SIMG-717-20052 Midterm Exam Solutions

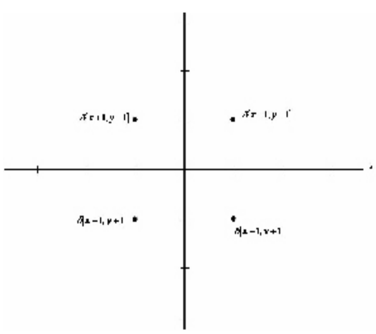
1. Consider the 2-D function:

$$f[x,y] = CYL\left(\frac{r}{2}\right) * \{\delta[x-1,y-1] + \delta[x+1,y+1]\}$$
$$+ RECT\left[\frac{x}{2},\frac{y}{2}\right]\{\delta[x-1,y+1] + \delta[x+1,y-1]\}$$

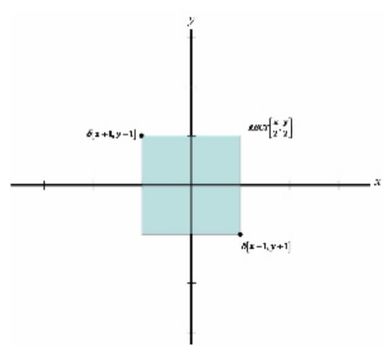
where the symbol "$*$" denotes the usual 2-D convolution operation.

(a) Sketch the "top view" of $f[x,y]$, including labels for amplitudes and coordinates.

$$RECT\left[\frac{x}{2},\frac{y}{2}\right]\cdot\delta[x-1,y+1] + RECT\left[\frac{x}{2},\frac{y}{2}\right]\delta[x+1,y+1]$$
$$= RECT\left[+\frac{1}{2},-\frac{1}{2}\right]\cdot\delta[x-1,y+1] + RECT\left[\frac{-1}{2},\frac{+1}{2}\right]\cdot\delta[x+1,y-1]$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\delta[x-1,y+1] + \frac{1}{2}\cdot\frac{1}{2}\cdot\delta[x+1,y-1]$$
$$= \frac{1}{4}\cdot(\delta[x-1,y+1] + \delta[x+1,y-1])$$

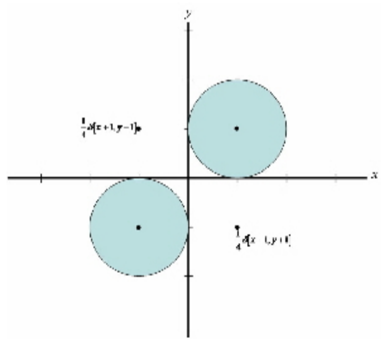


Dirac delta functions in $f[x,y]$



Scaling of Dirac delta functions by $RECT\left[\frac{x}{2},\frac{y}{2}\right]$

$$f[x,y] = CYL\left(\frac{r}{2}\right) * \{\delta[x-1,y-1] + \delta[x+1,y+1]\}$$
$$+ \frac{1}{4}\cdot(\delta[x-1,y+1] + \delta[x+1,y-1])$$
$$= CYL\left(\frac{\sqrt{(x-1)^2+(y-1)^2}}{2}\right) + CYL\left(\frac{\sqrt{(x+1)^2+(y+1)^2}}{2}\right)$$
$$+ \frac{1}{4}\cdot(\delta[x-1,y+1] + \delta[x+1,y-1])$$



$f[x,y]$

(b) Evaluate $\mathcal{F}_2\{f[x,y]\} \equiv F[\xi,\eta]$. You may use any known theorems or Fourier transforms of special functions without proof but specify what you are using.

