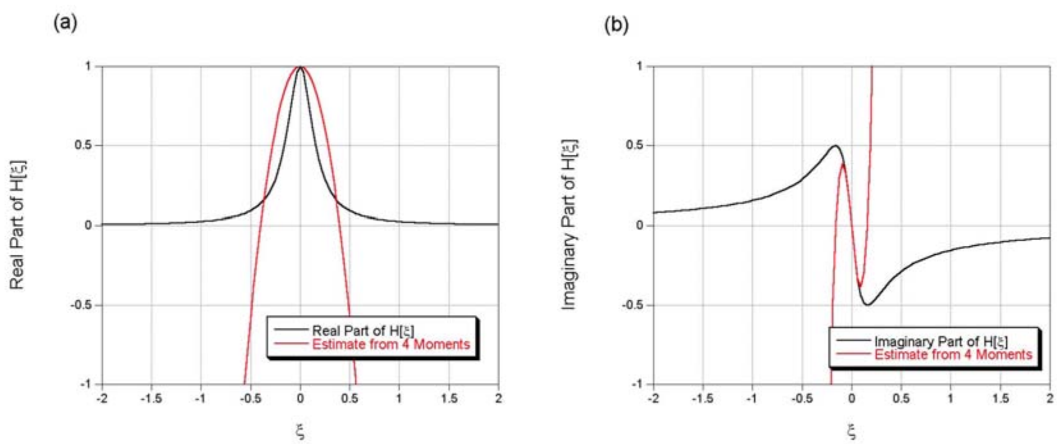
First, find the series solution for the transform

$$\begin{aligned} H[\xi] &= \frac{1}{1+2\pi i b\xi} = \frac{1}{1-(-2\pi i b\xi)} = \sum_{\ell=0}^{\infty} (-2\pi i b\xi)^\ell \\ &= 1 - 2\pi i b\xi + (2\pi i b\xi)^2 - (2\pi i b\xi)^3 \\ &= \left(1 - (2\pi b\xi)^2\right) + i\left(-2\pi b\xi + (2\pi b\xi)^3\right) \end{aligned}$$

The moments are:

$$\begin{aligned} \mathrm{m}_0 &= 1 \\ \mathrm{m}_1 &= b \\ \mathrm{m}_2 &= 2b^2 \\ \mathrm{m}_3 &= \left(\frac{1}{-2\pi i}\right)^3 48 i \pi^3 b^3 = 6b^3 \end{aligned}$$

$$
\begin{aligned}
H[\xi] &= \sum_{\ell=0}^{+\infty} \left(\frac{(-2\pi i)^{\ell} \ \mathrm{m}_{\ell}}{\ell!} \right) \xi^{\ell} \\
&\cong \sum_{\ell=0}^{+3} \left(\frac{(-2\pi i)^{\ell} \ \mathrm{m}_{\ell}}{\ell!} \right) \xi^{\ell} \\
&= 1 + (-2\pi i b\xi) + \frac{(-2\pi i)^2}{2!} \cdot \left(2b^2\right) \xi^2 + \frac{(-2\pi i)^3}{3!} \cdot 6b^3 \cdot \xi^3 \\
&= \left(1 - (2\pi b\xi)^2\right) + i \cdot \left(-2\pi b\xi + (2\pi b\xi)^3\right) \\
& \quad \textit{same as above}
\end{aligned}
$$



