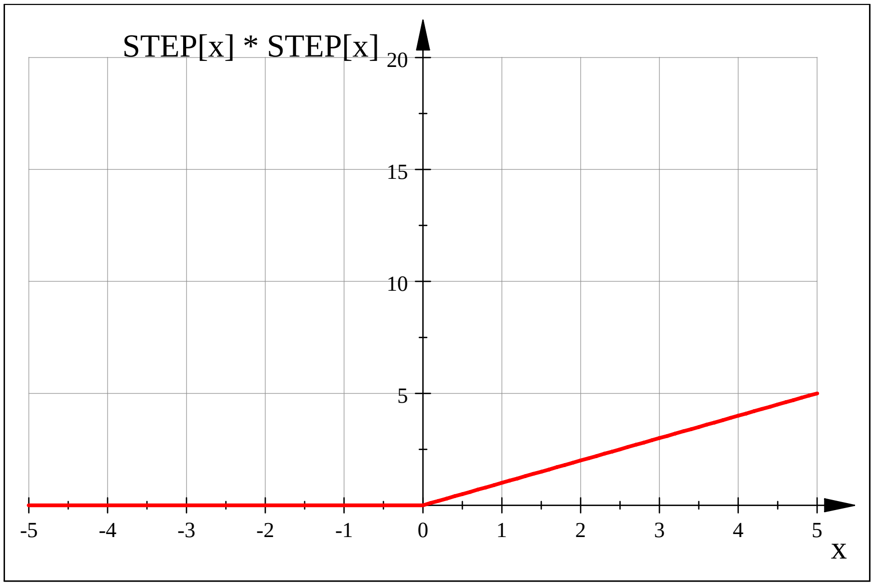
1051-716-20091 Amended Solution Set #4

1. Perform the following convolutions and sketch the result in each case:

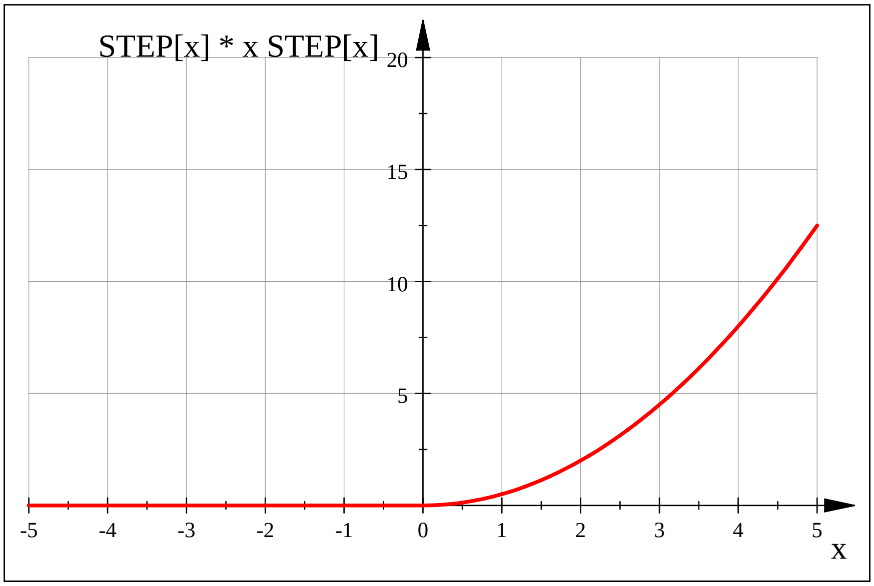
   (a) $STEP[x] * STEP[x]$

$$
\begin{aligned}
STEP[x] * STEP[x] &= \int_{-\infty}^{+\infty} STEP[\alpha] \cdot STEP[x-\alpha] \; d\alpha \\
&= \int_{0}^{+\infty} STEP[x-\alpha] \; d\alpha \\
&= \begin{cases} \int_0^x 1 \; d\alpha & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases} \\
&= \left(\int_0^x 1 \; d\alpha \right) \cdot STEP[x] = \boxed{x \cdot STEP[x] \equiv RAMP[x]}
\end{aligned}
$$



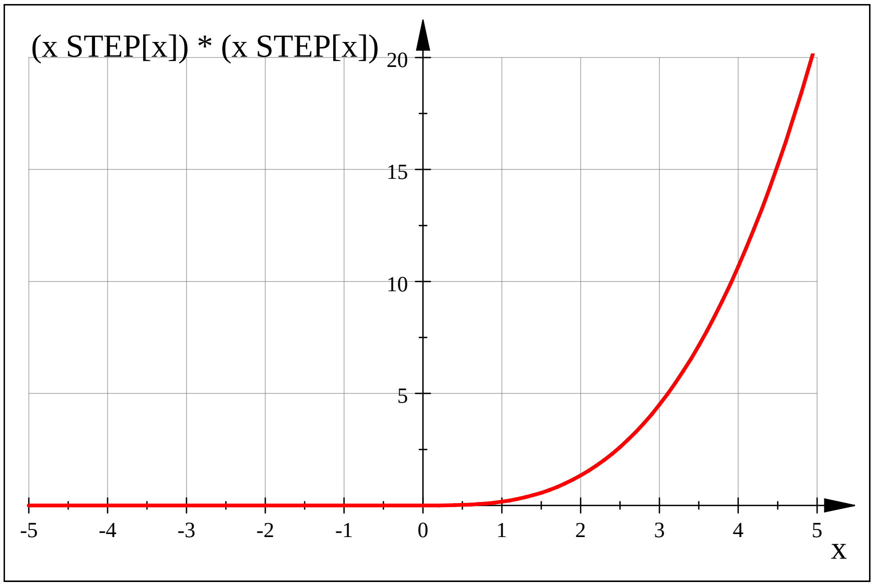
(b) $STEP[x] * (x \cdot STEP[x])$

$$\begin{aligned}
STEP[x] * (x \cdot STEP[x]) &= (x \cdot STEP[x]) * STEP[x] \quad \textit{by commutative property} \\
&= \int_{-\infty}^{+\infty} \alpha \cdot STEP[\alpha] \cdot STEP[x-\alpha] \; d\alpha \\
&= \int_{0}^{+\infty} \alpha \cdot STEP[x-\alpha] \; d\alpha \\
&= \begin{cases} \int_0^x \alpha \; d\alpha & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0 \end{cases} \\
&= \left(\int_0^x \alpha \; d\alpha \right) \cdot STEP[x] = \boxed{\frac{x^2}{2} \cdot STEP[x]}
\end{aligned}$$