$$F[\xi,\eta] = \mathcal{F}_2\left\{CYL\left(\frac{r}{2}\right)\right\} \cdot \mathcal{F}_2\{\delta[x-1,y-1]+\delta[x+1,y+1]\}$$
$$+\frac{1}{4}\mathcal{F}_2\{\delta[x-1,y+1]+\delta[x+1,y-1]\}$$
$$= \mathcal{F}_2\left\{CYL\left(\frac{r}{2}\right)\right\} \cdot \mathcal{F}_1\{\delta[x-1]\}\mathcal{F}_1\{\delta[y-1]\} + \mathcal{F}_1\{\delta[x+1]\}\mathcal{F}_1\{\delta[y+1]\}$$
$$+\frac{1}{4}\left(\mathcal{F}_1\{\delta[x-1]\}\mathcal{F}_1\{\delta[y+1]\} + \mathcal{F}_1\{\delta[x+1]\}\mathcal{F}_1\{\delta[y-1]\}\right)$$
$$= \frac{\pi}{4}\left(2^2\right) SOMB(2\rho)\cdot\left(e^{-2\pi i\xi}e^{-2\pi i\eta} + e^{+2\pi i\xi}e^{+2\pi i\eta}\right) + \frac{1}{4}\left(e^{-2\pi i\xi}e^{+2\pi i\eta} + e^{+2\pi i\xi}e^{-2\pi i\eta}\right)$$
$$= \pi SOMB(2\rho)\cdot\left(e^{-2\pi i(\xi+\eta)} + e^{+2\pi i(\xi+\eta)}\right) + \frac{1}{4}\left(e^{-2\pi i(\xi-\eta)} + e^{+2\pi i(\xi-\eta)}\right)$$
$$= \pi SOMB(2\rho)\cdot 2\cos[2\pi(\xi+\eta)] + \frac{1}{4}\left(2\cos[2\pi(\xi-\eta)]\right)$$
$$= 2\pi SOMB(2\rho)\cdot\cos[2\pi(\xi+\eta)] + \frac{1}{2}\cos[2\pi(\xi-\eta)]$$

(c) Evaluate the Radon transform $\mathcal{R}_2\{f[x,y]\} \equiv \lambda_f(p,\phi)$ for $\phi = 0$ and $\phi = +\frac{\pi}{2}$ radians. Again, you may use any known Radon transforms without proof but specify what that you are using.

The Radon transform of a cylinder function is:

$$\lambda_{CYL(r)}[p,\phi] = \left(\sqrt{1-4p^2}\right) RECT[p]\cdot 1[\phi]$$

If scaled so that the diameter is 2 units, the Radon transform is:

$$\lambda_{CYL\left(\frac{r}{2}\right)}(p,\phi) = \left(2\sqrt{1-4\left(\frac{p}{2}\right)^2}\right) RECT\left[\frac{p}{2}\right]\cdot 1[\phi]$$
$$= \left(2\sqrt{1-p^2}\right) RECT\left[\frac{p}{2}\right]\cdot 1[\phi]$$

The projection onto the x-axis is the sum of the Radon transforms with appropriate *translation and the projections of the Dirac delta functions:*

$$\lambda_f[p,\phi=0] = \left(2\sqrt{1-(p+2)^2}\right)\ RECT\left[\frac{p+2}{2}\right] + \left(2\sqrt{1-(p-2)^2}\right)\ RECT\left[\frac{p-2}{2}\right] + \frac{1}{4}\left(\delta[p+1]+\delta[p-1]\right)$$

The projection onto the y-axis is identical:

$$\lambda_f\left[p,\phi=+\frac{\pi}{2}\right] = \left(2\sqrt{1-(p+2)^2}\right)\ RECT\left[\frac{p+2}{2}\right] + \left(2\sqrt{1-(p-2)^2}\right)\ RECT\left[\frac{p-2}{2}\right] + \frac{1}{4}\left(\delta[p+1]+\delta[p-1]\right)$$

(d) Evaluate the 1-D Fourier transforms of the two projections at $\phi_1 = 0$ and at $\phi = +\frac{\pi}{2}$ radians and use these results to confirm (not "prove") the central-slice theorem.

Use the central-slice theorem; the 1-D transform is the central slice of the 2-D spectrum:

$$\begin{aligned}\mathcal{F}_1\{\lambda_f[p,\phi=0]\} &= F[\xi,0] \\ &= \left(2\pi SOMB\left(2\sqrt{\xi^2+\eta^2}\right)\cdot\cos[2\pi(\xi+\eta)] + \frac{1}{2}\cos[2\pi(\xi-\eta)]\right)\Big|_{\eta=0} \\ &= 2\pi SOMB(2|\xi|)\cdot\cos[2\pi\xi] + \frac{1}{2}\cos[2\pi\xi] \\ &= \left(2\pi SOMB(2\xi)+\frac{1}{2}\right)\cos[2\pi\xi]\end{aligned}$$

