1051-716-20111 Solution Set #8

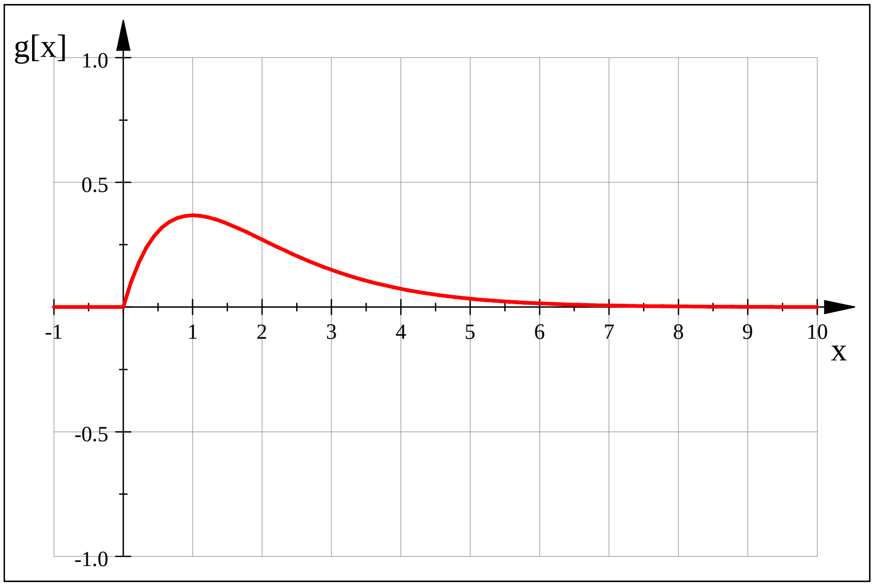
1. A signal $s[x] = \exp[-x] \cdot STEP[x]$ is the input to an LSI system.

   (a) Find and sketch the output $g[x]$ if $h[x] = s[x]$

$$g[x] = s[x] * s[x] = (\exp[-x] \cdot STEP[x]) * (\exp[-x] \cdot STEP[x])$$

We did this convolution in Chapter 8 – it is more easily done in the space domain:

$$\begin{aligned}
\int_{-\infty}^{+\infty} s[\alpha] \cdot s[x-\alpha]\ d\alpha &= \int_{-\infty}^{+\infty} (\exp[-\alpha] \cdot STEP[\alpha]) \cdot (\exp[-(x-\alpha)] \cdot STEP[x-\alpha])\ d\alpha \\
&= \int_{0}^{+\infty} \exp[-\alpha] \cdot \exp[-(x-\alpha)] \cdot STEP[x-\alpha]\ d\alpha \\
&= \exp[-x] \cdot \int_{0}^{+\infty} \exp[-\alpha] \cdot \exp[+\alpha] \cdot STEP[x-\alpha]\ d\alpha \\
&= \exp[-x] \cdot \int_{0}^{+\infty} STEP[x-\alpha]\ d\alpha \\
&= \exp[-x] \int_{0}^{+x} 1\ d\alpha\ \ \textit{if}\ \ x \geq 0 \\
&= x \cdot \exp[-x]\ \ \textit{if}\ \ x \geq 0 \\
&= x \cdot \exp[-x] \cdot STEP[x]
\end{aligned}$$



(b) Find and sketch the output $g[x]$ if $h[x] = s[-x]$

$$\begin{aligned} g[x] &= s[x] * s[-x] = s[x] \bigstar s[x] \\ &= (\exp[-x] \cdot STEP[x]) * (\exp[+x] \cdot STEP[-x]) \end{aligned}$$

$$\begin{aligned} g[x] &= \int_{-\infty}^{+\infty} (\exp[-\alpha] \cdot STEP[\alpha]) \cdot (\exp[+(x-\alpha)] \cdot STEP[-(x-\alpha)])\ d\alpha \\ &= \exp[+x] \int_{0}^{+\infty} \exp[-2\alpha] \cdot STEP[\alpha - x]\ d\alpha \end{aligned}$$

$STEP[\alpha - x]$ is $+1$ from x to ∞.
If $x > 0$:

$$\begin{aligned} g[x] &= \exp[+x] \cdot \left(\frac{\exp[-2\alpha]}{-2} \right) \Bigg|_{\alpha=x}^{\alpha=+\infty} = \frac{\exp[+x]}{-2} \cdot (\exp[-\infty] - \exp[-2x]) \\ &= \frac{\exp[+x]}{-2} \cdot (0 - \exp[-2x]) = \frac{\exp[+x]}{-2} \cdot (-\exp[-2x]) = +\frac{\exp[-x]}{2} \end{aligned}$$

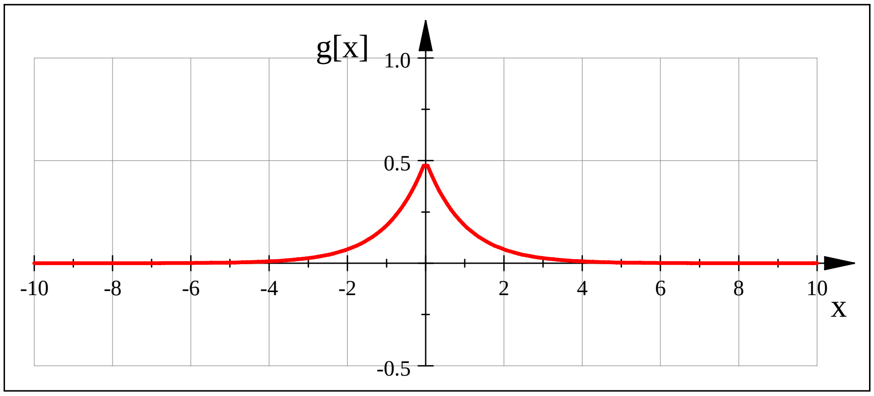
If $x < 0$, then

$$\begin{aligned} g[x] &= \exp[+x] \int_{0}^{+\infty} \exp[-2\alpha]\ d\alpha = \exp[+x] \cdot \left(\frac{\exp[-2\alpha]}{-2} \Bigg|_{\alpha=0}^{\alpha=\infty} \right) \\ &= \frac{\exp[+x]}{-2} \cdot (0 - 1) = \frac{\exp[+x]}{2} = \frac{\exp[-|x|]}{2} \quad \textit{if } x < 0 \end{aligned}$$

Combine these two expressions:

$$g[x] = s[x] * s[-x] = s[x] \bigstar s[x]$$

$$\boxed{g[x] = \frac{1}{2} \cdot \exp[-|x|] \ \textit{which is symmetric}}$$



(c) Describe the impulse response $h[x]$ and transfer function $H[\xi]$ of the matched filter that will maximize the output at $x = x_0$, i.e., at some arbitrary location.

The output will be the autocorrelation evaluated at $x - x_0$. *In the general case where* $s[x]$ *is complex:*

$$\begin{aligned} g[x] &= f[x] * h[x] = s[x] * s^*[-x] * \delta[x - x_0] \\ &\implies h[x] = s^*[-x] * \delta[x - x_0] = s^*[-(x - x_0)] = s^*[-x + x_0] \end{aligned}$$

$$\begin{aligned} H[\xi] &= \mathcal{F}_1\{m[x]\} = \mathcal{F}_1\{s^*[-x] * \delta[x - x_0]\} \\ &= S^*[\xi] \cdot \exp[-2\pi i x_0 \cdot \xi] \end{aligned}$$

In the special case where $s[x]$ *is real, then*

$$\boxed{h[x] = s[-x] * \delta[x - x_0] = s[-(x - x_0)] = s[-x + x_0]}$$

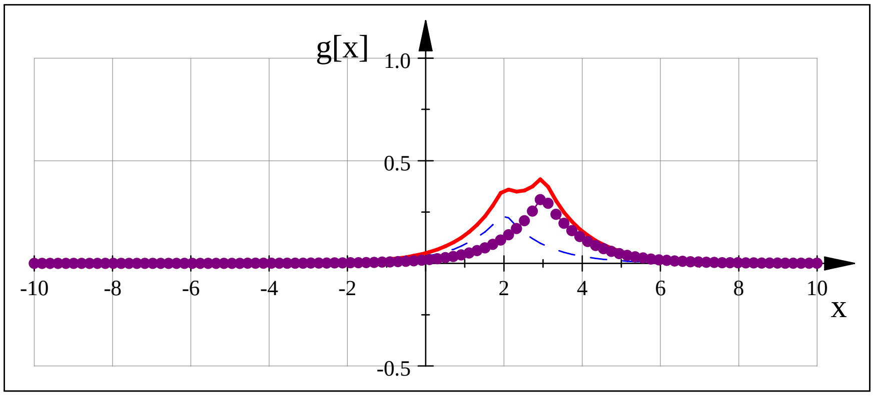
a reversed replica of $s[x]$ *centered at* $x = +x_0$

(d) Find and sketch the output if $h[x] = s[-x]$ and the input is

$$s[x] * \left(\frac{1}{2}\delta[x - 2] + \frac{1}{3}\delta[x + 3]\right)$$

$$\begin{aligned} g[x] &= \left(s[x] * \left(\frac{1}{2}\delta[x - 2] + \frac{1}{3}\delta[x + 3]\right)\right) * s[-x] \\ &\boxed{g[x] = (s[x] \bigstar s[-x]) * \left(\frac{1}{2}\delta[x - 2] + \frac{1}{3}\delta[x + 3]\right)} \end{aligned}$$

So we get replicas of the autocorrelation centered at $x = +2$ *and at* $x = -3$ *with respective scalings of* $\frac{1}{2}$ *and* $\frac{1}{3}$



$g[x] = (s[x] \bigstar s[-x]) * \left(\frac{1}{2}\delta[x - 2] + \frac{1}{3}\delta[x + 3]\right)$ *(red solid) with the individual terms shown as dashed blue and dotted purple.*

2. Repeat the four parts of #1 if $s[x] = \exp\left[+i\pi\frac{x^2}{2}\right]$

   (a) Find and sketch the output $g[x]$ if $h[x] = s[x]$

$$\begin{aligned}
g[x] &= \exp\left[+i\pi\frac{x^2}{2}\right] * \exp\left[+i\pi\frac{x^2}{2}\right] = \exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right] \\
&\implies \alpha_0 = \sqrt{2} \\
\mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\} &= |\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\alpha_0\xi\right)^2\right] = |\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\frac{\xi}{(\alpha_0)^{-1}}\right)^2\right] \\
\mathcal{F}_1\{s[x]\} &= \mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right]\right\} = S[\xi] = \left|\sqrt{2}\right|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right] \\
n.b.,\ G[\xi] &= S[\xi]\cdot S[\xi] \neq |S[\xi]|^2!!! \implies \Phi\{G[\xi]\} \neq 0[\xi] \\
G[\xi] &= 2\cdot\exp\left[+i\frac{\pi}{2}\right]\cdot\left(\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\right)^2
\end{aligned}$$