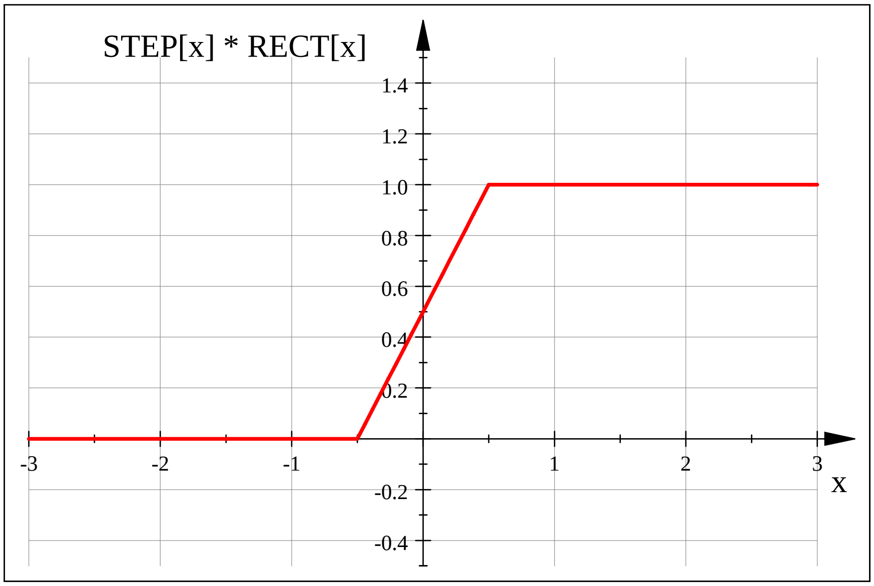
(c) $(x \cdot STEP[x]) * (x \cdot STEP[x])$

$$\begin{aligned}
(x \cdot STEP[x]) * (x \cdot STEP[x]) &= (x \cdot STEP[x]) * STEP[x] \;\; \textit{by commutative property} \\
&= \int_{-\infty}^{+\infty} \alpha \cdot STEP[\alpha] \cdot (x-\alpha) \cdot STEP[x-\alpha] \; d\alpha \\
&= \int_{0}^{+\infty} \alpha \cdot (x-\alpha) \cdot STEP[x-\alpha] \; d\alpha \\
&= \begin{cases} \int_0^x \alpha \cdot (x-\alpha) \; d\alpha & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases} \\
&= \left(\frac{x^3}{2} - \frac{x^3}{6}\right) \cdot STEP[x] = \boxed{\frac{x^3}{6} \cdot STEP[x]}
\end{aligned}$$



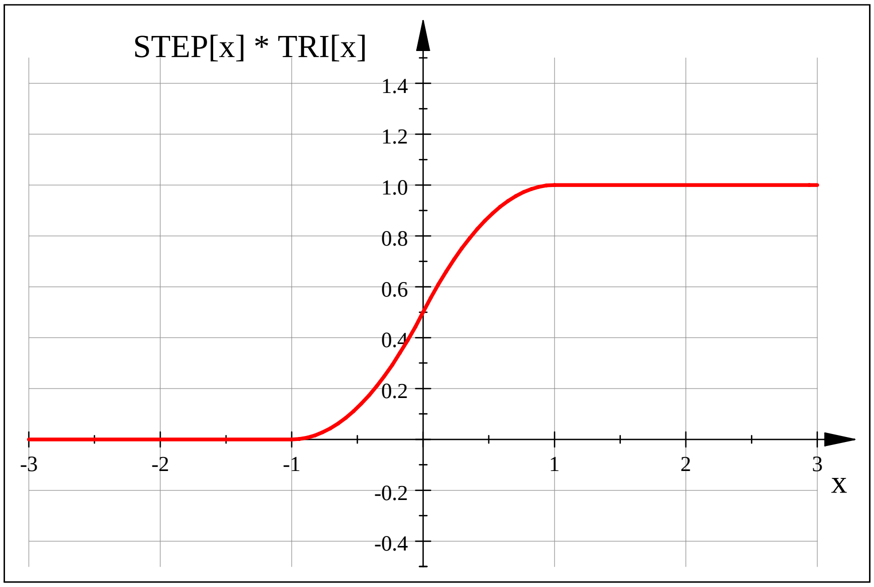
(d) $STEP[x] * RECT[x]$

$$\begin{aligned}
STEP[x] * RECT[x] &= \int_{-\infty}^{+\infty} STEP[\alpha] \cdot RECT[x-\alpha]\ d\alpha \\
&= \int_{0}^{+\infty} RECT[x-\alpha]\ d\alpha \\
&= \begin{cases} 1 & \text{if} \quad x > +\frac{1}{2} \\ \int_{-\frac{1}{2}}^{x} 1\ d\alpha = x - \left(-\frac{1}{2}\right) & \text{if} \quad -\frac{1}{2} \leq x \leq +\frac{1}{2} \\ 0 & \text{if} \quad x \leq -\frac{1}{2} \end{cases} \\
&= \boxed{\left(x + \frac{1}{2}\right) \cdot RECT[x] + STEP\left[x - \frac{1}{2}\right]}
\end{aligned}$$



(e) $STEP[x] * TRI[x]$

$$
\begin{aligned}
STEP[x] * TRI[x] &= TRI[x] * STEP[x] = \int_{-\infty}^{+\infty} TRI[\alpha] \cdot STEP[x-\alpha]\ d\alpha \\
&= \int_{-\infty}^{x} TRI[\alpha]\ d\alpha \\
&= \begin{cases} 1 & \text{if} \quad x > 1 \\ \int_{-1}^{0} (1+\alpha)\ d\alpha + \int_{0}^{x} (1-\alpha)\, d\alpha = \frac{1}{2} + \left(x - \frac{x^2}{2}\right) & \text{if} \quad 0 < x \leq 1 \\ \int_{-1}^{x} (1+\alpha)\ d\alpha = \frac{1}{2}(x+1)^2 & \text{if} \quad -1 \leq x \leq 0 \\ 0 & \text{if} \quad x \leq -1 \end{cases}
\end{aligned}
$$



(f) $s[x] = STEP[x] * (RECT[x+2] - RECT[x-2])$

$$\begin{aligned} & STEP[x] * (RECT[x+2] - RECT[x-2]) \\ = \; & (STEP[x] * RECT[x+2]) - (STEP[x] * RECT[x-2]) \\ = \; & (STEP[x] * (RECT[x] * \delta[x+2])) - (STEP[x] * (RECT[x] * \delta[x-2])) \\ = \; & (STEP[x] * RECT[x]) * \delta[x+2] - (STEP[x] * RECT[x]) * \delta[x-2] \end{aligned}$$

From (d), we know that:

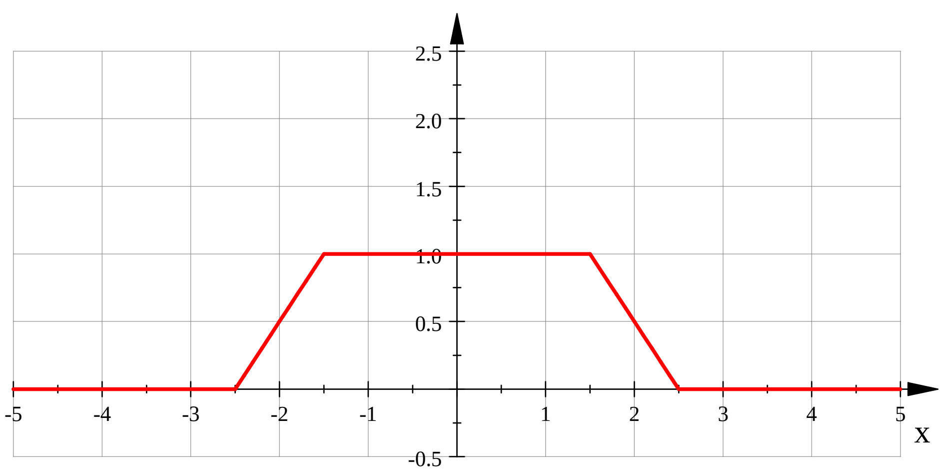
$$\begin{aligned} STEP[x] * RECT[x] &= \left(x + \frac{1}{2}\right) \cdot RECT[x] + STEP\left[x - \frac{1}{2}\right] \\ (STEP[x] * (RECT[x] * \delta[x+2])) &= \left(\left(x + \frac{1}{2}\right) \cdot RECT[x] + STEP\left[x - \frac{1}{2}\right]\right) * \delta[x+2] \\ &= \left((x+2) + \frac{1}{2}\right) \cdot RECT[x+2] + STEP\left[(x+2) - \frac{1}{2}\right] \\ &= \left(x + \frac{5}{2}\right) \cdot RECT[x+2] + STEP\left[x + \frac{3}{2}\right] \end{aligned}$$

$$\begin{aligned} (STEP[x] * (RECT[x] * \delta[x-2])) &= \left(\left(x + \frac{1}{2}\right) \cdot RECT[x] + STEP\left[x - \frac{1}{2}\right]\right) * \delta[x-2] \\ &= \left((x-2) + \frac{1}{2}\right) \cdot RECT[x-2] + STEP\left[(x-2) - \frac{1}{2}\right] \\ &= \left(x - \frac{3}{2}\right) \cdot RECT[x-2] + STEP\left[x - \frac{5}{2}\right] \end{aligned}$$

Add the two together:

$$\begin{aligned} STEP[x] * (RECT[x+2] - RECT[x-2]) = & \left(x + \frac{5}{2}\right) \cdot RECT[x+2] \\ & - STEP\left[x + \frac{3}{2}\right] + \left(x - \frac{3}{2}\right) \cdot RECT[x-2] \end{aligned}$$

$$= \begin{cases} 0 & \text{if} \quad x > \frac{5}{2} \\ \frac{5}{2} - x & \text{if} \quad \frac{3}{2} < x < \frac{5}{2} \\ 1 & \text{if} \quad \frac{3}{2} > x > -\frac{3}{2} \\ x + \frac{5}{2} & \text{if} \quad -\frac{5}{2} \le x \le -\frac{3}{2} \\ 0 & \text{if} \quad x \le -\frac{5}{2} \end{cases}$$

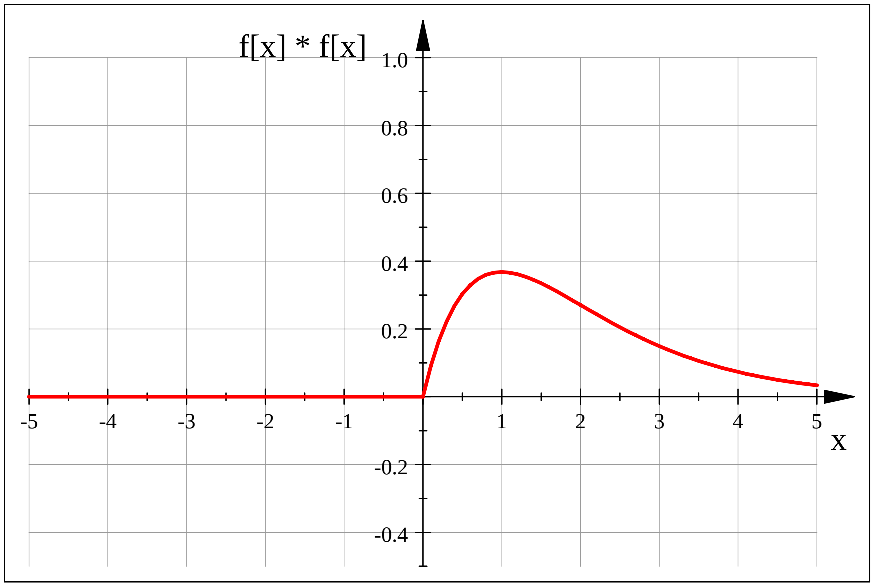


$STEP[x] * (RECT[x+2] + RECT[x-2])$

2. For $f[x] = e^{-x} \cdot STEP[x]$, evaluate and sketch the following convolutions:

(a) $f[x] * f[x]$

$$\begin{aligned}
\left(e^{-x} \cdot STEP[x]\right) * \left(e^{-x} \cdot STEP[x]\right) &= \int_{-\infty}^{+\infty} \left(e^{-\alpha} \cdot STEP[\alpha]\right) \cdot \left(e^{-(x-\alpha)} \cdot STEP[x-\alpha]\right) d\alpha \\
&= \int_{0}^{+\infty} e^{-\alpha} \cdot \left(e^{-(x-\alpha)} \cdot STEP[x-\alpha]\right) d\alpha \\
&= \begin{cases} \int_0^x e^{-\alpha} \cdot e^{-(x-\alpha)} d\alpha & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases} \\
&= \int_0^x e^{-\alpha} \cdot e^{-(x-\alpha)} d\alpha \cdot STEP[x] \\
&= e^{-x} \int_0^x d\alpha \cdot STEP[x] = \boxed{x \cdot e^{-x} \cdot STEP[x]}
\end{aligned}$$



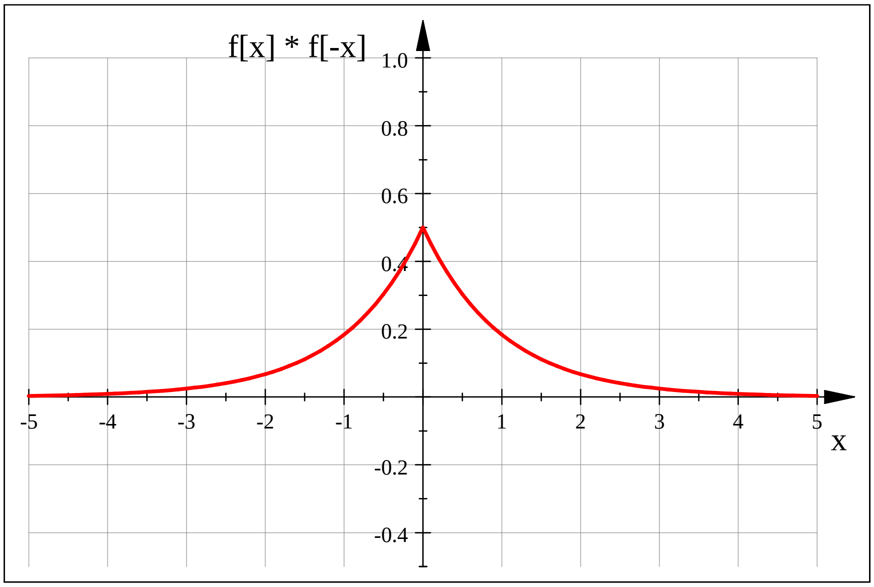
(b) $f[x] * f[-x]$

Note that since $f[x]$ is real valued, this is the autocorrelation of $f[x]$

$$f[x] * f[-x] = f[x] \bigstar f[+x] \text{ if } f[x] \text{ is real valued,}$$

so we expect the autocorrelation to be real and even with its maximum value at the origin:

$$\begin{aligned}
\left(e^{-x} \cdot STEP[x]\right) * \left(e^{-(-x)} \cdot STEP[-x]\right) &= \left(e^{-x} \cdot STEP[x]\right) * \left(e^{+x} \cdot STEP[-x]\right) \\
& \int_{-\infty}^{+\infty} \left(e^{-\alpha} \cdot STEP[\alpha]\right) \cdot \left(e^{+(x-\alpha)} \cdot STEP[-(x-\alpha)]\right) d\alpha \\
&= e^{x} \int_{0}^{+\infty} e^{-2\alpha} \cdot STEP[\alpha] \cdot STEP[\alpha - x] \; d\alpha \\
&= e^{x} \cdot \begin{cases} \int_{x}^{+\infty} e^{-2\alpha} \, d\alpha & \text{if} \quad x > 0 \\ \int_{0}^{+\infty} e^{-2\alpha} \, d\alpha & \text{if} \quad x \leq 0 \end{cases} \\
&= \begin{cases} \frac{1}{2} e^{-x} & \text{if} \quad x > 0 \\ \frac{1}{2} e^{+x} & \text{if} \quad x \leq 0 \end{cases} \\
&= \boxed{\frac{1}{2} \exp[-|x|]}
\end{aligned}$$



3. You may substitute any function $f[x]$ and any impulse response $h[x]$ in the following expression. State the conditions on $f[x]$ and $h[x]$ that ensure that the result is correct. Explain your reasoning.

$$f[x] * h[x] = f[-x]$$

We can think of this in either the space or frequency domains:

$$\begin{aligned}
f[x] * h[x] &= \int_{-\infty}^{+\infty} f[\alpha]\ \ h[x-\alpha]\ \ d\alpha \\
f[-x] &= f[-x] * \delta[x] = \int_{-\infty}^{+\infty} f[-\alpha]\ \ \delta[x-\alpha]\ \ d\alpha \\
h[x-\alpha] &= \delta[x-\alpha] \implies h[x] = \delta[x] \\
f[-\alpha] &= f[+\alpha] \implies f[x]\ \ \textit{is even}
\end{aligned}$$

In the frequency domain:

$$\begin{aligned}
F[\xi] \cdot H[x] &= F[-\xi] \implies H[\xi] = 1[\xi] \implies h[x] = \delta[x] \\
\textit{and}\ F[-\xi] &= F[\xi] \implies f[x] = f[-x]
\end{aligned}$$

So

$$\boxed{h[x] = \delta[x]\ \ \textit{and}\ \ f[x] = f[-x]}$$

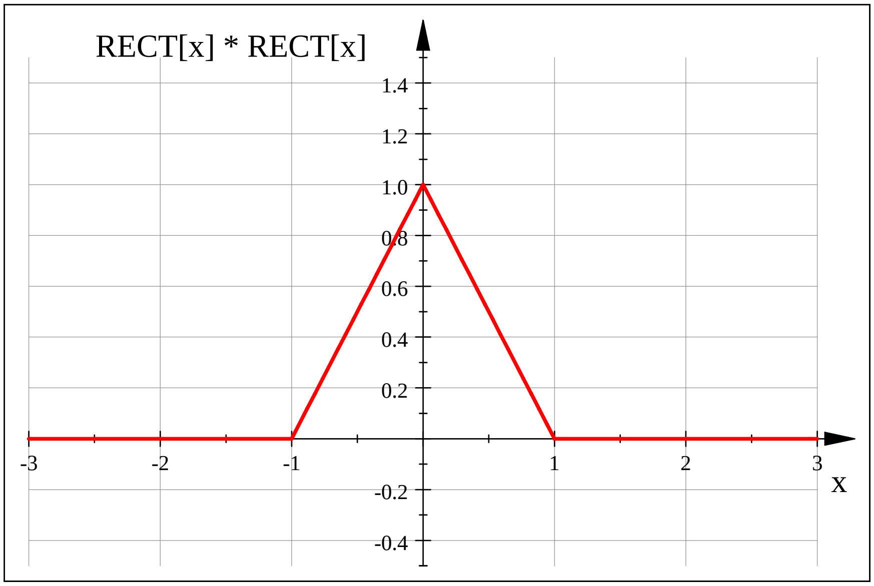
Note that you can also select $h[x] = 1[x]$ *for an even* $f[x]$, *so that the convolution calculates the constant area of* $f[x]$. *If* $f[x]$ *is odd, then the we can "reverse" the function if we also negate it, which means that:*

