

1051-716-20091 Amended Solution Set #5

1. Use the filter theorem to evaluate:

$$(a) \text{GAUS} \left[\frac{x}{3} \right] * \text{GAUS} \left[\frac{x}{4} \right] = \int_{-\infty}^{+\infty} \exp \left[-\pi \left(\frac{\alpha}{3} \right)^2 \right] \exp \left[-\pi \left(\frac{x-\alpha}{4} \right)^2 \right] d\alpha$$

Obviously, this is the same pair of convolutions given in HW#4, but now we can use the filter theorem to solve:

$$\begin{aligned} \text{GAUS} \left[\frac{x}{3} \right] * \text{GAUS} \left[\frac{x}{4} \right] &= \mathcal{F}_1^{-1} \left\{ \mathcal{F}_1 \left\{ \text{GAUS} \left[\frac{x}{3} \right] \right\} \cdot \mathcal{F}_1 \left\{ \text{GAUS} \left[\frac{x}{4} \right] \right\} \right\} \\ &= \mathcal{F}_1^{-1} \left\{ 3 \cdot \text{GAUS} [3\xi] \cdot 4 \cdot \text{GAUS} [4\xi] \right\} \\ &= \mathcal{F}_1^{-1} \left\{ 12 \cdot \exp \left[-\pi (3\xi)^2 \right] \cdot \exp \left[-\pi (4\xi)^2 \right] \right\} \\ &= \mathcal{F}_1^{-1} \left\{ 12 \cdot \exp \left[-\pi (9 + 16) \xi^2 \right] \right\} \\ &= \mathcal{F}_1^{-1} \left\{ 12 \cdot \exp \left[-\pi (5\xi)^2 \right] \right\} \\ &= \mathcal{F}_1^{-1} \left\{ 12 \cdot \text{GAUS} [5\xi] \right\} \end{aligned}$$

$$\boxed{\text{GAUS} \left[\frac{x}{3} \right] * \text{GAUS} \left[\frac{x}{4} \right] = \frac{12}{5} \text{GAUS} \left[\frac{x}{5} \right]}$$

which is “wider” than either input function.

$$(b) \text{SINC} [3x] * \text{SINC} [2x] = \int_{-\infty}^{+\infty} \left(\frac{\sin[3\pi\alpha]}{3\pi\alpha} \right) \left(\frac{\sin[2\pi(x-\alpha)]}{2\pi(x-\alpha)} \right) d\alpha$$

$$\begin{aligned} \text{SINC} [3x] * \text{SINC} [2x] &= \mathcal{F}_1^{-1} \left\{ \mathcal{F}_1 \left\{ \text{SINC} [3x] \right\} \cdot \mathcal{F}_1 \left\{ \text{SINC} [2x] \right\} \right\} \\ &= \mathcal{F}_1^{-1} \left\{ \frac{1}{3} \text{RECT} \left[\frac{\xi}{3} \right] \cdot \frac{1}{2} \text{RECT} \left[\frac{\xi}{2} \right] \right\} \\ &= \mathcal{F}_1^{-1} \left\{ \frac{1}{6} \text{RECT} \left[\frac{\xi}{2} \right] \right\} \\ &= \frac{1}{6} \cdot (2 \cdot \text{SINC} [2x]) \end{aligned}$$

$$\boxed{\text{SINC} [3x] * \text{SINC} [2x] = \frac{1}{3} \text{SINC} [2x]}$$

which is proportional to the “wider” SINC function.

2. Use the Fourier transform of $e^{+i\pi x^2}$ to derive the 2-D transform $\mathcal{F}_2 \{f[x, y]\} = \mathcal{F}_2 \{e^{+i\pi(x^2+y^2)}\} = \mathcal{F}_2 \{e^{+i\pi r^2}\}$; sketch the profiles along the x-axis of the real part, imaginary part, magnitude and phase.

$$\begin{aligned}
\mathcal{F}_2 \{e^{+i\pi(x^2+y^2)}\} &= \mathcal{F}_2 \{\exp [+i\pi x^2] \cdot \exp [+i\pi y^2]\} \\
&= \mathcal{F}_1 \{\exp [+i\pi x^2]\} \cdot \mathcal{F}_1 \{\exp [+i\pi y^2]\} \\
&= \left(\exp \left[+i\frac{\pi}{4} \right] \cdot \exp [-i\pi \xi^2] \right) \cdot \left(\exp \left[+i\frac{\pi}{4} \right] \cdot \exp [-i\pi \eta^2] \right) \\
&= \left(\exp \left[+i\frac{\pi}{4} \right] \right)^2 \cdot \left(\exp [-i\pi \xi^2] \cdot \exp [-i\pi \eta^2] \right) \\
&= \exp \left[+i\frac{\pi}{2} \right] \cdot \exp [-i\pi (\xi^2 + \eta^2)] \\
&= \boxed{\mathcal{F}_2 \{\exp [+i\pi (x^2 + y^2)]\} = i \cdot \exp [-i\pi \rho^2]} \\
&= i \cdot (\cos [\pi \rho^2] - i \sin [\pi \rho^2]) \\
&= \sin [\pi \rho^2] + i \cdot \cos [\pi \rho^2]
\end{aligned}$$

$$\boxed{\operatorname{Re} \{i \cdot \exp [-i\pi \rho^2]\} = \sin [\pi \rho^2]}$$

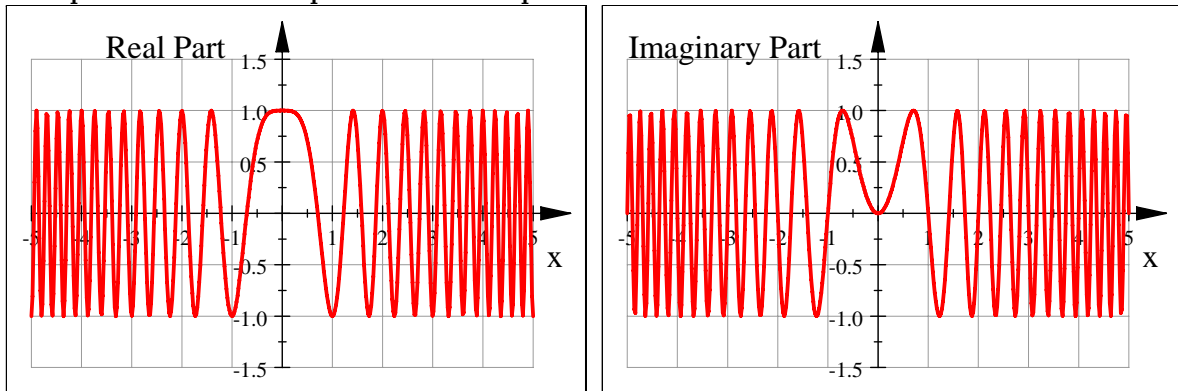
$$\boxed{\operatorname{Im} \{i \cdot \exp [-i\pi \rho^2]\} = \cos [\pi \rho^2]}; \text{ note that the real and imaginary parts "swap"}$$

$$\boxed{|i \cdot \exp [-i\pi \rho^2]| = 1 [\xi]}$$

$$\begin{aligned}
\Phi \{i \cdot \exp [-i\pi \rho^2]\} &= \tan^{-1} \left[\frac{\cos [\pi \rho^2]}{\sin [\pi \rho^2]} \right] = \tan^{-1} \left[\frac{-\sin [\pi \rho^2 - \frac{\pi}{2}]}{\cos [\pi \rho^2 - \frac{\pi}{2}]} \right] \\
&= \tan^{-1} \left[-\frac{\sin [\pi \rho^2 - \frac{\pi}{2}]}{\cos [\pi \rho^2 - \frac{\pi}{2}]} \right] = -\left(\pi \rho^2 - \frac{\pi}{2} \right)
\end{aligned}$$