$$\begin{aligned}\mathcal{F}_1\left\{\lambda_f\left[p,\phi=\frac{\pi}{2}\right]\right\} &= F[0,\eta] \\ &= \left(2\pi SOMB\left(2\sqrt{\xi^2+\eta^2}\right)\cdot\cos[2\pi(\xi+\eta)] + \frac{1}{2}\cos[2\pi(\xi-\eta)]\right)\Big|_{\xi=0} \\ &= 2\pi SOMB\left(2\sqrt{\eta^2}\right)\cdot\cos[2\pi(\eta)] + \frac{1}{2}\cos[2\pi(-\eta)] \\ &= 2\pi SOMB(2|\eta|)\cdot\cos[2\pi\eta] + \frac{1}{2}\cos[2\pi\eta] \\ &= \left(2\pi SOMB(2\eta)+\frac{1}{2}\right)\cos[2\pi\eta]\end{aligned}$$

2. For each of the following functions:

$$
\begin{aligned}
f_1[x] &= SINC[2x] \\
f_2[x] &= SINC^2[2x]
\end{aligned}
$$

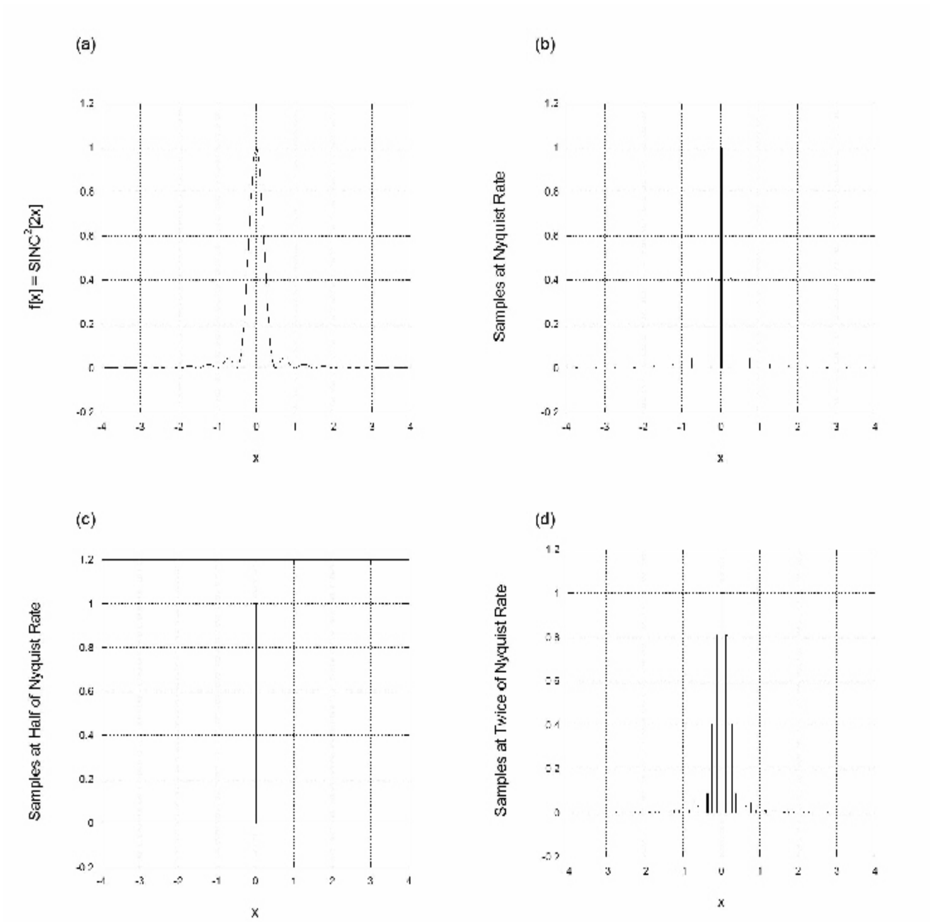
(a) Determine the Nyquist sampling frequencies ξ_{Nyquist}

$$
\begin{aligned}
\mathcal{F}_1\{SINC[2x]\} &= \frac{1}{2}RECT\left[\frac{\xi}{2}\right] \Longrightarrow \xi_{\max} = 1 \Longrightarrow \boxed{\xi_{\text{Nyquist}} = 2\frac{\text{cycles}}{\text{unit length}}} \\
\mathcal{F}_1\left\{SINC^2[2x]\right\} &= \frac{1}{2}TRI\left[\frac{\xi}{2}\right] \Longrightarrow \xi_{\max} = 2 \Longrightarrow \boxed{\xi_{\text{Nyquist}} = 4\frac{\text{cycles}}{\text{unit length}}}
\end{aligned}
$$