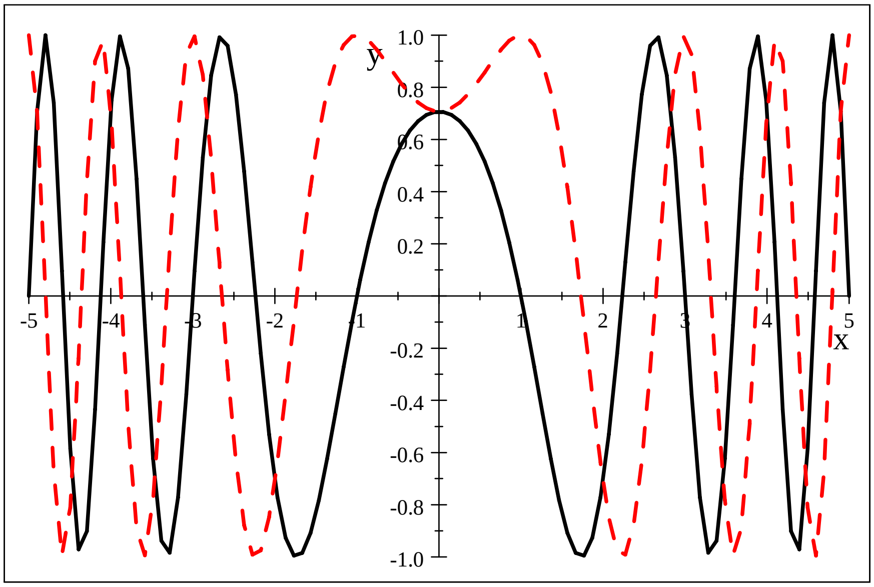
$$n.b.,\ \left(\exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\right)^2 = \exp\left[-i\pi\left(2\xi\right)^2\right]$$

$$\implies \boxed{G[\xi] = 2\cdot\exp\left[+i\frac{\pi}{2}\right]\cdot\exp\left[-i\pi(2\xi)^2\right] = 2\cdot i\cdot\exp\left[-i\pi(2\xi)^2\right]}$$

$$\begin{aligned}
g[x] &= \mathcal{F}_1^{-1}\left\{2\cdot i\cdot\exp\left[-i\pi(2\xi)^2\right]\right\} \\
&= 2\cdot i\cdot\mathcal{F}_1^{-1}\left\{\exp\left[-i\pi(2\xi)^2\right]\right\} \\
&= 2\cdot\exp\left[+i\frac{\pi}{2}\right]\cdot\left(\frac{1}{2}\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{x}{2}\right)^2\right]\right) \\
&= \exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{x}{2}\right)^2\right] \\
g[x] &= \left(\frac{1+i}{\sqrt{2}}\right)\cdot\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + i\cdot\sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right)
\end{aligned}$$

$$\begin{aligned}
g[x] &= \frac{1}{\sqrt{2}}\cdot\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + \frac{i^2}{\sqrt{2}}\cdot\sin\left[\pi\left(\frac{x}{2}\right)^2\right] + i\cdot\frac{1}{\sqrt{2}}\cdot\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) \\
&= \frac{1}{\sqrt{2}}\cdot\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] - \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) + i\cdot\frac{1}{\sqrt{2}}\cdot\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\{g[x]\} &= \frac{1}{\sqrt{2}}\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] - \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) \\
&= \cos\left[\pi\left(\frac{x}{2}\right)^2\right]\cos\left[\frac{\pi}{4}\right] - \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\sin\left[\frac{\pi}{4}\right] \\
&= \cos\left[\pi\left(\frac{x}{2}\right)^2 + \frac{\pi}{4}\right] \\
\mathrm{Im}\{g[x]\} &= \frac{1}{\sqrt{2}}\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) \\
&= \cos\left[\pi\left(\frac{x}{2}\right)^2\right]\cos\left[\frac{\pi}{4}\right] + \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\sin\left[\frac{\pi}{4}\right] \\
&= \cos\left[\pi\left(\frac{x}{2}\right)^2 - \frac{\pi}{4}\right]
\end{aligned}$$



$\mathrm{Re}\left\{g\left[x\right]\right\}$ *as solid black line and* $\mathrm{Im}\left\{g\left[x\right]\right\}$ *as dashed red line.*

(b) Find and sketch the output $g[x]$ if $h[x] = s[-x]$

In this case where the function $s[x]$ *is even,* $s[x] * s[-x] = s[x] * s[x]$*; the output is unchanged!*

$$g[x] = \left(\frac{1+i}{\sqrt{2}}\right) \cdot \left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + i \cdot \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right)$$

$$\boxed{g[x] = \cos\left[\pi\left(\frac{x}{2}\right)^2 + \frac{\pi}{4}\right] + i \cdot \cos\left[\pi\left(\frac{x}{2}\right)^2 - \frac{\pi}{4}\right]}$$

Note that if we were to make $h[x] = s^*[-x]$ *then the output is a Dirac delta function:*

$$\begin{aligned} g[x] &= \left(\exp\left[+i\pi\frac{x^2}{2}\right]\right) * \left(\exp\left[+i\pi\frac{(-x)^2}{2}\right]\right)^* \\ &= \left(\exp\left[+i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right]\right) * \left(\exp\left[-i\pi\left(\frac{x}{\sqrt{2}}\right)^2\right]\right) \end{aligned}$$

$$\begin{aligned} G[\xi] &= \left(\sqrt{2} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\left(\sqrt{2}\xi\right)^2\right]\right) \cdot \left(\sqrt{2} \cdot \exp\left[-i\frac{\pi}{4}\right] \cdot \exp\left[+i\pi\left(\sqrt{2}\xi\right)^2\right]\right) \\ &= 2 \cdot 1[\xi] \\ g[x] &= 2 \cdot \delta[x] \end{aligned}$$

(c) Describe the impulse response $h[x]$ and transfer function $H[\xi]$ of the matched filter that will maximize the output at $x = x_0$, i.e., at some arbitrary location.

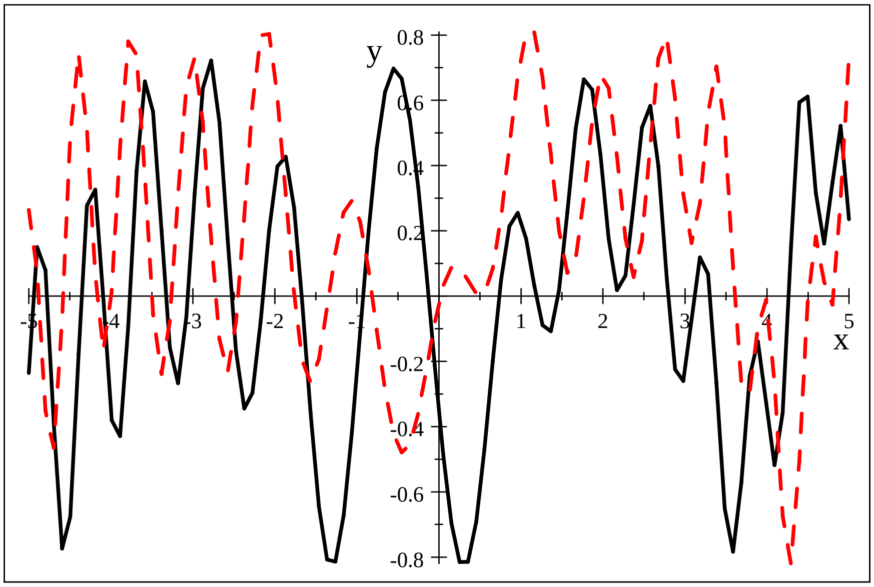
$$\begin{aligned} h[x] &= s^*[-x] \ \textit{centered at}\ x = +x_0 \\ h[x] &= s^*[-(x - x_0)] = s^*[-x + x_0] \end{aligned}$$

(d) Find and sketch the output if $h[x] = s[-x]$ and the input is

$$\begin{aligned} & s[x] * \left(\frac{1}{2}\delta[x-2] + \frac{1}{3}\delta[x+3]\right) \\ s[x] * \left(\frac{1}{2}\delta[x-2] + \frac{1}{3}\delta[x+3]\right) * s[-x] &= s[x] * s[-x] * \left(\frac{1}{2}\delta[x-2] + \frac{1}{3}\delta[x+3]\right) \\ &= \frac{1}{2} \cdot \left(s[x] * s[-x]\right)\Big|_{x \to x-2} + \frac{1}{3} \cdot \left(s[x] * s[-x]\right)\Big|_{x \to x+2} \end{aligned}$$

We get replicas of the output of part 2(a) centered at $x = +2$ *and at* $x = -3$ *with respective scalings of* $\frac{1}{2}$ *and* $\frac{1}{3}$.

$$\begin{aligned} \frac{1}{2} \cdot \left(s[x] * s[-x]\right)\Big|_{x \to x-2} &= \frac{1}{2} \cos\left[\pi\left(\frac{x-2}{2}\right)^2 + \frac{\pi}{4}\right] + i \cdot \cos\left[\pi\left(\frac{x-2}{2}\right)^2 - \frac{\pi}{4}\right] \\ \frac{1}{3} \cdot \left(s[x] * s[-x]\right)\Big|_{x \to x+3} &= \frac{1}{3} \cos\left[\pi\left(\frac{x+3}{2}\right)^2 + \frac{\pi}{4}\right] + i \cdot \cos\left[\pi\left(\frac{x+3}{2}\right)^2 - \frac{\pi}{4}\right] \end{aligned}$$



$\mathrm{Re}\{g[x]\}$ *as solid black line and* $\mathrm{Im}\{g[x]\}$ *as dashed red line.*

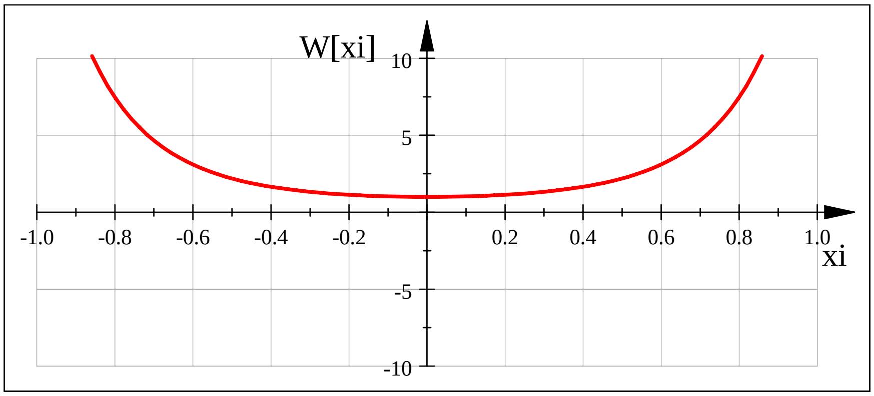
If the impulse response is the complex conjugate, we get Dirac delta functions at $x = +2$ *and at* $x = -3$ *with respective scalings of* $\frac{1}{2}$ *and* $\frac{1}{3}$.

$$\begin{aligned} \mathrm{Re}\{g[x]\} &= \frac{1}{\sqrt{2}}\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] - \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) \\ \mathrm{Im}\{g[x]\} &= \frac{1}{\sqrt{2}}\left(\cos\left[\pi\left(\frac{x}{2}\right)^2\right] + \sin\left[\pi\left(\frac{x}{2}\right)^2\right]\right) \end{aligned}$$

3. The transfer functions listed below describe the actions of different LSI systems. The goal of this problem is to find the corresponding inverse filter. In each case, sketch the transfer function of the inverse filter. ALSO *in those cases where it is possible*, evaluate and sketch the impulse response of the inverse filter. You may use reasonable approximations where appropriate – the sketches will be helpful here.