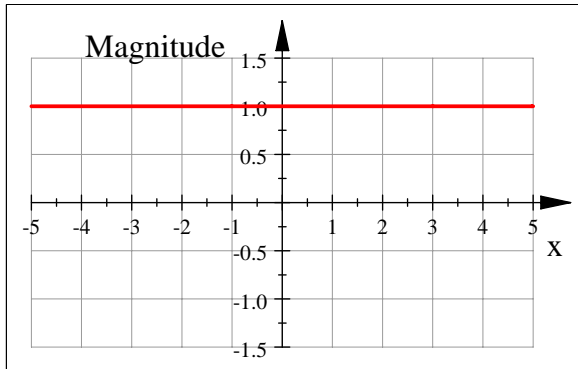
$$\boxed{\Phi \{i \cdot \exp [-i\pi \rho^2]\} = \frac{\pi}{2} - \pi \rho^2}$$

The profiles of both representations: space domain first:

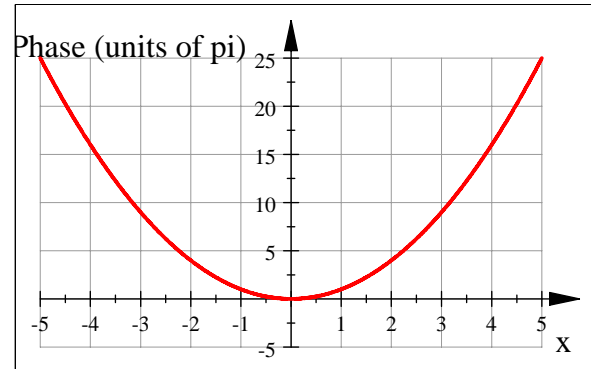


$$\text{Re} \{ \exp [+i\pi x^2] \} = \cos \pi x^2$$

$$\text{Im} \{ \exp [+i\pi x^2] \} = \sin \pi x^2$$

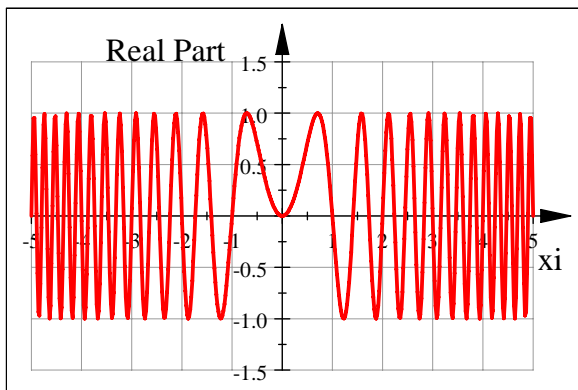


$$| \exp [+i\pi x^2] | = 1 [x]$$

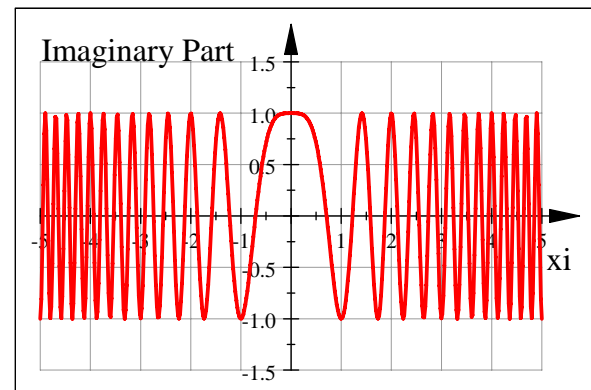


$$\Phi \{ \exp [+i\pi x^2] \} = \pi x^2$$

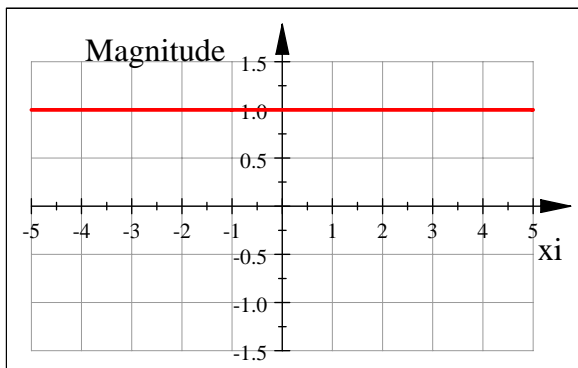
Frequency domain: plot along ξ -axis.



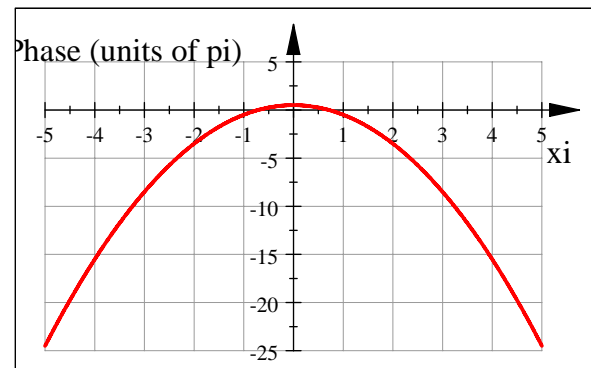
$$\text{Re} \{ i \exp [-i\pi \xi^2] \} = \sin [\pi \xi^2]$$



$$\text{Im} \{ i \exp [-i\pi \xi^2] \} = \cos [\pi \xi^2]$$



$$| i \exp [-i\pi \xi^2] | = 1 [\xi]$$



$$\Phi \{ i \cdot \exp [-i\pi \xi^2] \} = \frac{\pi}{2} - \pi \xi^2$$

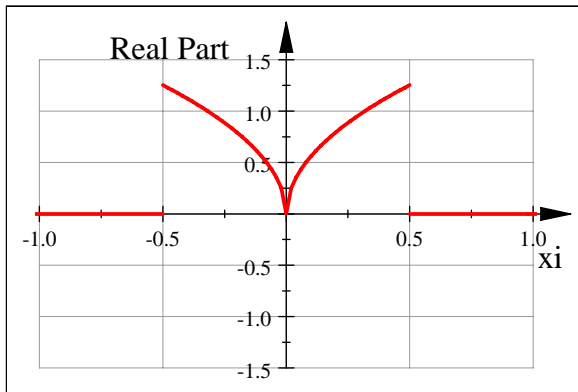
3. Use the general form of the derivative theorem in Eq.(9.143) to derive an expression for the “half-order” derivative of $f[x]$, i.e., $\left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right) f[x]$.

Derivative theorem:

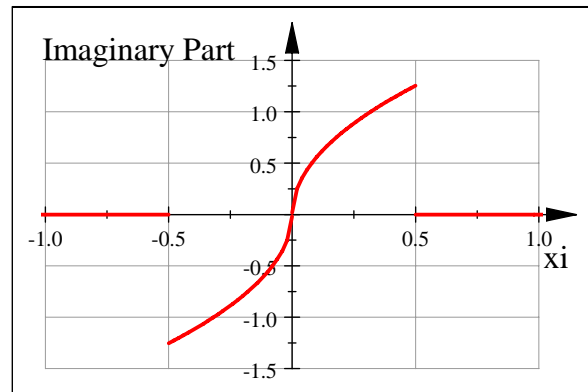
$$\begin{aligned} \mathcal{F}_1 \left\{ \frac{d^n f}{dx^n} \right\} &= (+2\pi i \xi)^n \cdot F[\xi] \\ \implies \mathcal{F}_1 \left\{ \frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} \right\} &= (+2\pi i \xi)^{\frac{1}{2}} \cdot F[\xi] \\ &= \sqrt{2\pi} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \sqrt{\xi} \cdot F[\xi] \end{aligned}$$

For example, if $f[x] = \text{SINC}[x] \implies F[\xi] = \text{RECT}[\xi]$:

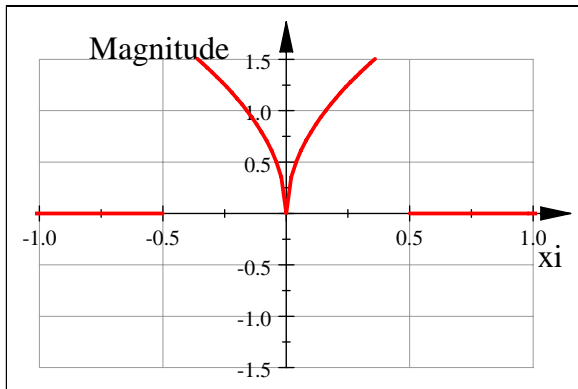
$$\begin{aligned} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \{ \text{SINC}[x] \} &= \sqrt{2\pi} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \sqrt{\xi} \cdot \text{RECT}[\xi] \\ &= \begin{cases} \sqrt{\pi} \cdot (1+i) \cdot \sqrt{\xi} & \text{if } |\xi| < \frac{1}{2} \\ 0 & \text{if } |\xi| > \frac{1}{2} \end{cases} \end{aligned}$$