$$\boxed{h[x] = -\delta[x]\ \ \textit{and}\ \ f[x] = -f[-x]}$$

4. Perform the following convolutions and sketch the result in each case:

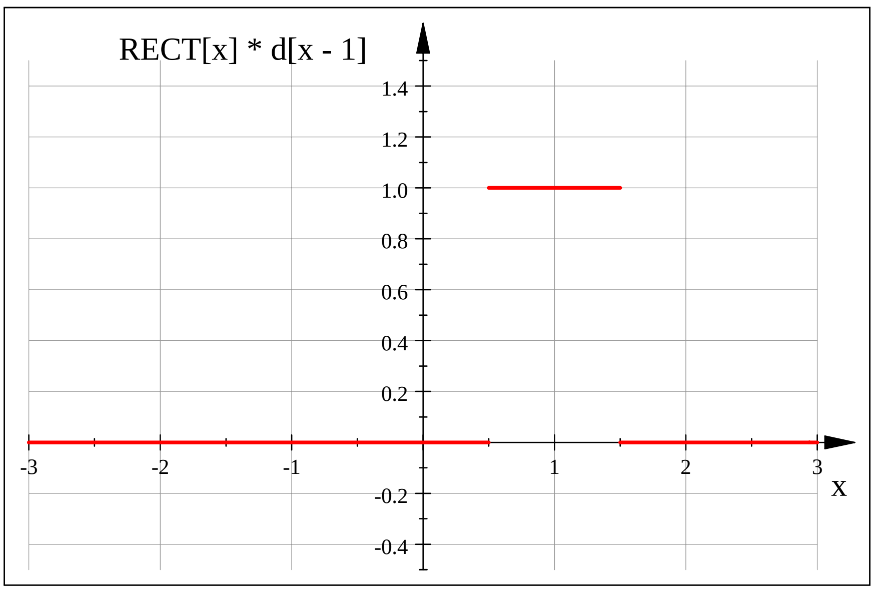
(a) $RECT\,[x] * RECT\,[x]$

$$\begin{aligned}
RECT\,[x] * RECT\,[x] &= \int_{-\infty}^{+\infty} RECT\,[\alpha] \cdot RECT\,[x-\alpha]\;\; d\alpha \\
&= \int_{-\frac{1}{2}}^{+\frac{1}{2}} RECT\,[x-\alpha]\;\; d\alpha \\
&= \begin{cases} 0 & \text{if} \quad x > +1 \\ \int_{x-\frac{1}{2}}^{+\frac{1}{2}} 1\; d\alpha & \text{if} \quad 0 < x < +1 \\ \int_{-\frac{1}{2}}^{x+\frac{1}{2}} 1\; d\alpha & \text{if} \quad -1 < x < 0 \\ 0 & \text{if} \quad x < -1 \end{cases} = \begin{cases} 0 & \text{if} \quad x > +1 \\ 1-x & \text{if} \quad 0 < x < +1 \\ 1+x & \text{if} \quad -1 < x < 0 \\ 0 & \text{if} \quad x < -1 \end{cases} \\
&= (1-|x|) \cdot RECT\left[\frac{x}{2}\right] = TRI\,[x]
\end{aligned}$$



(b) $RECT\,[x] * \delta\,[x-1]$

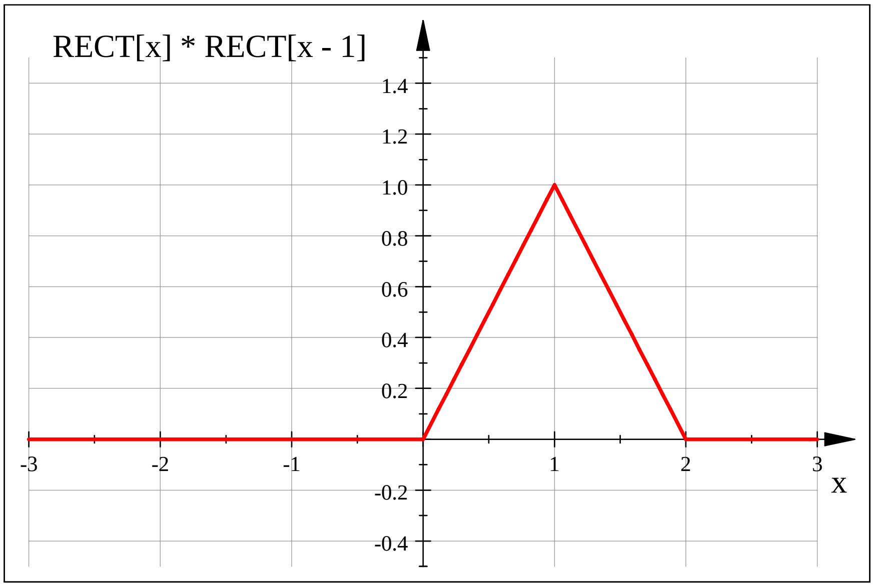
$$\begin{aligned}
RECT\,[x] * \delta\,[x-1] &= \int_{-\infty}^{+\infty} RECT\,[\alpha] \cdot \delta\,[x-1-\alpha]\;\; d\alpha \\
&= \int_{-\infty}^{+\infty} RECT\,[\alpha] \cdot \delta\,[x-(\alpha+1)]\;\; d\alpha \\
&= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \delta\,[x-(\alpha+1)]\;\; d\alpha \\
u &\equiv \alpha+1 \implies \alpha = u-1 \implies d\alpha = du \\
\alpha &= +\frac{1}{2} \implies u = +\frac{3}{2} \\
\alpha &= -\frac{1}{2} \implies u = +\frac{1}{2} \\
RECT\,[x] * \delta\,[x-1] &= \int_{-\frac{1}{2}}^{+\frac{3}{2}} \delta\,[x-u]\;\; du = \\
&= \begin{cases} 0 & \text{if} \quad x > +\frac{3}{2} \\ \frac{1}{2} & \text{if} \quad x = +\frac{3}{2} \\ 1 & \text{if} \quad +\frac{1}{2} < x < +\frac{3}{2} \\ \frac{1}{2} & \text{if} \quad x = +\frac{1}{2} \\ 0 & \text{if} \quad x < +\frac{1}{2} \end{cases} = RECT\,[x-1]
\end{aligned}$$



(c) $RECT[x] * RECT[x-1]$

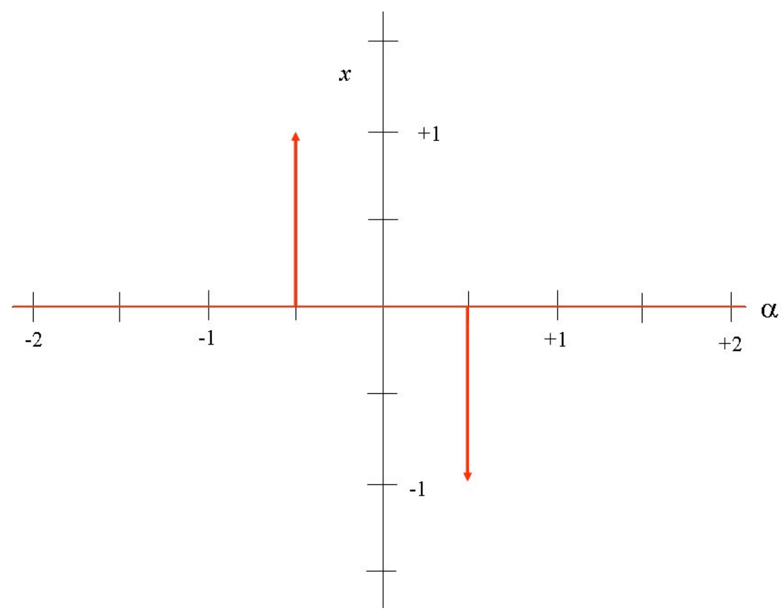
$$\begin{aligned} RECT[x] * RECT[x-1] &= RECT[x] * (RECT[x] * \delta[x-1]) \\ &= (RECT[x] * RECT[x]) * \delta[x-1] \\ &= TRI[x] * \delta[x-1] \;\; \textit{(from part a)} \\ &= TRI[x-1] \;\; \textit{(from part b)} \end{aligned}$$

$$\begin{cases} 0 & \text{if} \quad x > +2 \\ 2-x & \text{if} \quad 1 < x < +2 \\ x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x < 0 \end{cases}$$