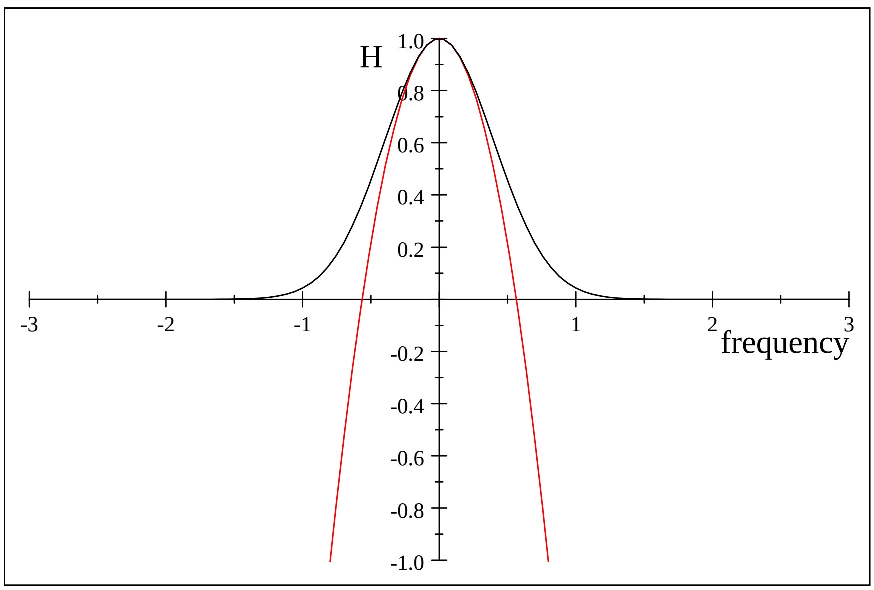
(d) $GAUS[x] = e^{-\pi x^2}$

The transform and its power series are:

$$GAUS[\xi] = e^{-\pi\xi^2} = \sum_{\ell=0}^{\infty} \frac{\left(-\pi\xi^2\right)^\ell}{\ell!} = 1 - \pi\xi^2 + \frac{\left(\pi\xi^2\right)^2}{2} - \frac{\left(\pi\xi^2\right)^3}{6}$$

$$\begin{aligned} \mathrm{m}_0 &= 1 \\ \mathrm{m}_1 &= 0 \\ \mathrm{m}_2 &= \frac{1}{2\pi} \\ \mathrm{m}_3 &= 0 \end{aligned}$$

$$\begin{aligned} \sum_{\ell=0}^{+\infty} \left(\frac{(-2\pi i)^\ell \ \mathrm{m}_\ell}{\ell!} \right) \xi^\ell &= 1 - \frac{2\pi i}{1!} \cdot 0 \cdot \xi + \frac{(-2\pi i)^2}{2!} \cdot \frac{1}{2\pi} \cdot \xi^2 + \frac{(-2\pi i)\, 3}{3!} \cdot 0 \cdot \xi^3 \\ &= 1 - \pi\xi^2 \end{aligned}$$



4. Find the algebraic expression for the moments of $f[x] = \delta[x - x_0]$ and show that the resulting power series are identical to the spectra for $x_0 = 0$ and $x_0 = 1$.

$$\begin{aligned} &\textit{First, the known spectra are} &:& \\ &\mathcal{F}_1\{\delta[x-0]\} &=& 1[\xi] \\ &\mathcal{F}_1\{\delta[x-1]\} &=& 1[\xi] \cdot e^{-2\pi i \xi} \end{aligned}$$

The definition of the moments:

$$\begin{aligned} \mathrm{m}_\ell &= \int_{-\infty}^{+\infty} \delta[x - x_0]\ x^\ell\ dx = \int_{-\infty}^{+\infty} \delta[x - x_0]\ x_0^\ell\ dx \\ &= x_0^\ell \int_{-\infty}^{+\infty} \delta[x - x_0]\ dx = x_0^\ell \end{aligned}$$

(a) *For* $x_0 = 0$, *the moments are:*

$$\mathrm{m}_0 = 0^0 = ?$$

To evaluate, return to the definition of moments:

$$\mathrm{m}_0 \equiv \int_{-\infty}^{+\infty} \delta[x]\ x^0\ dx = \int_{-\infty}^{+\infty} \delta[x]\ dx = 1$$

Lesson: check all expressions if uncertain.

$$\mathrm{m}_\ell = 0^\ell = 0 \textit{ for } \ell > 0$$

$$F[\xi] = \sum_{\ell=0}^{\infty} \frac{(-2\pi i)^\ell\, \mathrm{m}_\ell}{\ell!} \xi^\ell = \frac{(-2\pi i)^0}{0!} \mathrm{m}_0 \cdot \xi^0 = 1[\xi] \quad \textit{as required}$$

(b) *For* $x_0 = 1$, *the moments are:*

$$\begin{aligned} \mathrm{m}_\ell &= 1^\ell = 1 \\ F[\xi] &= \sum_{\ell=0}^{\infty} \frac{(-2\pi i)^\ell\, \mathrm{m}_\ell}{\ell!} \xi^\ell = \sum_{\ell=0}^{\infty} \frac{(-2\pi i \xi)^\ell \cdot (1)^\ell}{\ell!} = \sum_{\ell=0}^{\infty} \frac{(-2\pi i \xi)^\ell}{\ell!} = e^{-2\pi i \xi} \\ &\textit{from the known series for } e^{+i\theta} \end{aligned}$$