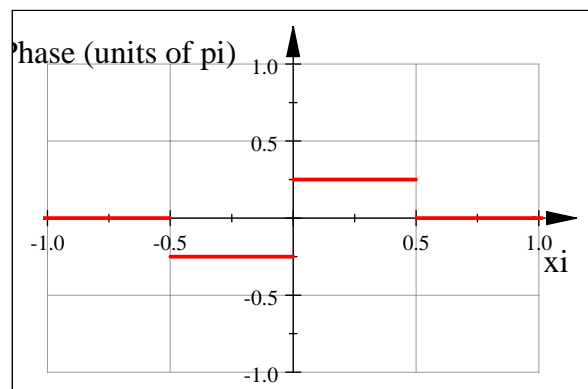
$\text{Re}\{\sqrt{\pi} \cdot (1+i) \cdot \sqrt{\xi}\} \cdot \text{RECT}[\xi]$



$\text{Im}\{\sqrt{\pi} \cdot (1+i) \cdot \sqrt{\xi}\} \cdot \text{RECT}[\xi]$



$|\sqrt{\pi} \cdot (1+i) \cdot \sqrt{\xi} \cdot \text{RECT}[\xi]|$



$\Phi\{\exp[+i\pi x^2]\} = \pi x^2$

4. Use Rayleigh's theorem to evaluate the areas of the following: (§9-5)

(a) $SINC [x] \cdot e^{-\pi x^2}$

$$\int_{-\infty}^{+\infty} (f [x] \cdot h^* [x]) dx = \int_{-\infty}^{+\infty} (F [\xi] \cdot H^* [\xi]) d\xi$$

$$\int_{-\infty}^{+\infty} (SINC [x] \cdot e^{-\pi x^2}) dx = \int_{-\infty}^{+\infty} (RECT [\xi] \cdot e^{-\pi \xi^2}) d\xi$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-\pi \xi^2} d\xi = 2 \cdot \int_0^{\frac{1}{2}} e^{-\pi \xi^2} d\xi$$

This is the partial area of a Gaussian function and may be evaluated in terms of the so-called "error function"

$$\text{erf} [x] \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$u \equiv \sqrt{\pi} \xi \implies d\xi = \frac{1}{\sqrt{\pi}} du$$

$$\xi = \frac{1}{2} \implies u = \frac{\sqrt{\pi}}{2}$$

$$2 \cdot \int_0^{\frac{1}{2}} e^{-\pi \xi^2} d\xi = 2 \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \left(\frac{2}{\sqrt{\pi}} \int_0^{\frac{\sqrt{\pi}}{2}} e^{-t^2} dt \right)$$

$$= \text{erf} \left[\frac{\sqrt{\pi}}{2} \right]$$

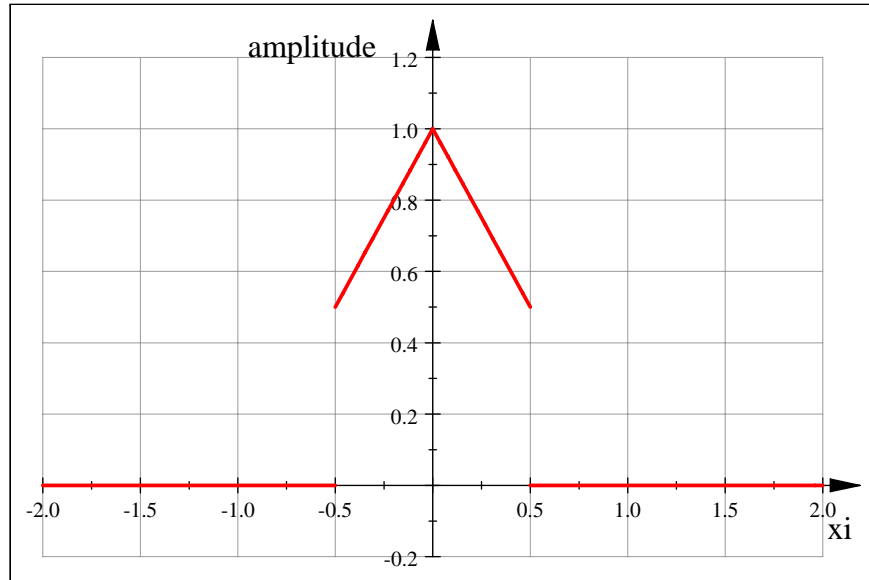
The error function is tabulated and you can look up the value:

$$\text{erf} \left[\frac{\sqrt{\pi}}{2} \right] \cong 0.78991$$

(b) $SINC^3[x]$

$$\begin{aligned}
 \int_{-\infty}^{+\infty} SINC^3[x] dx &= \int_{-\infty}^{+\infty} (SINC^2[x]) \cdot (SINC[x]) dx = \int_{-\infty}^{+\infty} (TRI[\xi]) \cdot (RECT[\xi]) d\xi \\
 &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} TRI[\xi] d\xi \\
 &= 2 \cdot \int_0^{+\frac{1}{2}} (1 - \xi) d\xi = 2 \cdot \left(\xi - \frac{\xi^2}{2} \right) \Big|_{\xi=0}^{\xi=\frac{1}{2}} = \frac{3}{4}
 \end{aligned}$$

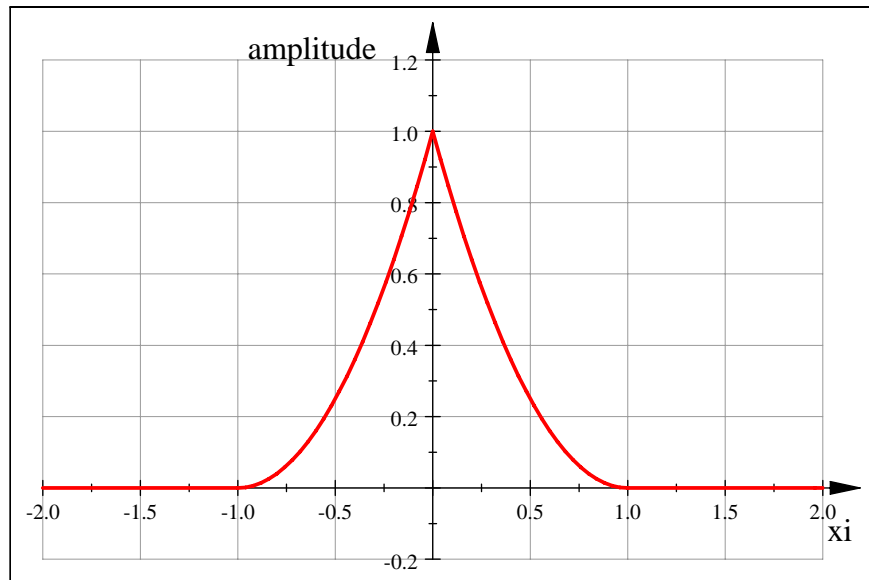
$$\int_{-\infty}^{+\infty} SINC^3[x] dx = \frac{3}{4}$$



Graph of $TRI[\xi] \cdot RECT[\xi]$, whose area is equal to the area of $(SINC[x])^3$.

(c) $SINC^4[x]$

$$\begin{aligned}
 \int_{-\infty}^{+\infty} SINC^4[x] dx &= \int_{-\infty}^{+\infty} (SINC^2[x]) \cdot (SINC^2[x]) dx \\
 &= \int_{-\infty}^{+\infty} (TRI[\xi]) \cdot (TRI[\xi]) d\xi \\
 &= \int_{-1}^{+1} (TRI[\xi])^2 d\xi \\
 &= 2 \cdot \int_0^{+1} (1-\xi)^2 d\xi = 2 \cdot \int_0^{+1} (1-2\xi+\xi^2) d\xi = 2 \cdot \left(\xi - 2 \cdot \frac{\xi^2}{2} + \frac{\xi^3}{3} \right) \Bigg|_{\xi=0}^{\xi=1} = \frac{2}{3} \\
 \boxed{\int_{-\infty}^{+\infty} SINC^4[x] dx = \frac{2}{3}} &< \frac{3}{4} = \int_{-\infty}^{+\infty} SINC^3[x] dx
 \end{aligned}$$



Graph of $(TRI[\xi])^2$, whose area is equal to the area of $(SINC[x])^4$.