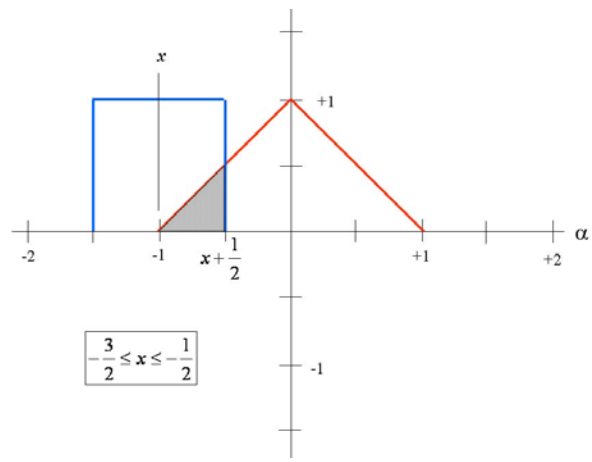
(d) $RECT[x] * \left(\frac{d}{dx}\delta[x]\right)$

$$\begin{aligned} RECT[x] * \frac{d}{dx}\delta[x] &= \int_{-\infty}^{+\infty} RECT[\alpha] \cdot \frac{d}{dx}\delta[x-\alpha] \;\; d\alpha \\ &= \frac{d}{dx}\int_{-\infty}^{+\infty} RECT[\alpha] \cdot \delta[x-\alpha] \;\; d\alpha \\ &= \frac{d}{dx}RECT[x] \\ &= \frac{d}{dx}\left(STEP\left[x+\frac{1}{2}\right] - STEP\left[x-\frac{1}{2}\right]\right) \\ &= \frac{d}{dx}STEP\left[x+\frac{1}{2}\right] - \frac{d}{dx}STEP\left[x-\frac{1}{2}\right] \\ &= \delta\left[x+\frac{1}{2}\right] - \delta\left[x-\frac{1}{2}\right] \end{aligned}$$

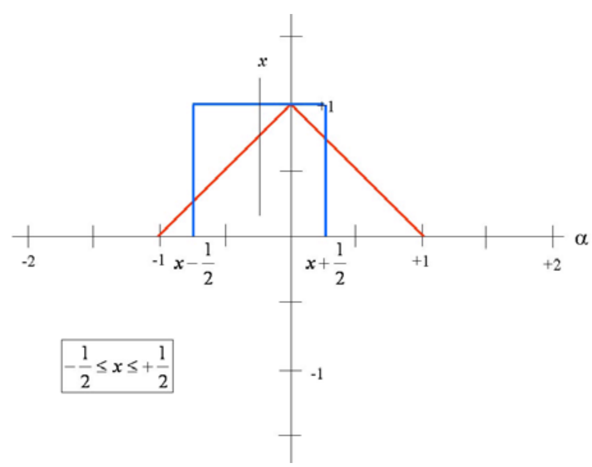


(e) $RECT[x] * TRI[x]$

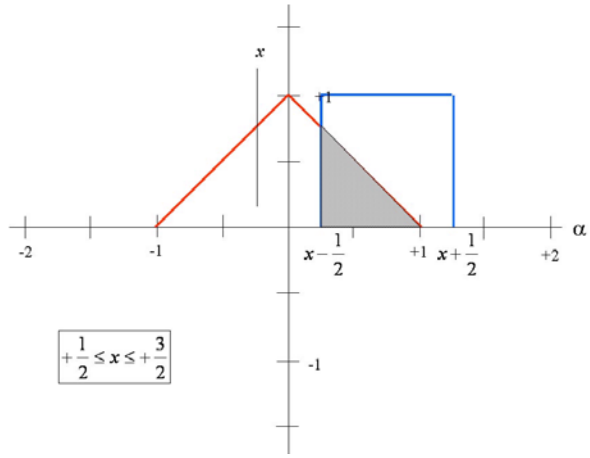
$$\begin{aligned} RECT[x] * TRI[x] &= TRI[x] * RECT[x] = \int_{-\infty}^{+\infty} TRI[\alpha] \cdot RECT[x-\alpha]\ d\alpha \\ &= \int_{-\infty}^{+\infty} (1-|\alpha|) \cdot RECT\left[\frac{\alpha}{2}\right] \cdot RECT[x-\alpha]\ d\alpha \\ &= 0\ \textit{if}\ x < -1 - \frac{1}{2} = -\frac{3}{2} \\ &= 0\ \textit{if}\ x > +1 + \frac{1}{2} = +\frac{3}{2} \end{aligned}$$



$$RECT[x] * TRI[x] = \int_{-1}^{x+\frac{1}{2}} (1+\alpha) \cdot 1\ d\alpha = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8}\ \textit{if}\ -\frac{3}{2} < x < -\frac{1}{2}$$

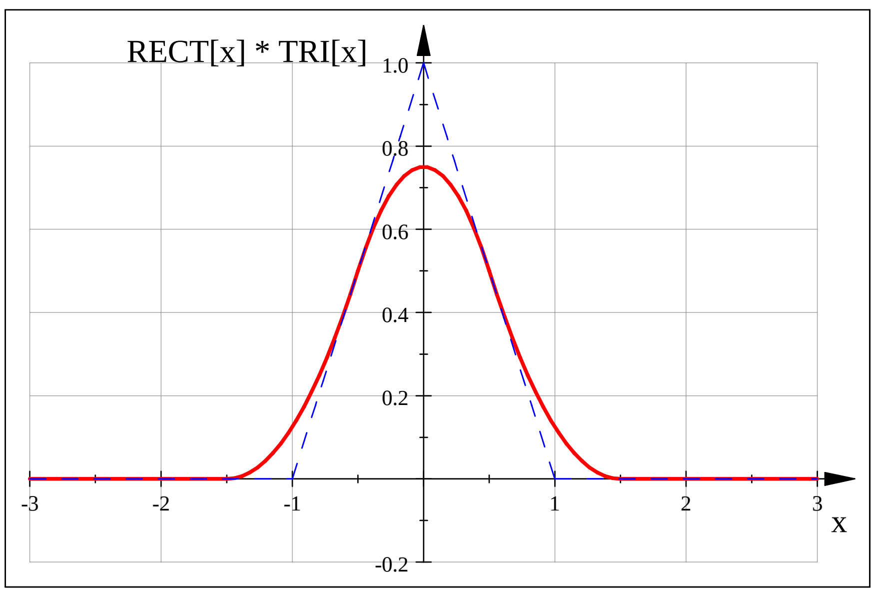


$$\begin{aligned} RECT[x] * TRI[x] &= \int_{x-\frac{1}{2}}^{0} (1+\alpha) \cdot 1\ d\alpha + \int_{0}^{x+\frac{1}{2}} (1-\alpha) \cdot 1\ d\alpha \textit{if}\ -\frac{1}{2} < x < +\frac{1}{2} \\ &= \left(-\frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{8}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{2}x + \frac{3}{8}\right) \\ &= \frac{3}{4} - x^2\ \textit{if}\ -\frac{1}{2} < x < +\frac{1}{2} \end{aligned}$$



$$
\begin{aligned}
RECT[x] * TRI[x] &= \int_{x-\frac{1}{2}}^{1} (1-\alpha)\cdot 1\, d\alpha \ \textit{if} \ +\frac{1}{2} < x < +\frac{3}{2} \\
&= \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8}
\end{aligned}
$$

$$
RECT[x] * TRI[x] = \begin{cases} 0 & \text{if} \quad x > +\frac{3}{2} \\ \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8} & \text{if} \quad +\frac{1}{2} \le x \le +\frac{3}{2} \\ \frac{3}{4} - x^2 & \text{if} \quad -\frac{1}{2} \le x \le +\frac{1}{2} \\ \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8} & \text{if} \quad -\frac{3}{2} \le x \le -\frac{1}{2} \\ 0 & \text{if} \quad x < -\frac{3}{2} \end{cases}
$$