   (a) $H[\xi] = GAUS[\xi]$

$$\begin{aligned} h[x] &= \mathcal{F}_1^{-1}\{GAUS[\xi]\} = GAUS[x] = \exp\left[-\pi x^2\right] \\ H[\xi] &= \exp\left[-\pi \xi^2\right] \Longrightarrow W[\xi] = \frac{1}{H[\xi]} = \exp\left[+\pi \xi^2\right] \end{aligned}$$

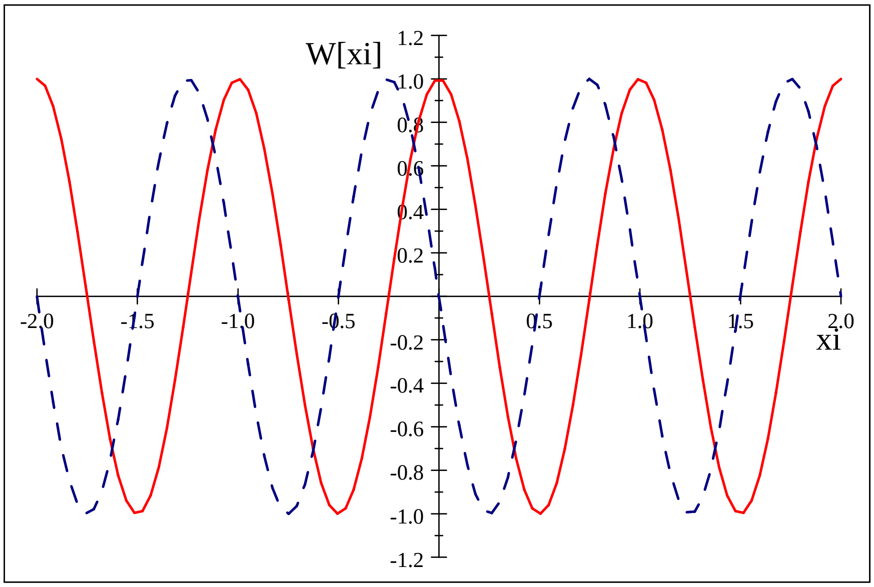


Since $W[\xi]$ has infinite amplitude and infinite area, $w[x]$ is not defined. Clearly $W[\xi]$ is a "high-boost filter."

(b) $H[\xi] = \exp[+2\pi i\xi]$

$$\begin{aligned} h[x] &= \mathcal{F}_1^{-1}\{\exp[+2\pi i\xi]\} = \delta[x+1] \\ W[\xi] &= \frac{1}{H[\xi]} = \frac{1}{\exp[+2\pi i\xi]} = \exp[-2\pi i\xi] = H^*[\xi] \end{aligned}$$

The inverse filter of the translation operator must perform the same translation in the opposite direction. Both $H[\xi]$ *and* $W[\xi]$ *are allpass linear-phase filters.*



Real part of $W[\xi]$ *(solid rad) and imaginary part of* $W[\xi]$ *(dashed blue)*

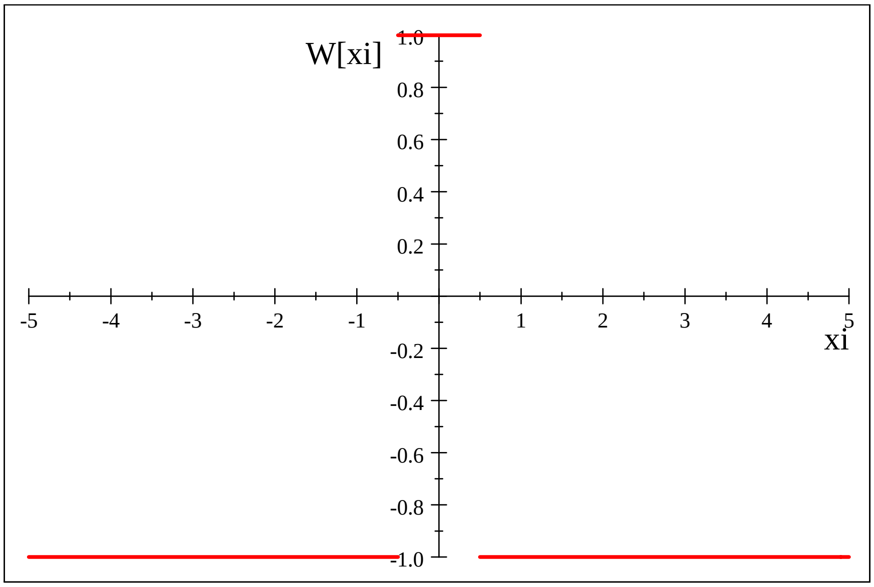
$$\begin{aligned} w[x] &= \mathcal{F}_1^{-1}\{H^*[\xi]\} = h^*[-x] = (\delta[-x+1])^* = \delta[-(x-1)] \\ &\boxed{w[x] = \delta[x-1]} \end{aligned}$$

(c) $H[\xi] = \exp[+i\pi(1 - RECT[\xi])]$

$$\begin{aligned} H[\xi] &= \begin{cases} e^0 = 1 & \text{if} \quad RECT[\xi] = 1 \\ e^{+i\frac{\pi}{2}} = +i & \text{if} \quad RECT[\xi] = \frac{1}{2} \\ e^{+i\pi} = -1 & \text{if} \quad RECT[\xi] = 0 \end{cases} \\ &= -1 + 2 \cdot RECT[\xi] \\ &= 2 \cdot RECT[\xi] - 1[\xi] \quad \textit{except for the endpoints of the rectangle.} \end{aligned}$$

$$h[x] = 2 \cdot SINC[x] - \delta[x]$$

$$\begin{aligned} W[\xi] &= \frac{1}{H[\xi]} = \begin{cases} \frac{1}{e^0} = 1 & \text{if} \quad RECT[\xi] = 1 \\ \frac{1}{e^{+i\frac{\pi}{2}}} = -i & \text{if} \quad RECT[\xi] = \frac{1}{2} \\ \frac{1}{e^{+i\pi}} = -1 & \text{if} \quad RECT[\xi] = 0 \end{cases} \\ &= -1 + 2 \cdot RECT[\xi] \\ &= 2 \cdot RECT[\xi] - 1[\xi] \end{aligned}$$



(except for the endpoints of the rectangle, which have no effect on the computation of the inverse Fourier transform). The impulse response is:

$$w[x] = h[x] = 2 \cdot SINC[x] - \delta[x]$$

So except for the endpoints of the rectangle, the transfer functions of the filter and inverse filter are identical.

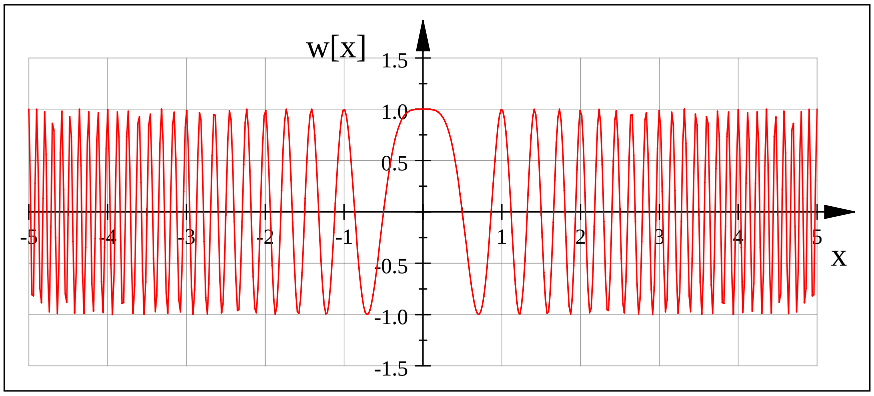
(d) $H[\xi] = \exp\left[+i\pi\frac{\xi^2}{2}\right]$

$$\begin{aligned}
H[\xi] &= \exp\left[+i\pi\frac{\xi^2}{2}\right] = \exp\left[+i\pi\left(\frac{\xi}{\sqrt{2}}\right)^2\right] \\
h[x] &= \mathcal{F}_1^{-1}\left\{\exp\left[+i\pi\left(\frac{\xi}{\sqrt{2}}\right)^2\right]\right\} = \sqrt{2}\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\sqrt{2}x\right)^2\right] \\
&= \sqrt{2}\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-2\pi i x^2\right]
\end{aligned}$$

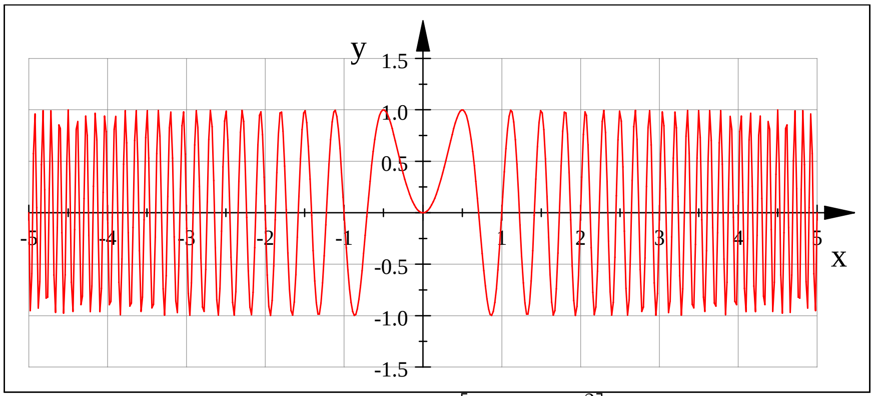
$$\begin{aligned}
W[\xi] &= (H[\xi])^{-1} = \left(\exp\left[+i\pi\frac{\xi^2}{2}\right]\right)^{-1} = \exp\left[-i\pi\left(\frac{\xi}{\sqrt{2}}\right)^2\right] \\
w[x] &= \mathcal{F}_1^{-1}\left\{\exp\left[-i\pi\left(\frac{\xi}{\sqrt{2}}\right)^2\right]\right\} = \sqrt{2}\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\sqrt{2}x\right)^2\right]
\end{aligned}$$

So the inverse filter is allpass quadratic phase, just like the original filter $h[x]$

$$\begin{aligned}
w[x] &= \sqrt{2}\cdot\left(\frac{1-i}{\sqrt{2}}\right)\cdot\cos\left[+\pi\left(\sqrt{2}x\right)^2\right] + i\cdot\sin\left[+\pi\left(\sqrt{2}x\right)^2\right] \\
\operatorname{Re}\{w[x]\} &= \cos\left[+\pi\left(\sqrt{2}x\right)^2\right]
\end{aligned}$$



$\operatorname{Re}\{w[x]\} = \cos\left[2\pi x^2\right]$



$\operatorname{Im}\{w[x]\} = \sin\left[+\pi\left(\sqrt{2}x\right)^2\right]$

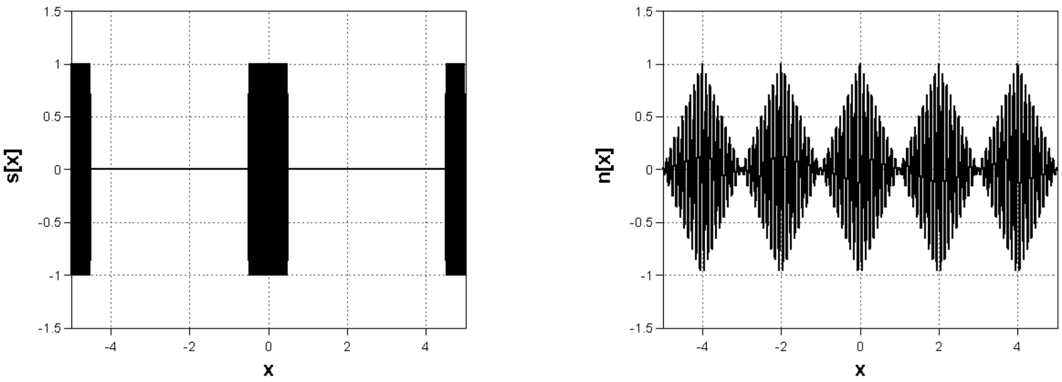
4. A desired signal $f[x]$ is corrupted by additive noise $n[x]$, and you are required to design a filter that can be used to recover the signal for the following cases. Find a transfer function $H[\xi]$ such that the output $g[x] \cong s[x]$ when the input is $f[x] = s[x] + n[x]$. Sketch $f[x]$ and $g[x]$.