(d) $SINC^5[x]$

$$\begin{aligned} \int_{-\infty}^{+\infty} SINC^5[x] dx &= \int_{-\infty}^{+\infty} (SINC^2[x]) \cdot (SINC^3[x]) dx \\ &= \int_{-\infty}^{+\infty} (TRI[\xi]) \cdot (RECT[\xi] * RECT[\xi] * RECT[\xi]) d\xi \end{aligned}$$

$$RECT[\xi] * RECT[\xi] * RECT[\xi] = TRI[\xi] * RECT[\xi]$$

From HW#4-4e:

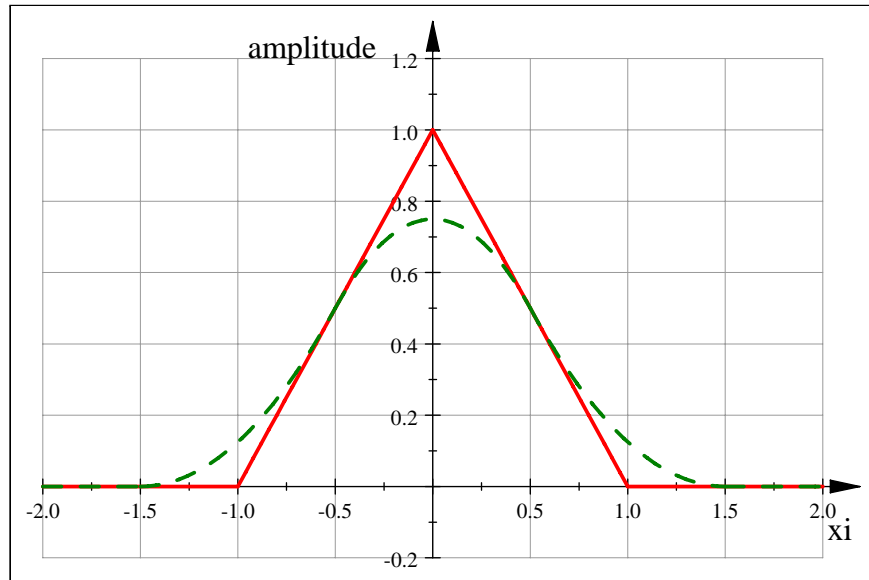
$$RECT[\xi] * TRI[\xi] = \begin{cases} 0 & \text{if } \xi > +\frac{3}{2} \\ \frac{1}{2}\xi^2 - \frac{3}{2}\xi + \frac{9}{8} & \text{if } +\frac{1}{2} \leq \xi \leq +\frac{3}{2} \\ \frac{3}{4} - \xi^2 & \text{if } -\frac{1}{2} \leq \xi \leq +\frac{1}{2} \\ \frac{1}{2}\xi^2 + \frac{3}{2}\xi + \frac{9}{8} & \text{if } -\frac{3}{2} \leq \xi \leq -\frac{1}{2} \\ 0 & \text{if } \xi < -\frac{3}{2} \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{+\infty} (TRI[\xi]) \cdot (RECT[\xi] * RECT[\xi] * RECT[\xi]) d\xi \\ = \int_{-\infty}^{+\infty} \left((1 - |\xi|) \cdot RECT\left[\frac{\xi}{2}\right] \right) \cdot (RECT[\xi] * RECT[\xi] * RECT[\xi]) d\xi \end{aligned}$$

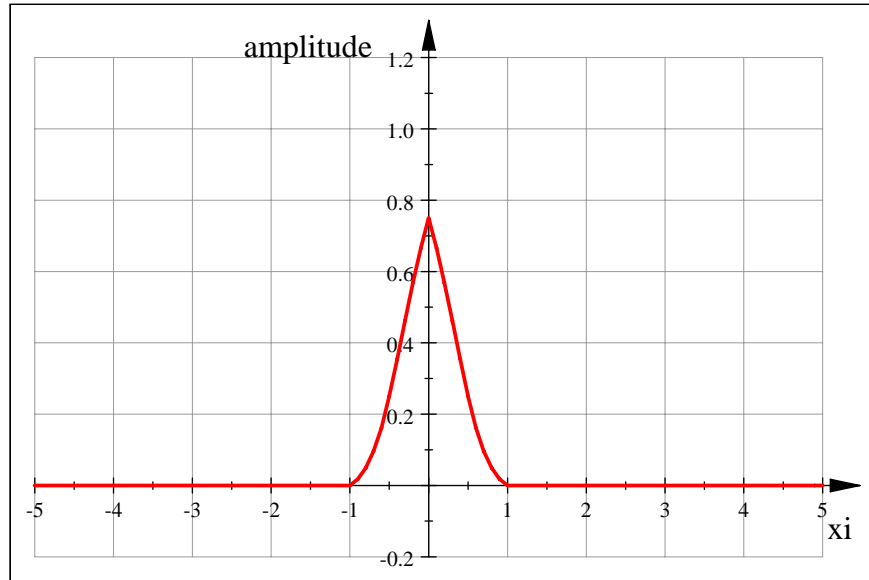
which is the sum of four integrals:

$$\begin{aligned} \int_{-1}^{-\frac{1}{2}} (1 + \xi) \cdot \left(\frac{1}{2}\xi^2 + \frac{3}{2}\xi + \frac{9}{8} \right) d\xi + \int_{-\frac{1}{2}}^0 (1 + \xi) \cdot \left(\frac{3}{4} - \xi^2 \right) d\xi \\ + \int_0^{\frac{1}{2}} (1 - \xi) \cdot \left(\frac{3}{4} - \xi^2 \right) d\xi + \int_{\frac{1}{2}}^1 (1 - \xi) \cdot \left(\frac{1}{2}\xi^2 - \frac{3}{2}\xi + \frac{9}{8} \right) d\xi \\ = \frac{17}{384} + \frac{49}{192} + \frac{49}{192} + \frac{17}{384} = \frac{115}{192} \cong 0.59896 \end{aligned}$$

$$\boxed{\int_{-\infty}^{+\infty} SINC^5[x] dx = \frac{115}{192}} < \frac{2}{3} = \int_{-\infty}^{+\infty} SINC^4[x] dx$$



Graphs of $TRI[\xi]$ (solid red) and $RECT[\xi] * TRI[\xi]$ (dashed green)



Graph of product $TRI[\xi] \cdot (RECT[\xi] * TRI[\xi])$, whose area is equal to that of $(SINC[\xi])^5$.

Note that the areas of $SINC^n[x]$ decrease with increasing n except for $n = 1, 2$:

$$\int_{-\infty}^{+\infty} SINC^1[x] dx = \int_{-\infty}^{+\infty} SINC^2[x] dx - 1$$

$$\int_{-\infty}^{+\infty} SINC^3[x] dx = \frac{3}{4}$$

$$\int_{-\infty}^{+\infty} SINC^4[x] dx = \frac{2}{3}$$

$$\int_{-\infty}^{+\infty} SINC^5[x] dx = \frac{115}{192}$$

5. Evaluate and sketch $f[x] = \text{SINC}[x] * \frac{1}{x}$.

$$\text{SINC}[x] * \frac{1}{x} = \mathcal{F}_1^{-1} \left\{ \mathcal{F}_1 \{ \text{SINC}[x] \} \cdot \mathcal{F}_1 \left\{ \frac{1}{x} \right\} \right\}$$

$$\mathcal{F}_1 \{ \text{SINC}[x] \} = \text{RECT}[\xi]$$

We know that:

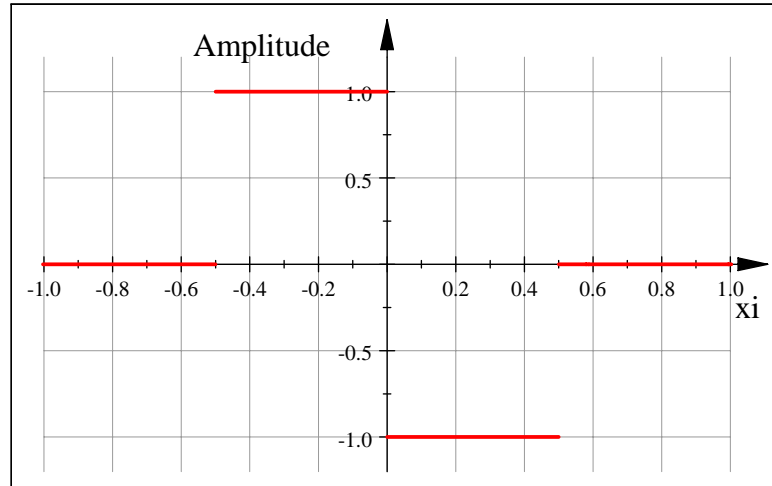
$$\mathcal{F}_1 \{ \text{SGN}[x] \} = \frac{1}{i\pi\xi}$$

The transform-of-theorem tells us that

$$\mathcal{F}_1 \left\{ \frac{1}{x} \right\} = i\pi \cdot \mathcal{F}_1 \left\{ \frac{1}{i\pi x} \right\} = i\pi \cdot \text{SGN}[-\xi]$$

so the convolution is the product of the rectangle and the signum function, which may be rewritten more conveniently:

$$\text{SINC}[x] * \frac{1}{x} = \mathcal{F}_1^{-1} \{ \text{RECT}[\xi] \cdot i\pi \cdot \text{SGN}[-\xi] \}$$



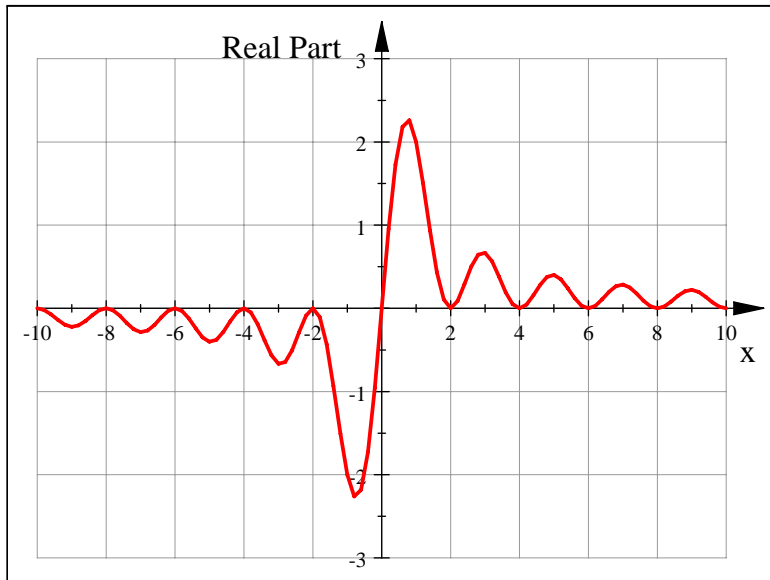
$$\text{RECT}[\xi] \cdot \text{SGN}[-\xi] = \text{RECT}[2\xi] * \left(\delta \left[\xi + \frac{1}{4} \right] - \delta \left[\xi - \frac{1}{4} \right] \right)$$

From the graph, we see that we can rewrite this spectrum in a more convenient form:

$$\begin{aligned} \text{SINC}[x] * \frac{1}{x} &= i\pi \cdot \mathcal{F}_1^{-1} \left\{ \text{RECT}[2\xi] * \left(\delta \left[\xi + \frac{1}{4} \right] - \delta \left[\xi - \frac{1}{4} \right] \right) \right\} \\ &= i\pi \cdot \left(\frac{1}{2} \text{SINC} \left[\frac{x}{2} \right] \cdot \frac{2}{i} \sin \left[2\pi x \cdot \frac{1}{4} \right] \right) \\ &= +\pi \cdot \left(\text{SINC} \left[\frac{x}{2} \right] \cdot \sin \left[\frac{2\pi x}{4} \right] \right) \\ &= +\pi \cdot \left(\frac{\sin \left[\frac{\pi x}{2} \right]}{\frac{\pi x}{2}} \right) \cdot \sin \left[\frac{\pi x}{2} \right] \\ &= +\pi^2 \cdot \frac{x}{2} \cdot \text{SINC}^2 \left[\frac{x}{2} \right] \end{aligned}$$

Note that:

$$+\pi^2 \cdot \frac{x}{2} \cdot \text{SINC}^2 \left[\frac{x}{2} \right] = \frac{1 - \cos[\pi x]}{x}$$



$$\text{SINC}[x] * \frac{1}{x} = +\pi^2 \cdot \frac{x}{2} \cdot \text{SINC}^2\left[\frac{x}{2}\right] = \frac{1 - \cos[\pi x]}{x}$$

6. Evaluate and sketch the following:

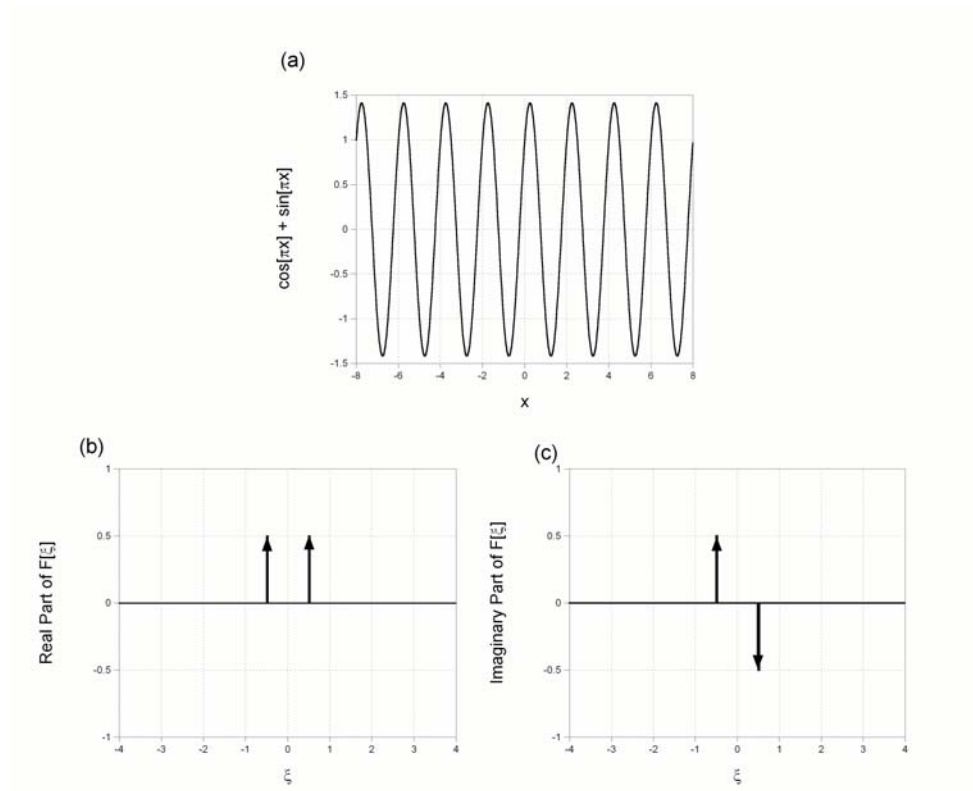
(a) $\mathcal{F}_1 \left\{ \cos \left(2\pi \frac{x}{2} \right) + \sin \left(2\pi \frac{x}{2} \right) \right\}$

$$\begin{aligned} \mathcal{F}_1 \left\{ \cos \left(2\pi \frac{x}{2} \right) + \sin \left(2\pi \frac{x}{2} \right) \right\} &= \mathcal{F}_1 \left\{ \cos \left(2\pi \frac{x}{2} \right) \right\} + \mathcal{F}_1 \left\{ \sin \left(2\pi \frac{x}{2} \right) \right\} \\ &= \frac{1}{2} \left(\delta \left[\xi + \frac{1}{2} \right] + \delta \left[\xi - \frac{1}{2} \right] \right) + i \cdot \frac{1}{2} \left(\delta \left[\xi + \frac{1}{2} \right] - \delta \left[\xi - \frac{1}{2} \right] \right) \\ &= \left(\frac{1+i}{2} \right) \cdot \delta \left[\xi + \frac{1}{2} \right] + \left(\frac{1-i}{2} \right) \cdot \delta \left[\xi - \frac{1}{2} \right] \end{aligned}$$