(b) For ONLY the single function $f_n[x]$ with the LARGER value of ξ_{Nyquist}, make sketches (with appropriate axis labels) of the following three cases in the space domain:

1. $f_n[x]$ and the result of ideal sampling at the Nyquist rate
2. $f_n[x]$ and the result of ideal sampling at half of the Nyquist rate,
3. $f_n[x]$ and the result of ideal sampling at twice the Nyquist rate.
   The interpolation function is

$$h[x] = SINC[4x]$$



(c) Evaluate the continuous functions that would be "reconstructed" from the sets of samples after interpolation with the appropriate functions.

The samples at the Nyquist rate (i) and at twice the Nyquist rate (iii) recover the original function, while that at half the Nyquist rate recovers $SINC\left[\frac{x}{2}\right]$

(d) Comment on the results of part c with respect to the Whittaker-Shannon sampling theorem.

Need to sample at better than the Nyquist rate.

3. Evaluate the Radon transform of

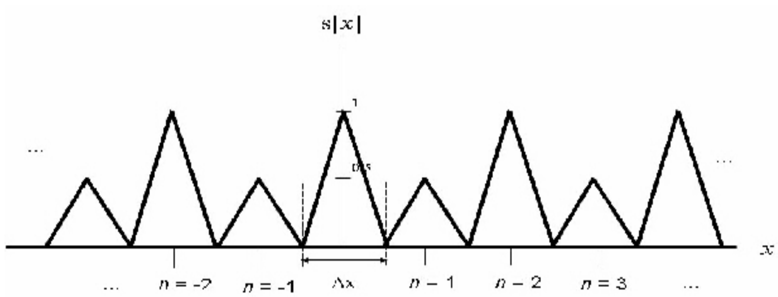
$$f[x,y] = \exp\left[+i\pi\frac{x^2+y^2}{a^2}\right]$$

$$\exp\left[+i\pi\frac{x^2+y^2}{a^2}\right] = \exp\left[+i\pi\frac{x^2}{a^2}\right]\exp\left[+i\pi\frac{y^2}{a^2}\right]$$

This is a circularly symmetric function and thus will have the same line-integral projections at all azimuth angles. Evaluate for $\phi = 0$

$$\begin{aligned}
\int_{-\infty}^{+\infty} f[x,y]\ dy &= \int_{-\infty}^{+\infty} \exp\left[+i\pi\frac{x^2}{a^2}\right]\exp\left[+i\pi\frac{y^2}{a^2}\right]\ dy \\
&= \exp\left[+i\pi\frac{x^2}{a^2}\right]\int_{-\infty}^{+\infty} \exp\left[+i\pi\frac{y^2}{a^2}\right]\ dy \\
&= \exp\left[+i\pi\frac{x^2}{a^2}\right]\cdot \mathcal{F}_1\left\{\exp\left[+i\pi\frac{y^2}{a^2}\right]\right\}\Bigg|_{\eta=0} \\
&= \exp\left[+i\pi\frac{x^2}{a^2}\right]\cdot\left(|a|\exp\left[+i\frac{\pi}{4}\right]\exp\left[-i\pi a^2\eta^2\right]\right)\Bigg|_{\eta=0} \\
&= \exp\left[+i\pi\frac{x^2}{a^2}\right]\cdot\left(|a|\exp\left[+i\frac{\pi}{4}\right]\right) \\
&= \boxed{|a|\exp\left[+i\tfrac{\pi}{4}\right]\exp\left[+i\pi\tfrac{x^2}{a^2}\right]}
\end{aligned}$$

4. A 1-D imaging sensor is composed of an infinite number of detector elements of width Δx. The "response" (sensitivity function) of a small portion of the sensor is shown below. Note that the response function continues with this form to $x = \pm\infty$.