(a)

$$\begin{aligned}
s[x] &= \left(\frac{1}{5}COMB\left[\frac{x}{5}\right] * RECT[x]\right) \cdot \cos[60\pi x] \\
&= \left(\frac{1}{5}COMB\left[\frac{x}{5}\right] * RECT[x]\right) \cdot \cos\left[2\pi\frac{x}{\left(\frac{1}{30}\right)}\right] \implies \xi_s = 30 \\
n[x] &= \left(\frac{1}{2}COMB\left[\frac{x}{2}\right] * TRI[x]\right) \cdot \cos[20\pi x] \\
&= \left(\frac{1}{2}COMB\left[\frac{x}{2}\right] * TRI[x]\right) \cdot \cos\left[2\pi\frac{x}{\left(\frac{1}{10}\right)}\right] \implies \xi_n = 10
\end{aligned}$$

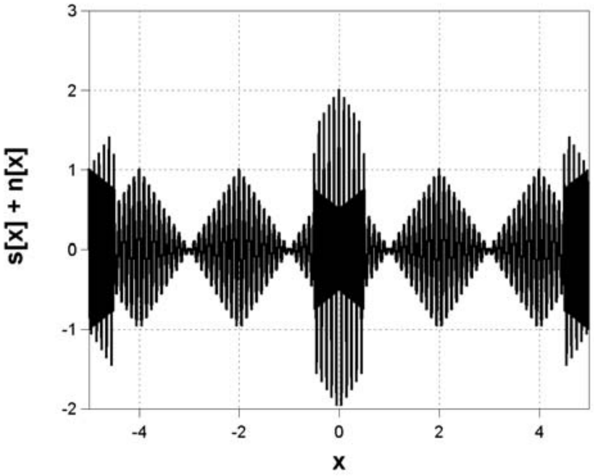
$$\begin{aligned}
S[\xi] &= (COMB[5\xi] \cdot SINC[\xi]) * \left(\frac{1}{2} \cdot (\delta[\xi+30] + \delta[\xi-30])\right) \\
&= \left(\frac{1}{2}SINC[\xi] \cdot \sum_{k=-\infty}^{+\infty} \delta[5\xi - k]\right) * (\delta[\xi+30] + \delta[\xi-30]) \\
&= \left(\frac{1}{2}SINC[\xi] \cdot \sum_{k=-\infty}^{+\infty} \delta\left[5\left(\xi - \frac{k}{5}\right)\right]\right) * (\delta[\xi+30] + \delta[\xi-30]) \\
&= \left(\frac{1}{2}SINC[\xi] \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{5}\delta\left[\xi - \frac{k}{5}\right]\right) * (\delta[\xi+30] + \delta[\xi-30]) \\
&= \left(\frac{1}{10}SINC[\xi] \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{5}\right]\right) * (\delta[\xi+30] + \delta[\xi-30])
\end{aligned}$$

$$\begin{aligned}
N[\xi] &= \mathcal{F}\left\{\left(\frac{1}{2}COMB\left[\frac{x}{2}\right] * TRI[x]\right) \cdot \cos\left[2\pi\frac{x}{\left(\frac{1}{10}\right)}\right]\right\} \\
&= \left(COMB[2\xi] \cdot SINC^2[\xi]\right) * \frac{1}{2}(\delta[\xi+10] + \delta[\xi-10]) \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta[2\xi - k] \cdot SINC^2[\xi]\right) * \frac{1}{2}(\delta[\xi+10] + \delta[\xi-10]) \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta\left[2 \cdot \left(\xi - \frac{k}{2}\right)\right] \cdot SINC^2[\xi]\right) * \frac{1}{2}(\delta[\xi+10] + \delta[\xi-10]) \\
&= \left(\sum_{k=-\infty}^{+\infty} \frac{1}{2}\delta\left[\xi - \frac{k}{2}\right] \cdot SINC^2[\xi]\right) * \frac{1}{2}(\delta[\xi+10] + \delta[\xi-10])
\end{aligned}$$



$s[x] = \left(\frac{1}{5}COMB\left[\frac{x}{5}\right] * RECT[x]\right) \cdot \cos[60\pi x]$ *and* $n[x] = \left(\frac{1}{2}COMB\left[\frac{x}{2}\right] * TRI[x]\right) \cdot \cos[20\pi x]$ *Note that the oscillation frequency of the signal is larger than that of the noise, so the oscillations on the left are "jammed tight."*

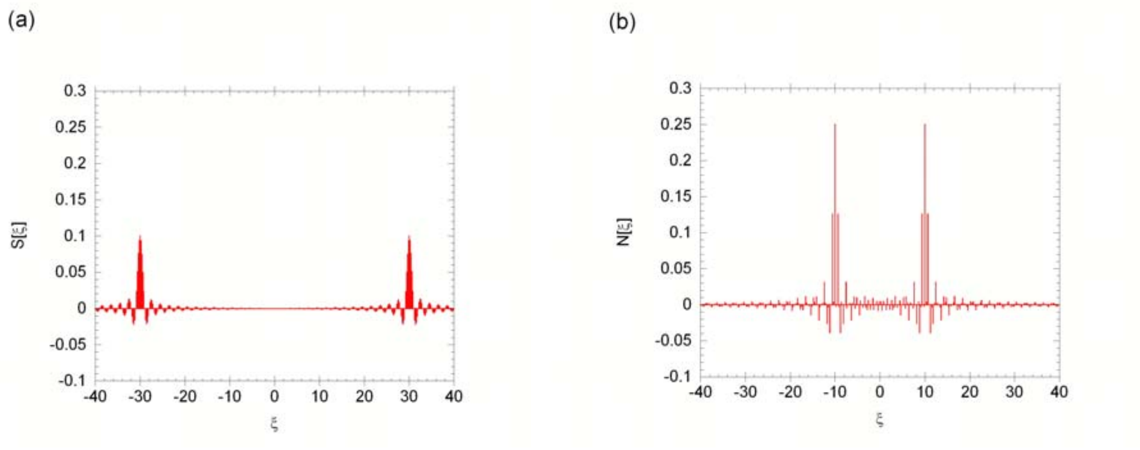
$$s[x]+n[x] = \left(\frac{1}{5}COMB\left[\frac{x}{5}\right] * RECT[x]\right)\cos\left[2\pi\frac{x}{\left(\frac{1}{30}\right)}\right]+\left(\frac{1}{2}COMB\left[\frac{x}{2}\right] * TRI[x]\right)\cos\left[2\pi\frac{x}{\left(\frac{1}{10}\right)}\right]$$



The sum of the signal and noise.

Evaluate the spectra of the signal and noise.

$$\begin{aligned}
S[\xi] &= (COMB[5\xi]\cdot SINC[\xi]) * \frac{1}{2}(\delta[\xi+30]+\delta[\xi-30]) \\
&= \left(\sum_{k=-\infty}^{+\infty}\delta[5\xi-k]\cdot SINC[\xi]\right) * \frac{1}{2}(\delta[\xi+30]+\delta[\xi-30]) \\
&= \left(\frac{1}{5}\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{5}\right]\cdot SINC[\xi]\right) * \frac{1}{2}(\delta[\xi+30]+\delta[\xi-30]) \\
&= \frac{1}{10}\left(\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{5}\right]\cdot SINC[\xi]\right) * (\delta[\xi+30]+\delta[\xi-30]) \\
N[\xi] &= \left(COMB[2\xi]\cdot SINC^2[\xi]\right) * \frac{1}{2}(\delta[\xi+10]+\delta[\xi-10]) \\
&= \left(\frac{1}{4}\sum_{k=-\infty}^{+\infty}\delta\left[\xi-\frac{k}{2}\right]\cdot SINC^2[\xi]\right) * (\delta[\xi+10]+\delta[\xi-10])
\end{aligned}$$



(a) $S[\xi]$*, (b)* $N[\xi]$ *plotted to same scale*

At this point, we need to choose the form of the transfer function of the filter. The "suboptimum" solution in Eq. 19.46 is:

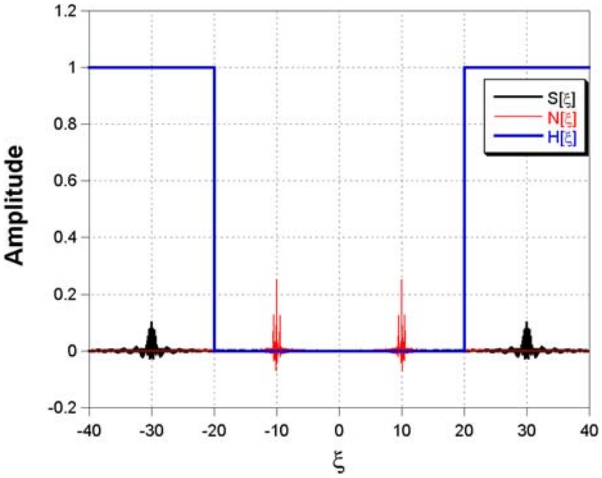
$$H[\xi] = \frac{S[\xi]}{S[\xi]+N[\xi]}$$

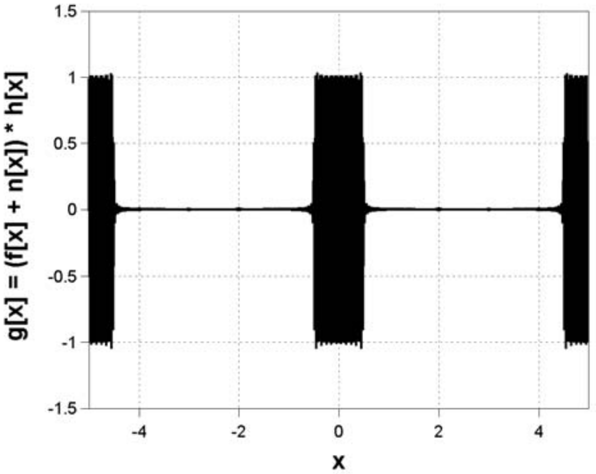
The optimum solution derived by Wiener (Eq. 19.79) is:

$$W[\xi] = \frac{|S[\xi]|^2}{|S[\xi]|^2+|N[\xi]|^2}$$

For these parameters, both are difficult to evaluate exactly, but a conceptual solution is straightforward from the graphs. An approximate solution is obtained by "cutting out" the noise spectra at the midpoints between the maxima of the noise and signal spectra. Since the maxima of the noise spectrum at located at $\xi = \pm 10$ *and those of the signal at* $\xi = \pm 30$*, a reasonable solution is a highpass filter with cutoff frequencies at* $\xi = \pm 20$*:*

$$\begin{aligned}
H[\xi] &\simeq 1 - RECT\left[\frac{\xi}{40}\right] \\
h[x] &\simeq \delta[x] - 40\ SINC\left[\frac{x}{\left(\frac{1}{40}\right)}\right]
\end{aligned}$$

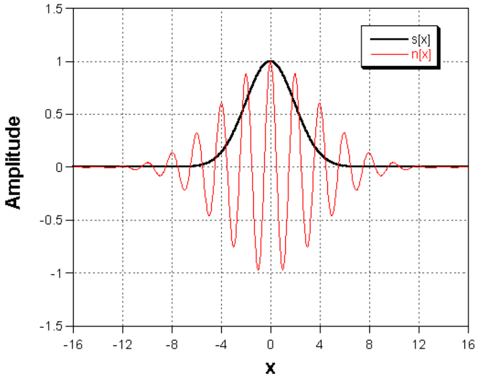


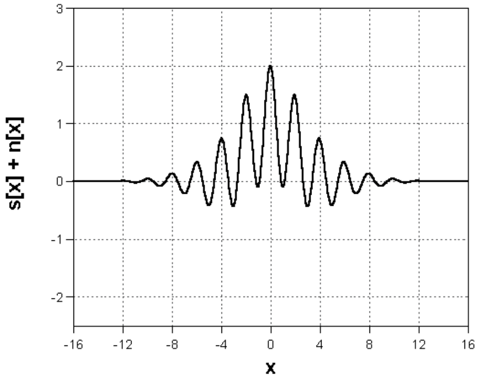


The difference between results obtained for the two different types of filters is quite small.