Note that we can rewrite $\cos \left(2\pi \frac{x}{2} \right) + \sin \left(2\pi \frac{x}{2} \right)$ as:

$$\begin{aligned} \cos \left(2\pi \frac{x}{2} \right) + \sin \left(2\pi \frac{x}{2} \right) &= \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} \cos \left(2\pi \frac{x}{2} \right) + \frac{1}{\sqrt{2}} \sin \left(2\pi \frac{x}{2} \right) \right) \\ &= \sqrt{2} \cdot \left(\cos \left[\frac{\pi}{4} \right] \cos \left(2\pi \frac{x}{2} \right) + \sin \left[\frac{\pi}{4} \right] \sin \left(2\pi \frac{x}{2} \right) \right) \\ &= \sqrt{2} \cdot \cos \left[2\pi \frac{x}{2} + \frac{\pi}{4} \right] \end{aligned}$$

which confirms that the sum of two sinusoids with the same amplitude and frequency is a sinusoid with that frequency and a (possibly) different amplitude and phase.



$$(b) \mathcal{F}_1 \left\{ \sin \left[\left(\frac{x}{2} \right)^2 \right] \right\}$$

$$\begin{aligned} \mathcal{F}_1 \left\{ \sin \left[\left(\frac{x}{2} \right)^2 \right] \right\} &= \mathcal{F}_1 \left\{ \sin \left[\frac{\pi}{\pi} \left(\frac{x}{2} \right)^2 \right] \right\} \\ &= \mathcal{F}_1 \left\{ \sin \left[\pi \left(\frac{x}{2\sqrt{\pi}} \right)^2 \right] \right\} \\ &= \mathcal{F}_1 \left\{ \sin \left[\pi \left(\frac{x}{\alpha} \right)^2 \right] \right\} \text{ where } \alpha = 2 \cdot \sqrt{\pi} \end{aligned}$$

Recall that the transform of a chirp function is:

$$\begin{aligned} \mathcal{F}_1 \{ \exp [+i\pi x^2] \} &= \exp \left[+i\frac{\pi}{4} \right] \cdot \exp [-i\pi \xi^2] \\ &= \frac{1+i}{\sqrt{2}} \cdot \exp [-i\pi \xi^2] \end{aligned}$$

The transform of a scaled chirp function is:

$$\begin{aligned} \mathcal{F}_1 \left\{ \exp \left[+i\pi \left(\frac{x}{\alpha} \right)^2 \right] \right\} &= |\alpha| \exp \left[+i\frac{\pi}{4} \right] \cdot \exp \left[-i\pi (\alpha \xi)^2 \right] \\ &= |\alpha| \exp \left[+i\frac{\pi}{4} \right] \cdot \exp \left[-i\pi \left(\frac{\xi}{\alpha^{-1}} \right)^2 \right] \\ \Rightarrow \mathcal{F}_1 \left\{ \exp \left[+i\pi \left(\frac{x}{2 \cdot \sqrt{\pi}} \right)^2 \right] \right\} &= |2 \cdot \sqrt{\pi}| \cdot \exp \left[+i\frac{\pi}{4} \right] \cdot \exp \left[-i\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] \end{aligned}$$

which is a chirp with rate $(2 \cdot \sqrt{\pi})^{-1} \cong 0.28209 < 1$, so this chirp oscillates "faster" than the space-domain chirp.

Break into real and imaginary parts:

$$\begin{aligned} \mathcal{F}_1 \left\{ \exp \left[+i\pi \left(\frac{x}{2 \cdot \sqrt{\pi}} \right)^2 \right] \right\} &= |2 \cdot \sqrt{\pi}| \cdot \left(\frac{1+i}{\sqrt{2}} \right) \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] - i \cdot \sin \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] \right) \\ &= \sqrt{\frac{\pi}{2}} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] + \sin \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] \right) \\ &\quad + i \cdot \sqrt{\frac{\pi}{2}} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] - \sin \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] \right) \end{aligned}$$

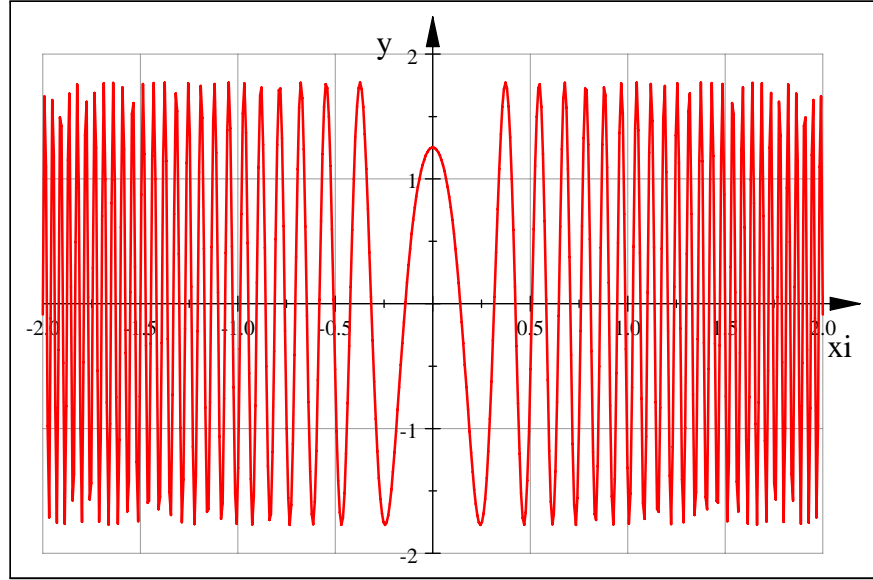
Now recall that:

$$\begin{aligned} \cos [A] \pm \sin [A] &= \sqrt{2} \cdot \left(\cos \left[A \mp \frac{\pi}{4} \right] \right) \\ \Rightarrow \cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] \pm \sin \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \right] &= \sqrt{2} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 \mp \frac{\pi}{4} \right] \right) \end{aligned}$$

Since real and imaginary parts are BOTH even functions, we can equate the real parts and the

imaginary parts of the transform:

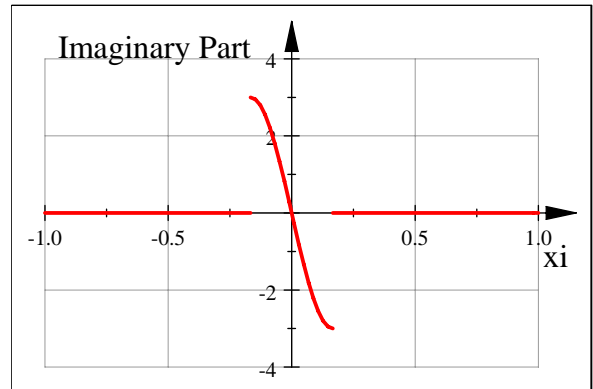
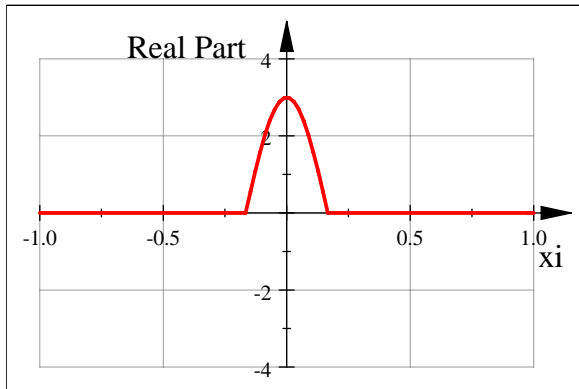
$$\begin{aligned}\mathcal{F}_1 \left\{ \cos \left[\left(\frac{x}{2} \right)^2 \right] \right\} &= \sqrt{\frac{\pi}{2}} \cdot \sqrt{2} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 - \frac{\pi}{4} \right] \right) \\ &= \sqrt{\pi} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 - \frac{\pi}{4} \right] \right) \\ \mathcal{F}_1 \left\{ \sin \left[\left(\frac{x}{2} \right)^2 \right] \right\} &= \sqrt{\pi} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 + \frac{\pi}{4} \right] \right)\end{aligned}$$