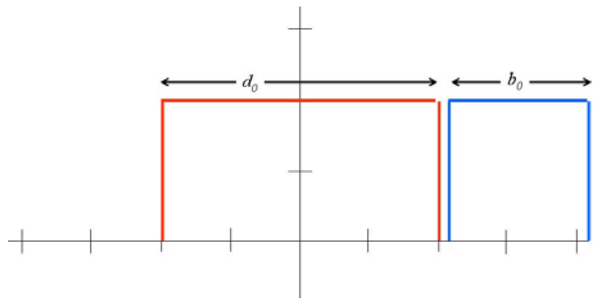


$TRI[x] * RECT[x]$ (red) and $TRI[x]$ (blue), showing the increased width and the "pushing" of the values towards the mean by to the convolution.

5. Show that the following expression is true for arbitrary real values d_0 and b_0:

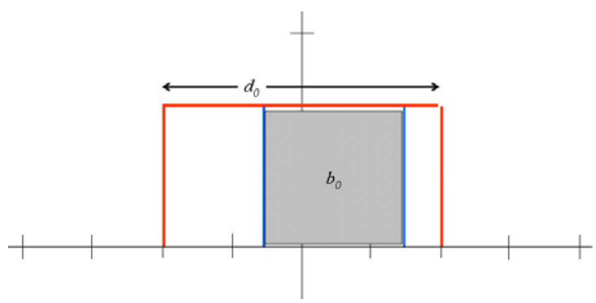
$$RECT\left[\frac{x}{d_0}\right] * RECT\left[\frac{x}{b_0}\right] = \left(\frac{d_0+b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+b_0}{2}\right)}\right] - \left(\frac{|d_0-b_0|}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{|d_0-b_0|}{2}\right)}\right]$$

First assume that $d_0 > b_0 > 0$ (the opposite case is easy to prove via the commutative property of convolution). It is helpful (as usual) to look at this graphically:



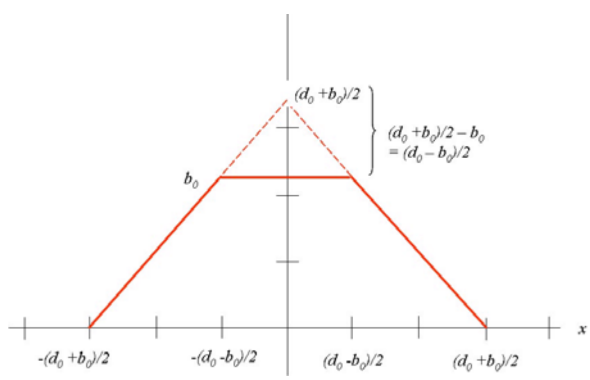
Sample rectangles of different widths $d_0 > b_0$

For translations where the narrower rectangle is enclosed by the wider rectangle, the overlap area is the area of the narrower rectangle:



If $-\frac{d_0-b_0}{2} < x < +\frac{d_0-b_0}{2}$, the narrower rectangle is enclosed by the wider rectangle and the overlap area is the area b_0 of the narrower rectangle.

For values of x where the smaller rectangle does not completely overlap the larger, the convolution decreases linearly with the offset until the two do not overlap, which occurs for $x < -\frac{d_0+b_0}{2}$ and $x > \frac{d_0+b_0}{2}$, as shown:



The slope of the side for $x > 0$ is

$$\frac{-b_0}{\left(\frac{d_0+b_0}{2}\right) - \left(\frac{d_0-b_0}{2}\right)} = \frac{-b_0}{b_0} = -1$$

and that for $x < 0$ is:

$$\frac{+b_0}{-\left(\frac{d_0-b_0}{2}\right)-\left(-\frac{d_0+b_0}{2}\right)}=\frac{+b_0}{b_0}=+1$$

So we could write consider the slanted sides to have been generated by a triangle with unit slope and height $\frac{d_0+b_0}{2}$:

$$\left(\frac{d_0+b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+b_0}{2}\right)}\right]$$

We now merely subtract off the top portion, which is a triangle of height $\frac{d_0+b_0}{2}-b_0$ and width parameter $\frac{d_0-b_0}{2}$:

$$\begin{aligned}RECT\left[\frac{x}{d_0}\right]*RECT\left[\frac{x}{b_0}\right] &= \left(\frac{d_0+b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+b_0}{2}\right)}\right]-\left(\frac{d_0+b_0}{2}-b_0\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0-b_0}{2}\right)}\right]\\ &= \left(\frac{d_0+b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+b_0}{2}\right)}\right]-\left(\frac{d_0-b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0-b_0}{2}\right)}\right]\end{aligned}$$

Note that this is true for the conditions given: that $d_0 > b_0$. If $d_0 < b_0$, the first ("wider") triangle is unaffected, but the width and height of the second become negative. We can remedy this by evaluating the absolute value:

$$\boxed{RECT\left[\frac{x}{d_0}\right]*RECT\left[\frac{x}{b_0}\right]=\left(\frac{d_0+b_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+b_0}{2}\right)}\right]-\left|\frac{d_0-b_0}{2}\right|\cdot TRI\left[\frac{x}{\left(\frac{d_0-b_0}{2}\right)}\right]}$$

Test for the known case of $d_0 = b_0$:

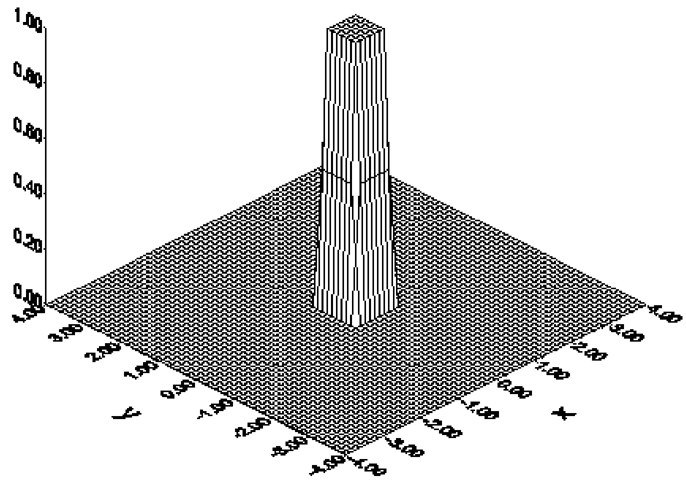
$$\begin{aligned}RECT\left[\frac{x}{d_0}\right]*RECT\left[\frac{x}{d_0}\right] &= \left(\frac{d_0+d_0}{2}\right)\cdot TRI\left[\frac{x}{\left(\frac{d_0+d_0}{2}\right)}\right]-\left|\frac{d_0-d_0}{2}\right|\cdot TRI\left[\frac{x}{\left(\frac{d_0-d_0}{2}\right)}\right]\\ &= d_0\cdot TRI\left[\frac{x}{d_0}\right]-0\cdot TRI\left[\frac{x}{0}\right]=d_0\cdot TRI\left[\frac{x}{d_0}\right]\ \textit{as it should be.}\end{aligned}$$

6. Evaluate the following 2-D convolutions and produce "appropriate" sketches (i.e., axial profiles, "top views", or perspective views).