(a) Write down a mathematical expression for the samples generated by this detector from a 1-D input signal $f[x]$. Assume that $f[x]$ is bandlimited so that $\xi_{\max} < \frac{1}{2(\Delta x)}$ (there are several ways to do this).

The input function will be averaged over the triangle (via convolution) that has width $\frac{\Delta x}{2}$ and then sampled by a COMB with elements separated by Δx but with unequal weights. You can write the COMB function as the sum of two parts: one with components with half-unit area separated by intervals of Δx, and one with components of half-unit area separated by intervals of $2 \cdot \Delta x$

$$\begin{aligned}
f_s[x;\Delta x] &= \left(f[x] * TRI\left[\frac{x}{\left(\frac{\Delta x}{2}\right)}\right]\right) \cdot \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(\delta[x - n\cdot\Delta x] + \delta[x - n\cdot(2\cdot\Delta x)]\right) \\
&= \left(f[x] * TRI\left[\frac{x}{\left(\frac{\Delta x}{2}\right)}\right]\right) \cdot \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\delta[x - n\cdot\Delta x] + \sum_{n=-\infty}^{+\infty}\delta[x - n\cdot(2\cdot\Delta x)]\right) \\
&= \left(f[x] * TRI\left[\frac{x}{\left(\frac{\Delta x}{2}\right)}\right]\right) \cdot \frac{1}{2}\left(\frac{1}{\Delta x}COMB\left[\frac{x}{\Delta x}\right] + \frac{1}{2\cdot\Delta x}COMB\left[\frac{x}{2\cdot\Delta x}\right]\right)
\end{aligned}$$

(b) Evaluate the Fourier transform of this sampled signal. You may assume any form for $F[\xi]$ within the band limit.

$$\begin{aligned}
\mathcal{F}\{f_s[x;\Delta x]\} &= \left(F[\xi]\cdot\frac{\Delta x}{2}\cdot SINC^2\left[\frac{\Delta x}{2}\cdot\xi\right]\right) \\
&\quad * \frac{1}{2}\left(COMB[\Delta x\cdot\xi]+COMB[2\cdot\Delta x\cdot\xi]\right) \\
&= \frac{\Delta x}{2}\cdot\left(F[\xi]\cdot SINC^2\left[\frac{\Delta x}{2}\cdot\xi\right]\right) \\
&* \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\delta\left[\Delta x\cdot\left(\xi-\frac{n}{\Delta x}\right)\right]+\sum_{m=-\infty}^{+\infty}\delta\left[2\cdot\Delta x\cdot\left(\xi-\frac{m}{2\cdot\Delta x}\right)\right]\right) \\
&= \frac{\Delta x}{2}\cdot\left(F[\xi]\cdot SINC^2\left[\frac{\xi}{\left(\frac{2}{\Delta x}\right)}\right]\right) \\
&* \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\frac{1}{\Delta x}\cdot\delta\left[\xi-\frac{n}{\Delta x}\right]+\sum_{m=-\infty}^{+\infty}\frac{1}{2\cdot\Delta x}\cdot\delta\left[\xi-\frac{m}{2\cdot\Delta x}\right]\right) \\
&= \frac{\Delta x}{2}\cdot\left(F[\xi]\cdot SINC^2\left[\frac{\xi}{\left(\frac{2}{\Delta x}\right)}\right]\right) \\
&* \frac{1}{4\cdot\Delta x}\left(\sum_{n=-\infty}^{+\infty}2\cdot\delta\left[\xi-\frac{n}{\Delta x}\right]+\sum_{m=-\infty}^{+\infty}\delta\left[\xi-\frac{m}{2\cdot\Delta x}\right]\right) \\
&= \frac{1}{8}\left(F[\xi]\cdot SINC^2\left[\frac{\xi}{\left(\frac{2}{\Delta x}\right)}\right]\right) \\
&\quad * \left(\sum_{n=-\infty}^{+\infty}2\cdot\delta\left[\xi-\frac{n}{\Delta x}\right]+\sum_{m=-\infty}^{+\infty}\delta\left[\xi-\frac{m}{2\cdot\Delta x}\right]\right)
\end{aligned}$$

The area of the Dirac delta function at the origin ($\xi = 0$) is $\frac{1}{8\cdot\Delta x}(2+1) = \frac{3}{8\cdot\Delta x}$; the area of the "next" Dirac delta function is $\frac{1}{8\cdot\Delta x}$ and is located at $\xi = \pm\frac{1}{2}\left(\frac{1}{\Delta x}\right)$.