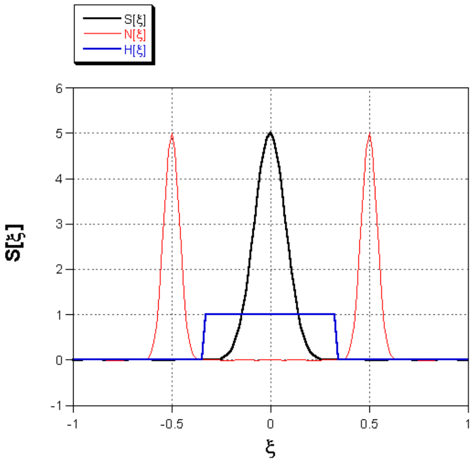
(b)

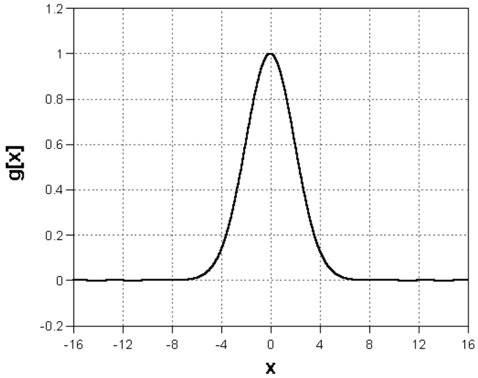
$$\begin{aligned} s\left[x\right] &= GAUS\left[\frac{x}{5}\right] \\ n\left[x\right] &= GAUS\left[\frac{x}{10}\right]\cdot\cos\left[\pi x\right] = GAUS\left[\frac{x}{10}\right]\cos\left[2\pi\frac{x}{2}\right] \end{aligned}$$





$$\begin{aligned} S\left[\xi\right] &= 5\cdot GAUS\left[5\xi\right] = 5\cdot GAUS\left[\frac{\xi}{\left(\frac{1}{5}\right)}\right] \\ N\left[\xi\right] &= 10\ GAUS\left[10\xi\right] * \frac{1}{2}\left(\delta\left[\xi+\frac{1}{2}\right]+\delta\left[\xi-\frac{1}{2}\right]\right) \\ &= 5\cdot\left(GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]+GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right) \end{aligned}$$





Using the "nonoptimum" solution:

$$\begin{aligned}
H[\xi] &= \frac{S[\xi]}{S[\xi]+N[\xi]} = \frac{GAUS\left[\frac{\xi}{\left(\frac{1}{5}\right)}\right]}{GAUS\left[\frac{\xi}{\left(\frac{1}{5}\right)}\right] + \left(GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10}\right)}\right] + GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right)} \\
&\simeq \begin{cases} 1 & \text{if} \quad GAUS[5\xi] > GAUS[10(\xi \pm 0.5)] \\ 0 & \qquad\qquad \textit{otherwise} \end{cases} \\
&\Longrightarrow e^{-25\pi\xi^2} - e^{-100\pi\left(\xi-\frac{1}{2}\right)^2} = 0 \\
&\Longrightarrow e^{-75\pi\xi^2+100\pi\xi-25\pi} = 1 \\
&\Longrightarrow 75\pi\xi^2 - 100\pi\xi + 25\pi = 0 \Longrightarrow 3\xi^2 - 4\xi + 1 = 0 \Longrightarrow \xi = \frac{1}{3} \textit{ and } \xi = +1
\end{aligned}$$

Obviously, the desired solution is $\xi = \frac{1}{3}$*, so:*

$$\begin{aligned}
H[\xi] &= \frac{S[\xi]}{S[\xi]+N[\xi]} \simeq RECT\left[\frac{\xi}{\left(\frac{2}{3}\right)}\right] \\
h[x] &\simeq \frac{2}{3} SINC\left[\frac{x}{\left(\frac{3}{2}\right)}\right]
\end{aligned}$$

$$f[x] = \left. \begin{array}{cc} +1 & \text{if} \quad -\frac{1}{3} < x < +\frac{1}{3} \\ 0 & \text{if} \quad x \geq +\frac{1}{3} \end{array} \right|$$

The result obtained for the optimum solution is:

$$\begin{aligned}
W[\xi] &= \frac{|S[\xi]|^2}{|S[\xi]|^2 + |N[\xi]|^2} = \frac{\left|GAUS\left[\frac{\xi}{\left(\frac{1}{5}\right)}\right]\right|^2}{\left|GAUS\left[\frac{\xi}{\left(\frac{1}{5}\right)}\right]\right|^2 + \left|GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10}\right)}\right] + GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right|^2} \\
&= \frac{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right]}{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right] + \left|GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10}\right)}\right] + GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right|^2}
\end{aligned}$$

Since the Gaussians in the noise spectrum are "almost" disjoint, we can approximate the squared magnitude of the sum as the sum of the squared magnitudes:

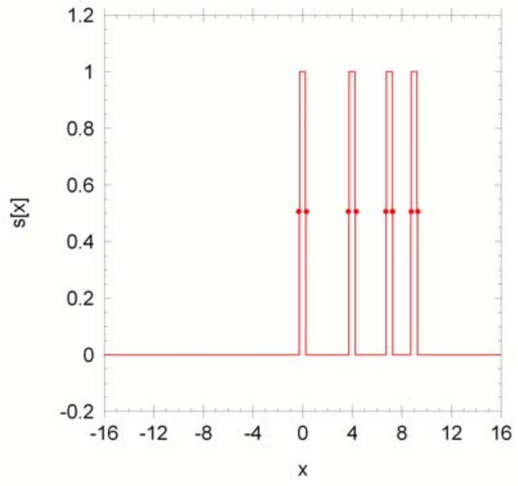
$$\begin{aligned}
W[\xi] &\cong \frac{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right]}{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right] + \left|GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right|^2 + \left|GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10}\right)}\right]\right|^2} \\
&= \frac{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right]}{GAUS\left[\frac{\xi}{\left(\frac{1}{5\sqrt{2}}\right)}\right] + GAUS\left[\frac{\xi+\frac{1}{2}}{\left(\frac{1}{10\sqrt{2}}\right)}\right] + GAUS\left[\frac{\xi-\frac{1}{2}}{\left(\frac{1}{10\sqrt{2}}\right)}\right]}
\end{aligned}$$

In words, the Gaussian components of the transfer function are "narrower" in this case than in the suboptimum case, so the prescription for the filter is approximately the same

5. The input to an LSI system is:

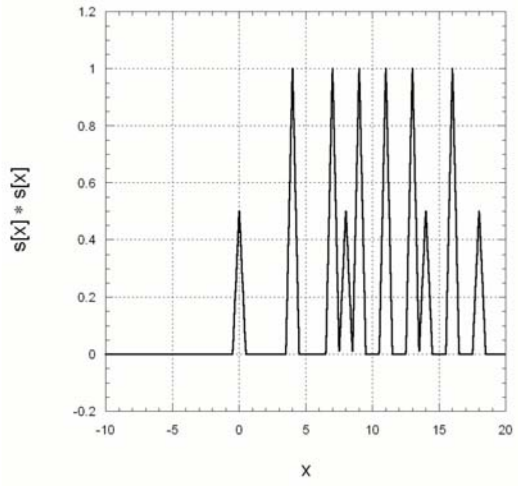
$$s[x] = RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])$$

often useful to sketch the input:



(a) Find and sketch the output $g[x]$ when $h[x] = s[x]$

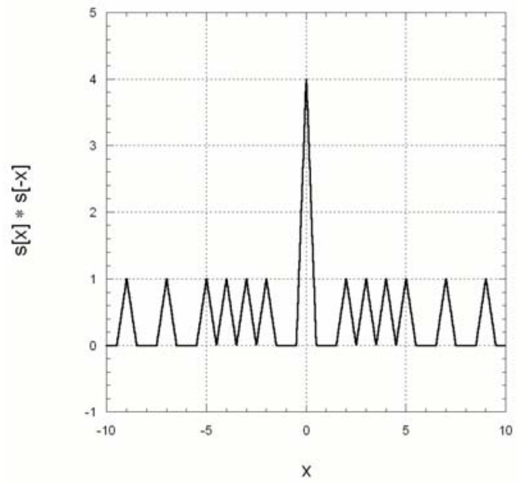
$$\begin{aligned}
g[x] &= (RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])) \\
&\quad * (RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])) \\
&= RECT[2x] * RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\
&\quad * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\
&= \frac{1}{2} TRI[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\
&\quad * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\
&= (\delta[x] + 2 \cdot \delta[x-4] + 2 \cdot \delta[x-7] + \delta[x-8] + 2 \cdot \delta[x-9] \\
&\quad + 2\delta[x-11] + 2\delta[x-13] + \delta[x-14] + 2\delta[x-16] + \delta[x-18]) * \frac{1}{2} TRI[2x]
\end{aligned}$$



$s[x] * s[x]$

(b) Find and sketch the output $g[x]$ when $h[x] = s[-x]$

$$\begin{aligned}
g[x] &= (RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])) \\
&\quad * (RECT[2x] * (\delta[x] + \delta[x+4] + \delta[x+7] + \delta[x+9])) \\
&= RECT[2x] * RECT[2x] * (\delta[x+9] + \delta[x+7] + \delta[x+5] + \delta[x+4] + \delta[x+3] \\
&\quad + \delta[x+2] + 4\delta[x] + \delta[x-2] + \delta[x-3] + \delta[x-4] + \delta[x-5] + \delta[x-7] + \delta[x-9]) \\
&= \frac{1}{2} TRI[2x] * (\delta[x+9] + \delta[x+7] + +\delta[x+5] + \delta[x+4] + \delta[x+3] + \delta[x+2] \\
&\quad + 4\delta[x] + \delta[x-2] + \delta[x-3] + \delta[x-4] + \delta[x-5] + \delta[x-7] + \delta[x-9])
\end{aligned}$$



$s[x] * s[-x] = s[x] \bigstar s[x]$ *(note change of scales compared to graph of* $s[x] * s[x]$*).*

This is the autocorrelation of the real-valued function $s[x]$ *which is guaranteed to be even and have its maximum value at the origin.*

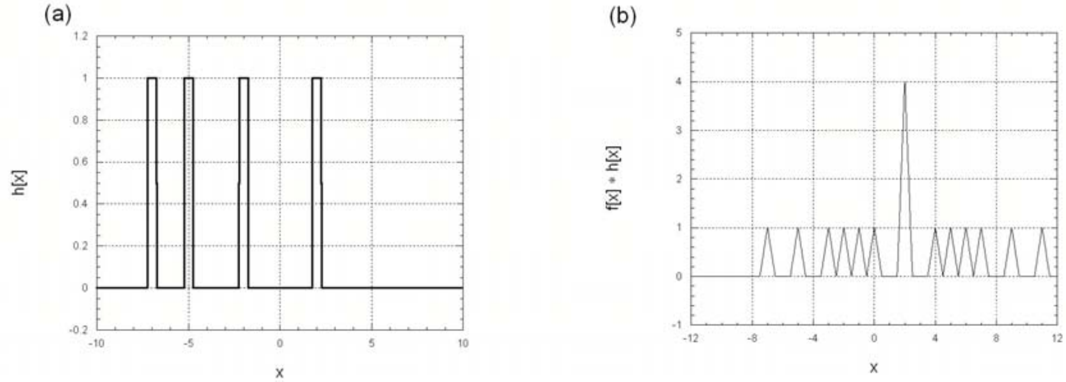
(c) Describe the impulse response $h[x]$ and transfer function $H[\xi]$ (assume that $H[0] = 1$) of the matched filter that will maximize the output at $x = 2$. Sketch the output.