$$\sqrt{\pi} \cdot \left(\cos \left[\pi \left(\frac{\xi}{(2 \cdot \sqrt{\pi})^{-1}} \right)^2 + \frac{\pi}{4} \right] \right)$$

$$(c) \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{x}{3} - \frac{1}{2} \right] \right\}$$

$$\begin{aligned} \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{x}{3} - \frac{1}{2} \right] \right\} &= \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{2x-3}{6} \right] \right\} \\ &= \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{x-\frac{3}{2}}{3} \right] \right\} = \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{x}{3} \right] * \delta \left[x - \frac{3}{2} \right] \right\} \\ &= \mathcal{F}_1 \left\{ \text{SINC} \left[\frac{x}{3} \right] * \delta \left[x - \frac{3}{2} \right] \right\} \\ &= 3 \cdot \text{RECT} [3\xi] \cdot \exp \left[-2\pi i \xi \cdot \frac{3}{2} \right] \\ &= 3 \cdot \text{RECT} \left[\frac{\xi}{3^{-1}} \right] \cdot \exp [-3i\pi\xi] \\ &= 3 \cdot \text{RECT} \left[\frac{\xi}{3^{-1}} \right] \cdot (\cos [3\pi\xi] - i \sin [3\pi\xi]) \\ &= \begin{cases} 3 \cos [3\pi x] & \text{if } |x| < \frac{1}{6} \\ 0 & \text{if } |x| > \frac{1}{6} \end{cases} \end{aligned}$$



7. Use the theorems of the Fourier transform to prove that:

$$r[x - x_0] \star r[x - x_0] = r[x] \star r[x]$$

$$\begin{aligned}
r[x] \star r[x] &= r[x] * r^*[-x] \\
r[x - x_0] \star r[x - x_0] &= (r[x] * \delta[x - x_0]) \star (r[x] * \delta[x - x_0]) \\
&= (r[x] * \delta[x - x_0]) * (r[-x] * \delta[-x - x_0])^* \\
&= (r[x] * \delta[x - x_0]) * (r^*[-x] * \delta[-x - x_0]) \\
&= r[x] * r^*[-x] * \delta[x - x_0] * \delta[-(x + x_0)] \\
&= (r[x] * r^*[-x]) * (\delta[x - x_0] * \delta[x + x_0]) \\
&= (r[x] \star r[x]) * (\delta[x] * \delta[x + x_0 - x_0]) \\
&= (r[x] \star r[x]) * \delta[x] \\
&= r[x] \star r[x]
\end{aligned}$$

By the filter theorem:

$$\begin{aligned}
r[x - x_0] \star r[x - x_0] &= r[x - x_0] * r^*[-x - x_0] \\
&= \mathcal{F}_1^{-1} \{R[\xi] \cdot \exp[-2\pi i \xi x] \cdot R^*[\xi] \cdot \exp[-2\pi i \xi x]\} \\
&= \mathcal{F}_1^{-1} \{R[\xi] \cdot R^*[\xi]\} \\
&= \mathcal{F}_1^{-1} \{|R[\xi]|^2\} \\
&= r[x] \star r[x]
\end{aligned}$$

In words, the autocorrelation is not affected by translation of the input function.

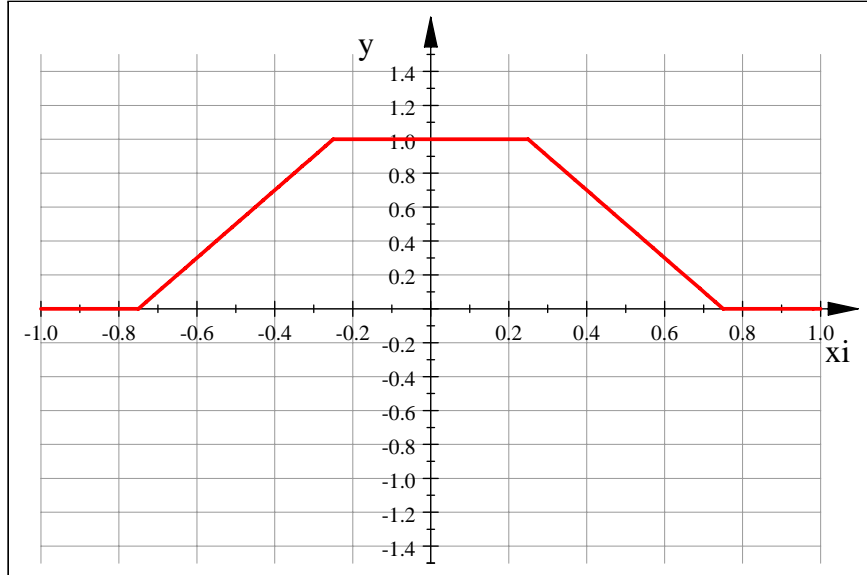
8. Evaluate the following Fourier transforms and sketch them as real-and-imaginary parts and as magnitude-phase:

(a) $SINC [x] \cdot SINC \left[\frac{x}{2} \right]$

$$\begin{aligned} \mathcal{F}_1 \left\{ SINC [x] \cdot SINC \left[\frac{x}{2} \right] \right\} &= RECT [\xi] * 2 \cdot RECT [2\xi] \\ &= 2 \cdot (RECT [\xi] * RECT [2\xi]) \end{aligned}$$

Recall from HW#4:

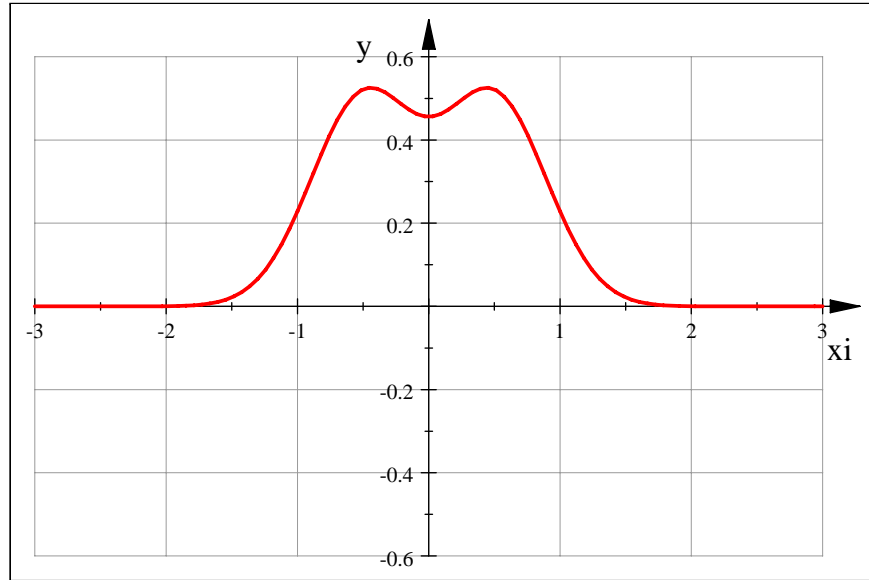
$$\begin{aligned} RECT \left[\frac{x}{d_0} \right] * RECT \left[\frac{x}{b_0} \right] &= \left(\frac{d_0 + b_0}{2} \right) \cdot TRI \left[\frac{x}{\left(\frac{d_0 + b_0}{2} \right)} \right] - \left(\frac{|d_0 - b_0|}{2} \right) \cdot TRI \left[\frac{x}{\left(\frac{|d_0 - b_0|}{2} \right)} \right] \\ \Rightarrow 2 \cdot (RECT [\xi] * RECT [2\xi]) &= 2 \cdot \left(\left(\frac{1 + \frac{1}{2}}{2} \right) \cdot TRI \left[\frac{\xi}{\left(\frac{1 + \frac{1}{2}}{2} \right)} \right] - \left(\frac{|1 - \frac{1}{2}|}{2} \right) \cdot TRI \left[\frac{\xi}{\left(\frac{|1 - \frac{1}{2}|}{2} \right)} \right] \right) \\ &= \frac{3}{2} \cdot TRI \left[\frac{\xi}{\frac{3}{4}} \right] - \frac{1}{2} \cdot TRI \left[\frac{\xi}{\frac{1}{4}} \right] \end{aligned}$$