(c) Determine if it is *theoretically* possible to recover $f[x]$ from these samples. If not, give reasons. If so, then list the necessary steps in the procedure.

Since $F[\xi]$ has compact support within the range $|\xi| < \frac{1}{2\cdot\Delta x}$, the "width" of $F[\xi] < \frac{1}{\Delta x}$. The separation between the Dirac delta functions is only $\frac{1}{2}\left(\frac{1}{\Delta x}\right)$, so the replicas "overlap" and the spectrum of the continuous function cannot be segmented without aliasing.

5. Consider $f[x] = \delta[x-2]$

(a) Derive the general expression for the ℓ^{th} moment of $f[x]$.

$$
\begin{aligned}
\mathrm{m}_\ell \{f[x]\} &\equiv \int_{-\infty}^{+\infty} x^\ell \; f[x] \; dx \\
\mathrm{m}_\ell \{\delta[x-2]\} &= \int_{-\infty}^{+\infty} x^\ell \; \delta[x-2] \; dx = \int_{-\infty}^{+\infty} 2^\ell \; \delta[x-2] \; dx = 2^\ell \int_{-\infty}^{+\infty} \delta[x-2] \; dx \\
&= \boxed{2^\ell = m_\ell \{\delta[x-2]\}} \\
\mathrm{m}_0 \{\delta[x-2]\} &= 2^0 = 1 \\
\mathrm{m}_1 \{\delta[x-2]\} &= 2^1 = 2 \\
\mathrm{m}_2 \{\delta[x-2]\} &= 2^2 = 4
\end{aligned}
$$

(b) Write down an approximate expression for $F[\xi]$ that would be valid for $|\xi| \simeq 0$ and confirm it by comparison to $\mathcal{F}\{\delta[x-2]\}$

$$
\mathcal{F}\{\delta[x-2]\} = \exp[-2\pi i\xi \cdot 2] = \exp[-4\pi i\xi] = \sum_{\ell=0}^{\infty} \frac{(-4\pi i\xi)^\ell}{\ell!}
$$

$$
\begin{aligned}
F[\xi] &= \sum_{\ell=0}^{+\infty} \left(\frac{(-2\pi i)^\ell \; \mathrm{m}_\ell}{\ell!} \right) \xi^\ell \\
&= \sum_{\ell=0}^{+\infty} \left(\frac{(-2\pi i)^\ell \; 2^\ell}{\ell!} \right) \xi^\ell = \sum_{\ell=0}^{+\infty} \left(\frac{(-4\pi i)^\ell}{\ell!} \right) \xi^\ell
\end{aligned}
$$

so the two expressions are exactly equal...

6. Consider the convolution of an arbitrary input function $f[x,y]$ with the 2-D function $J_0(2\pi r\rho_n)\cdot 1(\theta)$ (where ρ_n is a specific radial spectral frequency indexed by n).

$$g_n[x,y] = f[x,y] * (J_0(2\pi r\rho_n)\cdot 1(\theta))$$

(a) Derive the expression for the spectrum of the output function $g[x,y]$.

$$G[\xi,\eta] = F[\xi,\eta]\cdot\frac{\delta(\rho-\rho_n)}{2\pi\rho_n}$$

which is the product of the original spectrum and a ring delta function, which acts as a transfer function that passes the ring and rejects components at all other spatial frequencies. in short it is a "bandpass filter."