The goal is to maximize the output at $x = 2$*, NOT NECESSARILY to generate a Dirac delta function at* $x = +2$*. We can evaluate the autocorrelation from part (b), which is guaranteed to have its maximum value at* $x = 0$*, and then translate the output to the desired location of* $x = +2$*. The desired output would be:*

$$\begin{aligned}
g[x] &= (s[x] \bigstar s[x]) * \delta[x-2] \\
&= (s[x] * s^*[-x]) * \delta[x-2] \\
&= s[x] * (s^*[-x] * \delta[x-2]) \\
&= s[x] * (s[-x] * \delta[x-2]) \;\; \textit{because } s[x] = s^*[-x] \\
&\Longrightarrow \boxed{h[x] = s[-x] * \delta[x-2] = s[-(x-2)] = s[-x+2]} \\
h[x] &= (RECT[-2x] * (\delta[-x] + \delta[-x-4] + \delta[-x-7] + \delta[-x-9])) * \delta[x-2] \\
&= (RECT[-2x] * (\delta[+x] + \delta[x+4] + \delta[x+7] + \delta[x+9])) * \delta[x-2] \\
& \boxed{h[x] = RECT[2x] * (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7])}
\end{aligned}$$

The transfer function is:

$$\begin{aligned}
H[\xi] &= \mathcal{F}_1\{RECT[2x] * (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7])\} \\
&= \frac{1}{2}SINC\left[\frac{x}{2}\right] \cdot (\exp[-2\pi\xi \cdot (+2)] + \exp[-2\pi\xi \cdot (-2)] + \exp[-2\pi\xi \cdot (-5)] + \exp[-2\pi\xi \cdot (-7)])
\end{aligned}$$



(d) For this $s[x]$, does the transfer function $H[\xi]$ exist such that $g[x] = s[x] * h[x] = \delta[x-2]$? Explain your answer.

The spectrum of $s[x]$ *is:*

$$S[\xi] = \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \cdot (1[\xi] + \exp[-2\pi \cdot 4 \cdot \xi] + \exp[-2\pi \cdot 7 \cdot \xi] + \exp[-2\pi \cdot 9 \cdot \xi])$$

which has zeros at nonzero integer multiples of $\xi = 2$ *because the SINC function has zeros there. The only way we can get a Dirac delta function at* $x = +2$ *is to construct a matched filter transfer function that is the reciprocal of* $S[\xi]$*. The zeros in* $S[\xi]$ *preclude this, so the answer is* $\boxed{NO}$.

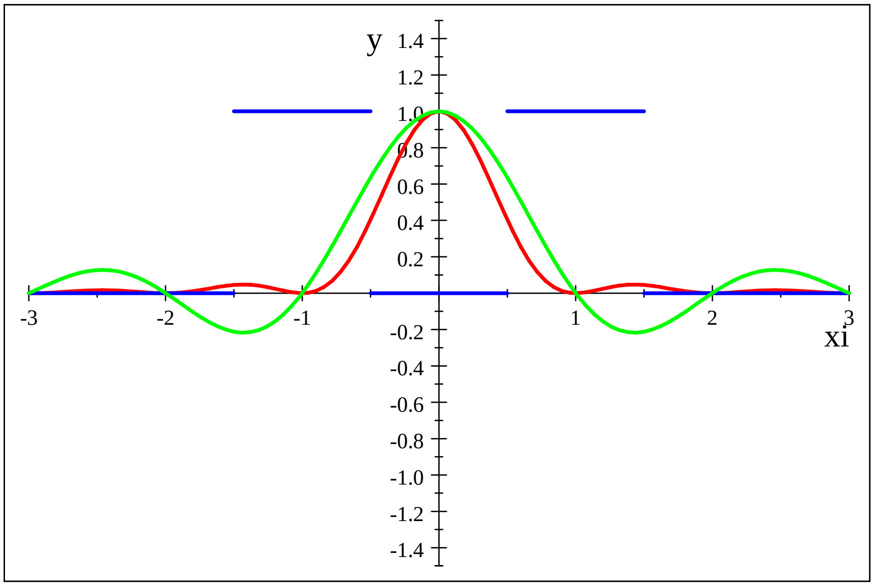
6. Design the Wiener-Helstrom filter for the following input signals, impulse responses, and noise power spectra.

   The transfer function of the Wiener-Helstrom filter is

$$W[\xi] = \frac{H^*[\xi]}{|H[\xi]|^2 + \frac{|N[\xi]|^2}{|F[\xi]|^2}}$$

   (a) $f[x] = RECT[x]$, $h[x] = RECT[x]$, $|N[\xi]|^2 = RECT[\xi+1] + RECT[\xi-1]$

$$\begin{aligned} f[x] &= RECT[x] \implies F[\xi] = SINC[\xi] \implies |F[\xi]|^2 = SINC^2[\xi] \\ h[x] &= RECT[x] \implies H[\xi] = SINC[\xi] \implies H^*[\xi] = SINC[\xi] \implies |H[\xi]|^2 = SINC^2[\xi] \\ |N[\xi]|^2 &= RECT[\xi+1] + RECT[\xi-1] \end{aligned}$$



$|F[\xi]|^2$ *in red,* $H[\xi]$ *in green, and* $|N[\xi]|^2$ *in blue showing that the noise is large for* $+\frac{1}{2} < |\xi| < +\frac{3}{2}$

$$W[\xi] = \frac{SINC[\xi]}{SINC^2[\xi] + \frac{RECT[\xi+1] + RECT[\xi-1]}{SINC^2[\xi]}}$$

   i. *for* $\xi < -\frac{3}{2}$, $|F[\xi]|^2 = SINC^2[\xi], H^*[\xi] = SINC[\xi], |N[\xi]|^2 = 0 \implies W[\xi] = \frac{SINC[\xi]}{|SINC[\xi]|^2+0} = (SINC[\xi])^{-1}$

   ii. *for* $\xi = -\frac{3}{2}$, $\left|F\left[-\frac{3}{2}\right]\right|^2 = SINC^2\left[-\frac{3}{2}\right] = \left(\frac{2}{3\pi}\right)^2, H^*\left[\frac{3}{2}\right] = \left(-\frac{2}{3\pi}\right)$, $|N[\xi]|^2 = \frac{1}{2} \implies W\left[-\frac{3}{2}\right] = \frac{\left(-\frac{2}{3\pi}\right)}{\left(\frac{2}{3\pi}\right)^2 + \frac{\frac{1}{2}}{\left(\frac{2}{3\pi}\right)^2}} \cong -1.9 \times 10^{-2}$

   iii. *for* $-\frac{3}{2} < \xi < -\frac{1}{2}$, $|F[\xi]|^2 = SINC^2[\xi], |N[\xi]|^2 = 1 \implies W[\xi] = \frac{SINC[\xi]}{|SINC[\xi]|^2 + \frac{1}{|SINC[\xi]|^2}} = \frac{(SINC[\xi])^3}{(SINC[\xi])^4+1}$

   iv. *for* $\xi = -\frac{1}{2}$, $\left|F\left[-\frac{1}{2}\right]\right|^2 = SINC^2\left[-\frac{1}{2}\right] = \left(\frac{2}{3}\right)^2, H^*\left[\frac{1}{2}\right] = \left(\frac{2}{\pi}\right)$, $|N[\xi]|^2 = \frac{1}{2} \implies W\left[-\frac{1}{2}\right] = \frac{\left(\frac{2}{\pi}\right)}{\left(\frac{2}{\pi}\right)^2 + \frac{\frac{1}{2}}{\left(\frac{2}{\pi}\right)^2}} \cong +0.388$

   v. *for* $-\frac{1}{2} < \xi < +\frac{1}{2}$, $|F[\xi]|^2 = SINC^2[\xi], H^*[\xi] = SINC[\xi], |N[\xi]|^2 = 0 \implies W[\xi] = \frac{SINC[\xi]}{SINC^2[\xi]+0} = (SINC[\xi])^{-1}$

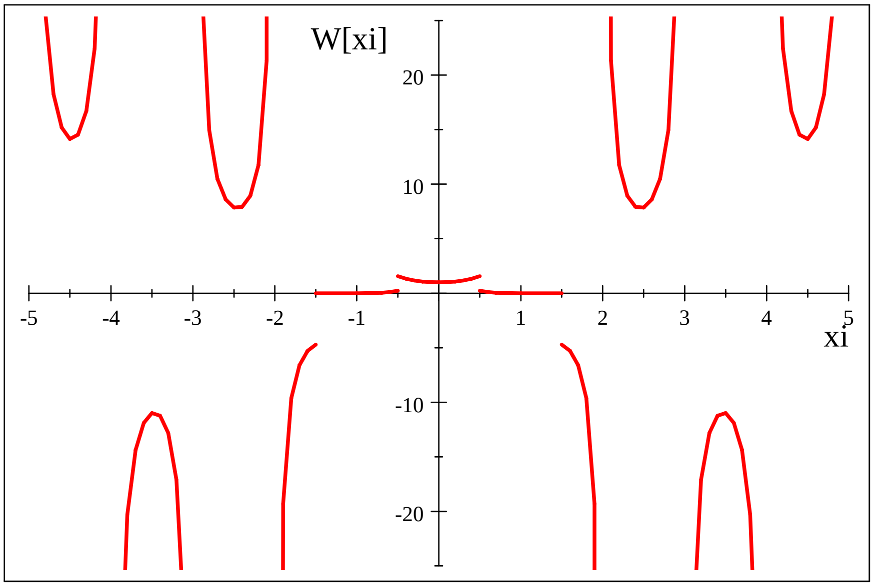
vi. *for* $\xi = +\frac{1}{2}$, $\left|F\left[\frac{1}{2}\right]\right|^2 = SINC^2\left[\frac{1}{2}\right] = \left(\frac{2}{3}\right)^2, H^*\left[\frac{1}{2}\right] = \left(\frac{2}{\pi}\right)$, $|N[\xi]|^2 = \frac{1}{2} \implies W\left[-\frac{1}{2}\right] = \frac{\left(\frac{2}{\pi}\right)}{\left(\frac{2}{\pi}\right)^2 + \frac{\frac{1}{2}}{\left(\frac{2}{\pi}\right)^2}} \cong +0.388$

vii. *for* $+\frac{1}{2} < \xi < +\frac{3}{2}$, $|F[\xi]|^2 = SINC^2[\xi], |N[\xi]|^2 = 1 \implies W[\xi] = \frac{SINC[\xi]}{|SINC[\xi]|^2 + \frac{1}{|SINC[\xi]|^2}} = \frac{(SINC[\xi])^3}{(SINC[\xi])^4 + 1}$

viii. *for* $\xi = +\frac{3}{2}$, $\left|F\left[\frac{3}{2}\right]\right|^2 = SINC^2\left[\frac{3}{2}\right] = \left(\frac{2}{3\pi}\right)^2, H^*\left[\frac{3}{2}\right] = \left(-\frac{2}{3\pi}\right)$, $|N[\xi]|^2 = \frac{1}{2} \implies W\left[-\frac{3}{2}\right] = \frac{\left(-\frac{2}{3\pi}\right)}{\left(\frac{2}{3\pi}\right)^2 + \frac{\frac{1}{2}}{\left(\frac{2}{3\pi}\right)^2}} \cong -1.9 \times 10^{-2}$

ix. *for* $+\frac{1}{2} < \xi < 1$, $|F[\xi]|^2 = 0, |N[\xi]|^2 > 1 - \xi \implies W[\xi] = 0$