Since this is a real-valued and nonnegative function, the imaginary part is $0[\xi]$, the magnitude is identical to the real part, and the phase is $0[\xi]$,

(b) $\cos[\pi x] \cdot GAUS[x]$

$$\begin{aligned}
 \mathcal{F}_1 \{ \cos[\pi x] \cdot GAUS[x] \} &= \mathcal{F}_1 \left\{ \cos \left[2\pi \frac{x}{2} \right] \cdot GAUS[x] \right\} \\
 &= \mathcal{F}_1 \left\{ \cos \left[2\pi \frac{x}{2} \right] \right\} * \mathcal{F}_1 \{ GAUS[x] \} \\
 &= \frac{1}{2} \left(\delta \left[\xi + \frac{1}{2} \right] + \delta \left[\xi - \frac{1}{2} \right] \right) * GAUS[\xi] \\
 &= \frac{1}{2} \left(GAUS \left[\xi + \frac{1}{2} \right] + GAUS \left[\xi - \frac{1}{2} \right] \right) \\
 &= \frac{1}{2} \left(\exp \left[-\pi \left(\xi + \frac{1}{2} \right)^2 \right] + \exp \left[-\pi \left(\xi - \frac{1}{2} \right)^2 \right] \right)
 \end{aligned}$$



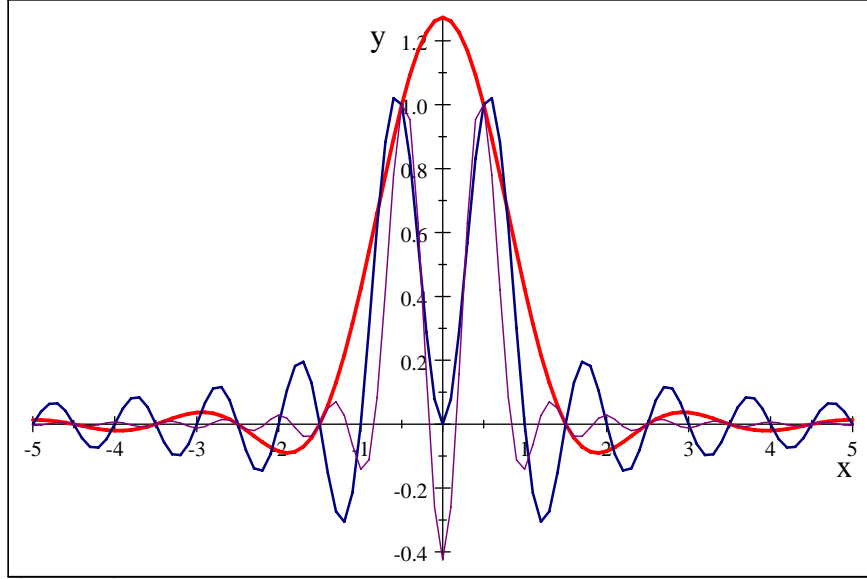
also a nonnegative real function, so the magnitude is identical to the real part and the imaginary part and phase are both 0[ξ]

(c) $\cos [2\pi\xi_0x] \cdot \text{RECT} \left[\frac{x}{b} \right]$

$$\begin{aligned} \mathcal{F}_1 \left\{ \cos [2\pi\xi_0x] \cdot \text{RECT} \left[\frac{x}{b} \right] \right\} &= \mathcal{F}_1 \{ \cos [2\pi\xi_0x] \} * \mathcal{F}_1 \left\{ \text{RECT} \left[\frac{x}{b} \right] \right\} \\ &= \frac{1}{2} \left(\delta \left[\xi + \frac{1}{2} \right] + \delta \left[\xi - \frac{1}{2} \right] \right) * |b| \cdot \text{SINC} [b\xi] \\ &= \frac{|b|}{2} \left(\text{SINC} \left[b \left(\xi + \frac{1}{2} \right) \right] + \text{SINC} \left[b \left(\xi - \frac{1}{2} \right) \right] \right) \end{aligned}$$

so that the Dirac delta functions in the spectrum of the cosine are now SINC functions of width b^{-1} and amplitude b .

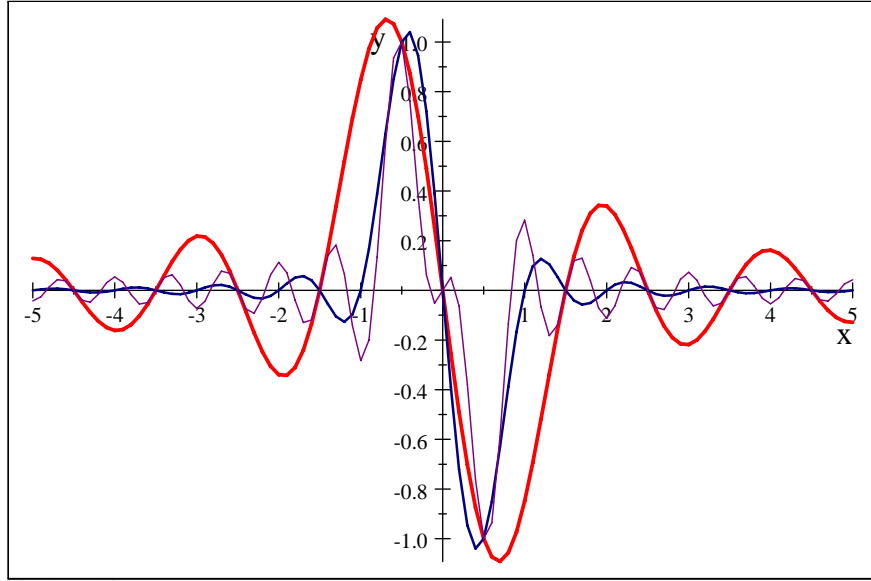
$$\frac{\sin \left[\pi \left(x + \frac{1}{2} \right) \right]}{\pi \left(x + \frac{1}{2} \right)} + \frac{\sin \left[\pi \left(x - \frac{1}{2} \right) \right]}{\pi \left(x - \frac{1}{2} \right)}$$



$\frac{\sin \left[\pi b \left(x + \frac{1}{2} \right) \right]}{\pi b \left(x + \frac{1}{2} \right)} + \frac{\sin \left[\pi b \left(x - \frac{1}{2} \right) \right]}{\pi b \left(x - \frac{1}{2} \right)}$ for $b = 1$ (red), for $b = 2$ (blue), and for $b = 3$ (purple). Note that the “widths” of the SINC functions decrease with increasing b .

(d) $\sin [2\pi\xi_0x] \cdot \text{RECT} \left[\frac{x}{b} \right]$

$$\begin{aligned} \mathcal{F}_1 \left\{ \sin [2\pi\xi_0x] \cdot \text{RECT} \left[\frac{x}{b} \right] \right\} &= \mathcal{F}_1 \{ \sin [2\pi\xi_0x] \} * \mathcal{F}_1 \left\{ \text{RECT} \left[\frac{x}{b} \right] \right\} \\ &= \frac{i}{2} \left(\delta \left[\xi + \frac{1}{2} \right] - \delta \left[\xi - \frac{1}{2} \right] \right) * |b| \cdot \text{SINC} [b\xi] \\ &= i \cdot \frac{|b|}{2} \left(\text{SINC} \left[b \left(\xi + \frac{1}{2} \right) \right] - \text{SINC} \left[b \left(\xi - \frac{1}{2} \right) \right] \right) \end{aligned}$$



$\frac{\sin[\pi b(x+\frac{1}{2})]}{\pi b(x+\frac{1}{2})} - \frac{\sin[\pi b(x-\frac{1}{2})]}{\pi b(x-\frac{1}{2})}$ for $b = 1$ (red), for $b = 2$ (blue), and for $b = 3$ (purple). Note that the “widths” of the SINC functions decrease with increasing b .