(b) If the same filtering process is realized for the same input function $f[x,y]$ but using different values of the spectral frequency ρ_n, describe the differences among the resulting output functions

The output images will include only those spatial frequencies $\rho_n = \sqrt{\xi_n^2+\eta_n^2}$ *and thus are "orthogonal" (meaning that the convolutions of two distinct output images evaluate to zero.*

7. Do FOUR (4) of the following short-answer questions:

(a) Evaluate the volume of $RECT\left(\frac{x}{2},\frac{y}{2}\right) * CYL\left(\frac{r}{2}\right)$.

$$\begin{aligned} F[\xi,\eta]\cdot H[\xi,\eta] &= G[\xi,\eta] \\ &= 4SINC[2\xi,2\eta]\cdot\frac{\pi}{4}\left(2^2\right)SOMB(2\rho) \\ G[0,0] &= 4\cdot\left(\frac{\pi}{4}\cdot 4\right) = \boxed{4\pi} \end{aligned}$$

(b) Determine the regions of support of $e^{-i\pi x^2} * e^{+i\pi x^2}$ and of $e^{+i\pi x^2} * e^{+i\pi x^2}$

$$\begin{aligned} e^{-i\pi x^2} * e^{+i\pi x^2} &= \mathcal{F}_1^{-1}\left(e^{-i\frac{\pi}{4}}e^{+i\pi\xi^2}\cdot e^{+i\frac{\pi}{4}}e^{-i\pi\xi^2}\right) = \mathcal{F}_1^{-1}(1) = \delta[x] \\ &\Longrightarrow \boxed{\textit{infinitesimal support}} \\ e^{+i\pi x^2} * e^{+i\pi x^2} &= \mathcal{F}_1^{-1}\left(e^{+i\frac{\pi}{4}}e^{-i\pi\xi^2}\cdot e^{+i\frac{\pi}{4}}e^{-i\pi\xi^2}\right) = \mathcal{F}_1^{-1}\left(e^{+i\frac{\pi}{2}}e^{-i\pi\left(\sqrt{2}\xi\right)^2}\right) \\ &= e^{+i\frac{\pi}{4}}\mathcal{F}_1^{-1}\left(e^{+i\frac{\pi}{4}}e^{-i\pi\left(\sqrt{2}\xi\right)^2}\right) = e^{+i\frac{\pi}{4}}e^{+i\pi\left(\frac{x}{\sqrt{2}}\right)^2} \\ &\Longrightarrow \boxed{\textit{infinite support}} \end{aligned}$$

(c) Evaluate $TRI\left[2x,\frac{y}{2}\right] * (1[x]\cdot\delta[y])$

$$\begin{aligned} TRI\left[2x,\frac{y}{2}\right] * (1[x]\cdot\delta[y]) &= (TRI[2x] * 1[x])\cdot\left(TRI\left[\frac{y}{2}\right] * \delta[y]\right) \\ &= \left(\int_{-\infty}^{+\infty} TRI[2x]\;dx\right)\cdot TRI\left[\frac{y}{2}\right] \\ &= \boxed{\tfrac{1}{2}\cdot TRI\left[\tfrac{y}{2}\right]} \end{aligned}$$

(d) Evaluate $(1[x]\cdot\delta[y]) * (\delta[x]\cdot 1[y])$

(duh!)

$$\begin{aligned} (1[x]\cdot\delta[y]) * (\delta[x]\cdot 1[y]) &= (1[x] * \delta[x])\cdot(\delta[y] * 1[y]) \\ &= 1[x]\cdot 1[y] = \boxed{1[x,y]} \end{aligned}$$

(e) Find the general expression for the line-integral projections of the 2-D function $\delta[x-1,y-2]$

$$\delta[x-1,y-2] * \left(\delta\left[\underline{\mathbf{r}}\bullet\underline{\hat{\mathbf{p}}}\right]\;1\left[\underline{\mathbf{r}}\bullet\underline{\hat{\mathbf{p}}}^{\perp}\right]\right)$$

$$= (\delta[x-1]\;\;\delta[y-2]) * \delta[x\cos\phi + y\sin\phi]$$

$$= \iint_{-\infty}^{+\infty}\delta[x-1,y-2]\cdot\delta[p-(x\;\cos[\phi]+y\;\sin[\phi])]\;dx\;dy$$

$$= \iint_{-\infty}^{+\infty}\delta[x-1,y-2]\cdot\delta[p-(1\;\cos[\phi]+2\;\sin[\phi])]\;dx\;dy$$

$$= \delta[p-(\cos[\phi]+2\;\sin[\phi])]\cdot\left(\iint_{-\infty}^{+\infty}\delta[x-1,y-2]\;dx\;dy\right)$$

$$= \delta\left[p-\sqrt{5}\;\cos\left[\phi-\tan^{-1}\left[\frac{2}{1}\right]\right]\right]$$

.