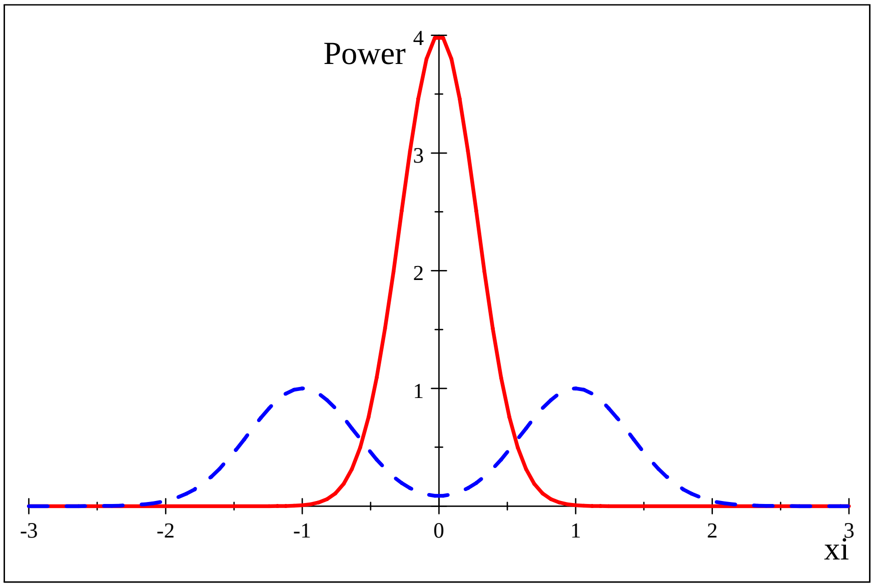
x. *for* $\xi > +\frac{3}{2}$, $|F[\xi]|^2 = SINC^2[\xi], H^*[\xi] = SINC[\xi], |N[\xi]|^2 = 0 \implies W[\xi] = \frac{SINC[\xi]}{|SINC[\xi]|^2 + 0} = (SINC[\xi])^{-1}$



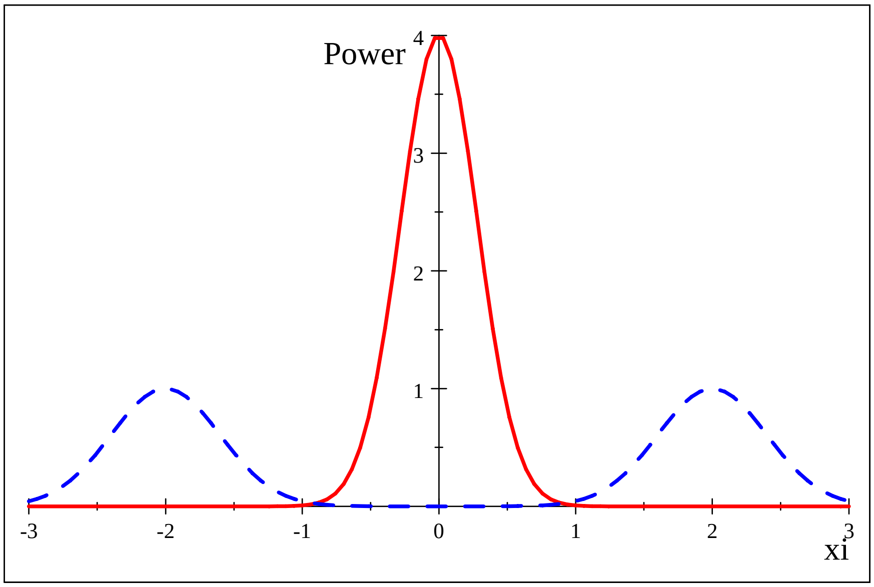
transfer function of Wiener-Helstrom filter showing "mild" amplification out to $|\xi| = \frac{1}{2}$ *cycle per unit length, then passes little information for* $\frac{1}{2} < |\xi| < \frac{3}{2}$ *where the noise is large, and finally acts like the inverse filter for* $|\xi| > \frac{3}{2}$ *where there is no noise.*

(b) $f[x] = 2\, GAUS[x]$, $h[x] = \delta[x]$, $|N[\xi]|^2 = GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0]$

$$\begin{aligned}
f[x] &= 2\cdot GAUS[x] \implies F[\xi] = 2\cdot GAUS[\xi] \\
&\implies |F[\xi]|^2 = 4\cdot(GAUS[\xi])^2 = 4\cdot\left(\exp\left[-\pi\xi^2\right]\right)^2 = 4\cdot\exp\left[-\pi\xi^2\cdot 2\right] = 4\cdot GAUS\left[\sqrt{2}\cdot\xi\right] \\
h[x] &= \delta[x] \implies H[\xi] = 1[\xi] \implies H^*[\xi] = 1[\xi] \implies |H[\xi]|^2 = 1[\xi] \\
|N[\xi]|^2 &= GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0]
\end{aligned}$$



Power spectra of signal and noise for $\xi_0 = 1, |F[\xi]|^2$ *as red solid line,* $|N[\xi]|^2$ *as blue dashed line*



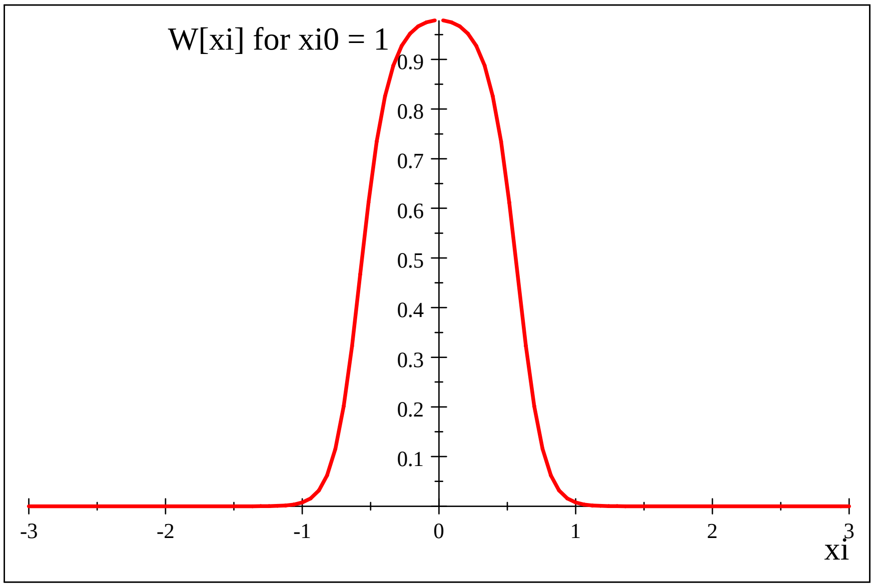
Power spectra of signal and noise for $\xi_0 = 2, |F[\xi]|^2$ *as red solid line,* $|N[\xi]|^2$ *as blue dashed line*

The fact that $h[x] = \delta[x]$ *means that this is really a Wiener filter rather than a Wiener-Helstrom filter:*

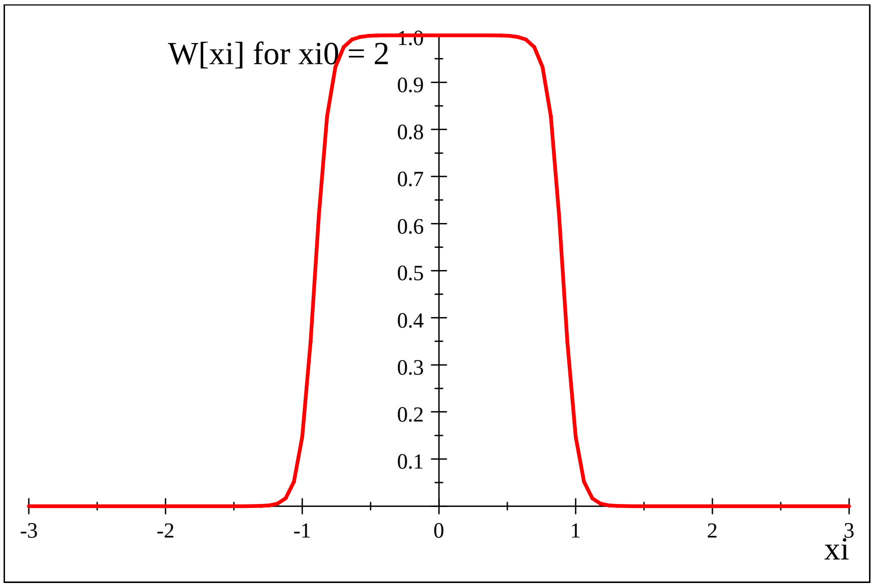
$$W[\xi] = \frac{1}{1+\frac{|N[\xi]|^2}{|F[\xi]|^2}} = \frac{|F[\xi]|^2}{|F[\xi]|^2 + |N[\xi]|^2}$$

$$
\begin{aligned}
f[x] &= 2\cdot GAUS[x] \implies F[\xi] = 2\cdot GAUS[\xi] = 2\cdot\exp\left[-\pi\xi^2\right] \\
&\implies |F[\xi]|^2 = 4\cdot GAUS^2[\xi] = 4\cdot\left(\exp\left[-\pi\xi^2\right]\right)^2 = 4\cdot\left(\exp\left[-2\pi\xi^2\right]\right) \\
&= 4\cdot\left(\exp\left[-\pi\left(\frac{\xi}{\frac{1}{\sqrt{2}}}\right)^2\right]\right) = 4\cdot GAUS\left[\sqrt{2}\cdot\xi\right]
\end{aligned}
$$

$$
\begin{aligned}
W[\xi] &= \frac{4\cdot\left(\exp\left[-\pi\left(\frac{\xi}{\frac{1}{\sqrt{2}}}\right)^2\right]\right)}{4\cdot\left(\exp\left[-\pi\left(\frac{\xi}{\frac{1}{\sqrt{2}}}\right)^2\right]\right)+2\cdot\cos\left[2\pi\xi\xi_0\right]\cdot\exp\left[-\pi\xi_0^2\right]\cdot\exp\left[-\pi\xi^2\right]} \\
&= \frac{2\cdot\left(\exp\left[-\pi\left(\frac{\xi}{\frac{1}{\sqrt{2}}}\right)^2\right]\right)}{2\cdot\left(\exp\left[-\pi\left(\frac{\xi}{\frac{1}{\sqrt{2}}}\right)^2\right]\right)+\left(\exp\left[-\pi\left(\xi+\xi_0\right)^2\right]+\exp\left[-\pi\left(\xi-\xi_0\right)^2\right]\right)}
\end{aligned}
$$



Wiener filter for $\xi_0 = 1$ showing fairly rapid falloff of transfer function with increasing frequency because of the location of the noise power spectrum.



Wiener filter for $\xi_0 = 2$ *showing "slower" falloff of transfer function with increasing frequency because the noise is relatively "farther out."*

Note that the transfer function of the filter gets "squarer" with increasing center frequency of the noise; put another way, the more disjoint the signal and noise power spectra, the shorter the "transition" of the filter.

7. A 1-D image $g[x]$ has been created by a double exposure of the original object $f[x]$. The original scene was translated between the exposures by the known distance $+b_0$. The object was stationary during the time that each image was collected, and the exposure time was the same in both cases.

For an input $f[x]$, the output image may be conveniently written in two forms:

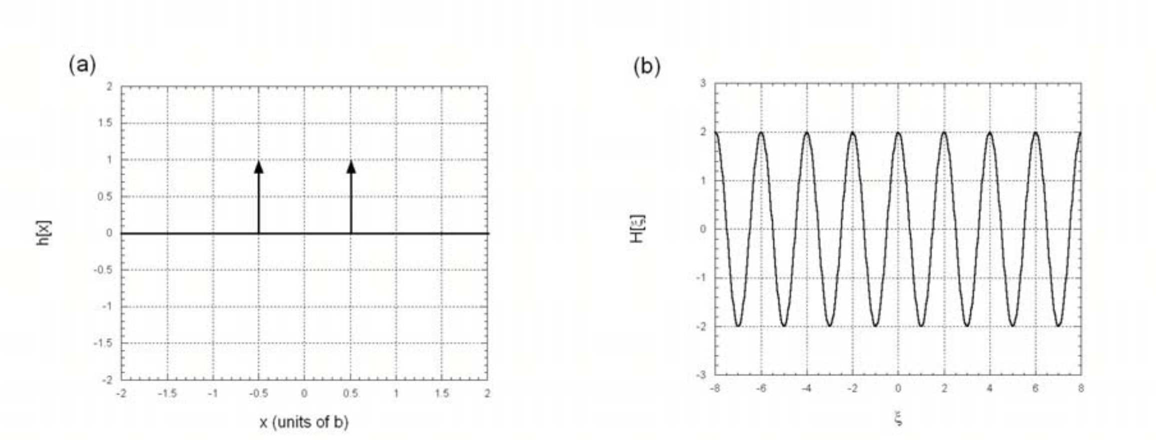
$$g[x] = f[x] + f[x - b_0] \;\; or \;\; f\left[x + \frac{b_0}{2}\right] + f\left[x + \frac{b_0}{2}\right]$$

The corresponding impulse responses are:

$$h[x] = \delta\left[x + \frac{b_0}{2}\right] + \delta\left[x + \frac{b_0}{2}\right]$$

(you could also use $h[x] = \delta[x] + \delta[x - b_0]$, but this adds a linear phase term that merely adds complexity without value. The corresponding transfer function is:

$$\begin{aligned} H[\xi] &= \mathcal{F}_1\left\{\delta\left[x + \frac{b_0}{2}\right] + \delta\left[x + \frac{b_0}{2}\right]\right\} \\ &= 2 \cdot \cos\left[2\pi \frac{b_0}{2} \xi\right] = 2 \cdot \cos[\pi b \xi] \end{aligned}$$



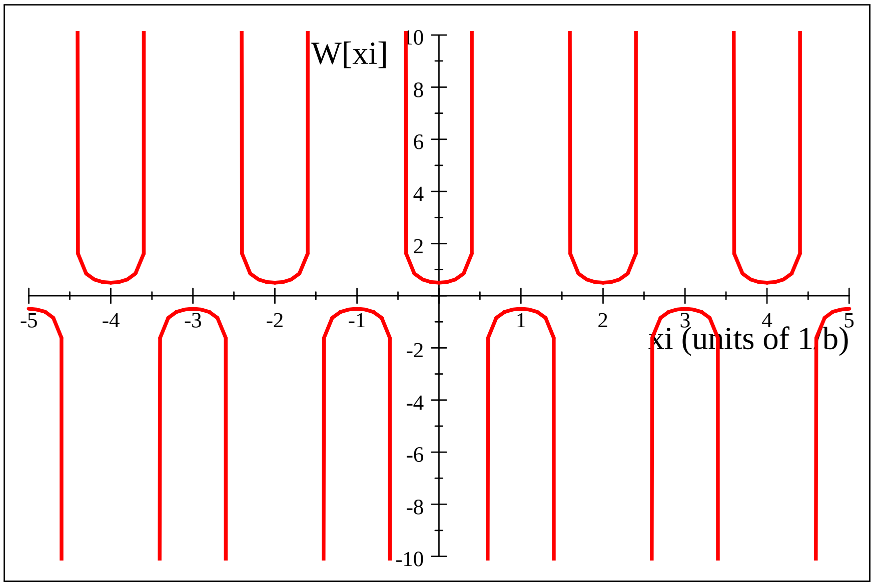
(a) impulse response $h[x] = \delta\left[x + \frac{b}{2}\right] + \delta\left[x - \frac{b}{2}\right]$; (b) transfer function $H[\xi] = 2 \cdot \cos\left[2\pi \cdot \frac{b}{2} \cdot \xi\right]$

(the only difference between the transfer functions is the linear phase term, which I shall subsequently ignore) We can see that both transfer functions have zeros at regularly spaced intervals of $\Delta\xi = \frac{2}{b_0}$, which means that the inverse filter does not exist in a strict sense.

(a) Design the inverse filter for this system in the frequency domain. Comment about the potential of success of the deblurring process, particularly if noise is present.

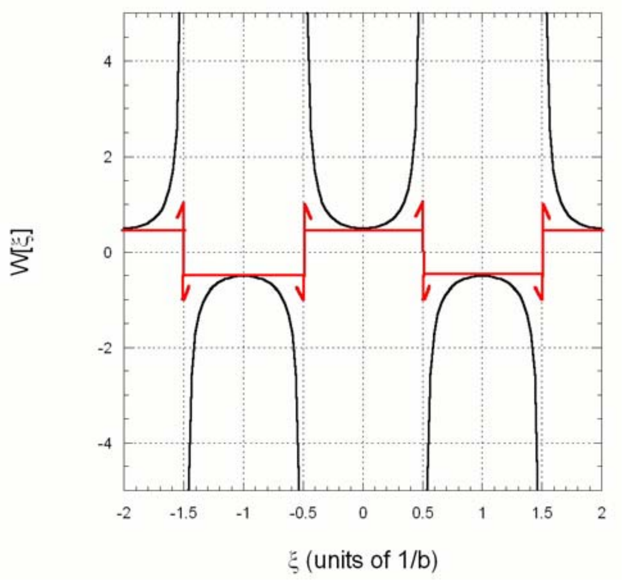
The ideal inverse filter would have the form:

$$\begin{aligned} W[\xi] &= (H_2[\xi])^{-1} \\ &= \frac{1}{2 \cdot \cos[\pi b \xi]} = \frac{1}{2} \sec[\pi b \xi] \end{aligned}$$



(b) Find an exact or approximate expression for the inverse filter in the space domain.

There is no easily derived analytical expression for the inverse Fourier transform of the trigonometric secant, but the shape of the function in the figure suggests an approximation by a square wave and added to appropriately positioned derivatives of Dirac delta functions:



$$\begin{aligned}
W_2[\xi] &\cong \textit{square wave with period } \frac{2}{b_0} + \textit{derivative of Dirac delta at each transition} \\
&\cong \left(RECT[b\xi] + \delta'\left[\xi - \frac{1}{2b}\right] - \delta'\left[\xi + \frac{1}{2b}\right]\right) * (|2b| \cdot COMB[2b\xi]) - \frac{1}{2} \cdot 1[\xi] \\
&\cong \left(RECT[b\xi] * (|2b| \cdot COMB[2b\xi]) - \frac{1}{2} \cdot 1[\xi]\right) \\
&\quad + \left(\delta'\left[\xi - \frac{1}{2b}\right] - \delta'\left[\xi + \frac{1}{2b}\right]\right) * (|2b| \cdot COMB[2b\xi]) \\
&= \frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right] \cdot \left(COMB\left[\frac{x}{2b}\right] - \frac{1}{2} \cdot \delta[x]\right) \\
&\quad + \left(2\pi i(-x) \cdot \exp\left[+2\pi i \cdot \frac{1}{2b} \cdot x\right] - \left(2\pi i(-x) \cdot \exp\left[-2\pi i \cdot \frac{1}{2b} \cdot x\right]\right)\right) \cdot COMB\left[\frac{x}{2b}\right] \\
&= \frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right] \cdot \left(COMB\left[\frac{x}{2b}\right] - \frac{1}{2} \cdot \delta[x]\right) \\
&\quad + (-2\pi i x)\left(\exp\left[+2\pi i \cdot \frac{1}{2b} \cdot x\right] - \exp\left[-2\pi i \cdot \frac{1}{2b} \cdot x\right]\right) \cdot COMB\left[\frac{x}{2b}\right] \\
&= \frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right] \cdot \left(COMB\left[\frac{x}{2b}\right] - \frac{1}{2} \cdot \delta[x]\right) + (-2\pi i x) \cdot 2i \cdot \sin\left[2\pi \frac{x}{2b}\right] \cdot COMB\left[\frac{x}{2b}\right] \\
&= \left(\frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right] + 4\pi x \cdot \sin\left[\pi \frac{x}{b_0}\right]\right) \cdot COMB\left[\frac{x}{2b}\right] - \frac{1}{2b} \cdot \delta[x]
\end{aligned}$$

$$\begin{aligned}
W_2[\xi] &\cong \delta'[\xi] * \left(\delta\left[\xi - \frac{1}{2b}\right] + \delta\left[\xi - \frac{3}{2b}\right] + \delta\left[\xi - \frac{5}{2b}\right] + \cdots + \delta\left[\xi - \frac{2k+1}{2b}\right]\right) \\
&\quad + \delta'[\xi] * \left(-\delta\left[\xi + \frac{1}{2b}\right] + \delta\left[\xi + \frac{3}{2b}\right] - \delta\left[\xi + \frac{5}{2b}\right] + \cdots - \delta\left[\xi + \frac{2k+1}{2b}\right]\right) \\
W_2[\xi] &= SGN[\xi] \cdot |b_0| \, COMB[b\xi] * \delta\left[\xi - \frac{b_0}{2}\right] * \delta'[\xi]
\end{aligned}$$

$$
\begin{aligned}
w_2[x] &\cong \frac{1}{i\pi(-x)} * COMB\left[\frac{x}{b_0}\right] \cdot \left(\exp[+2\pi i x] \cdot 2\pi i(-x)\right) \\
&= \frac{-2\pi i}{-i\pi}\left(\frac{1}{x} * COMB\left[\frac{x}{b_0}\right]\right) \cdot \left(\exp[+2\pi i x] \cdot x\right) \\
&= 2 \cdot \left(\frac{1}{x} * COMB\left[\frac{x}{b_0}\right]\right) \cdot \left(x \cdot \exp[+2\pi i x]\right)
\end{aligned}
$$