   (a) $(RECT[x]\ \ \delta[y]) * (\delta[x]\ \ RECT[y])$

   The functions are separable so the problem simplifies to the product of two 1-D convolutions:

$$\begin{aligned}(RECT[x]\ \ \delta[y]) * (\delta[x]\ \ RECT[y]) &= (RECT[x] * \delta[x])\ \ (\delta[y] * RECT[y]) \\ &= RECT[x]\ \ RECT[y] = RECT[x, y]\end{aligned}$$



$(RECT[x]\ \ \delta[y]) * (\delta[x]\ \ RECT[y])$

(b) $COR[x,y] * COR[x,y]$

$$COR[x,y] = \left(\delta\left[x+\frac{1}{2}\right] + \delta\left[x-\frac{1}{2}\right]\right) RECT[y] + RECT[x]\left(\delta\left[y+\frac{1}{2}\right] + \delta\left[y-\frac{1}{2}\right]\right)$$

The convolution has 16 component terms, BUT many are identical except for location:

$$\begin{aligned}
COR[x,y] * COR[x,y] = & \left\{\left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right) * \left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right) * \left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right) * \left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right) * \left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right) * \left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right) * \left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right) * \left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right) * \left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right) * \left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right) * \left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right) * \left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right) * \left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right) * \left(\delta\left[x+\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right) * \left(\delta\left[x-\frac{1}{2}\right] RECT[y]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right) * \left(RECT[x]\,\delta\left[y+\frac{1}{2}\right]\right)\right\} \\
& + \left\{\left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right) * \left(RECT[x]\,\delta\left[y-\frac{1}{2}\right]\right)\right\}
\end{aligned}$$

So we see that there are two basic forms of convolution – the first is:

$$(\delta[x-x_1]\,RECT[y]) * (\delta[x-x_2]\,RECT[y])$$

$$or$$

$$(RECT[x]\,\delta[y-y_1]) * (RECT[x]\,\delta[y-y_1])$$

which may be evaluated easily:

$$\begin{aligned}
& (\delta[x-x_1]\,RECT[y]) * (\delta[x-x_2]\,RECT[y]) \\
= & (\delta[x-x_1] * \delta[x-x_2]) \cdot (RECT[y] * RECT[y]) \\
= & \delta[x-(x_1+x_2)] \cdot TRI[y]
\end{aligned}$$

This is a 2-D function that is triangle along one axis and a delta along the other. In words, it produces a "wall" function modulated by a triangle.
The second convolution is of the form:

$$(\delta[x-x_1]\ RECT[y]) * (RECT[x]\ \delta[y-y_1])$$
$$or$$
$$(RECT[x]\ \delta[y-y_1]) * (\delta[x-x_1]\ RECT[y])$$

And, again, the first of these yields:

$$\begin{gathered}(\delta[x-x_1]\,RECT[y]) * (RECT[x]\,\delta[y-y_1]) \\ = (\delta[x-x_1] * RECT[x])\ \ (RECT[y] * \delta[y-y_1]) \\ = RECT[x-x_1]\ \ RECT[y-y_1]\end{gathered}$$

which is a 2-D rectangle function centered at $[x_1, y_1]$. So it's just a matter of placing these terms in their proper locations. The sum of the 16 component convolutions yields:

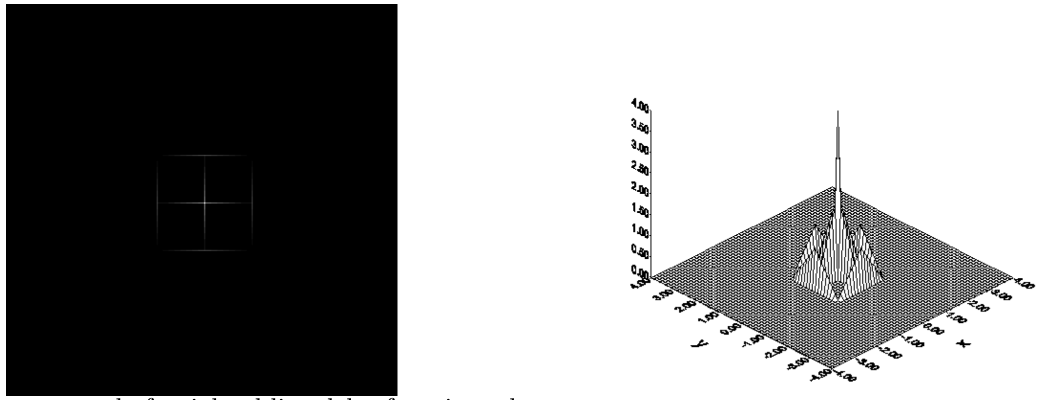
$$\begin{gathered}COR[x,y] * COR[x,y] = \delta[x+1]\ \ TRI[y] + \delta[x]\ \ TRI[y] \\ +RECT\left[x+\frac{1}{2}\right]\ \ RECT\left[y+\frac{1}{2}\right] + RECT\left[x+\frac{1}{2}\right]\ \ RECT\left[y-\frac{1}{2}\right] \\ +\delta[x]\ \ TRI[y] + \delta[x-1]\ \ TRI[y] \\ +RECT\left[x-\frac{1}{2}\right] RECT\left[y+\frac{1}{2}\right] + RECT\left[x-\frac{1}{2}\right] RECT\left[y-\frac{1}{2}\right] \\ +RECT\left[x+\frac{1}{2}\right] RECT\left[y+\frac{1}{2}\right] + RECT\left[x-\frac{1}{2}\right] RECT\left[y+\frac{1}{2}\right] \\ +TRI[x]\,\delta[y+1] + TRI[x]\,\delta[y] \\ +RECT\left[x+\frac{1}{2}\right] RECT\left[y-\frac{1}{2}\right] + RECT\left[x-\frac{1}{2}\right] RECT\left[y-\frac{1}{2}\right] \\ +TRI[x]\,\delta[y] + TRI[x]\,\delta[y-1]\end{gathered}$$

The terms may be combined to give a simpler expression:

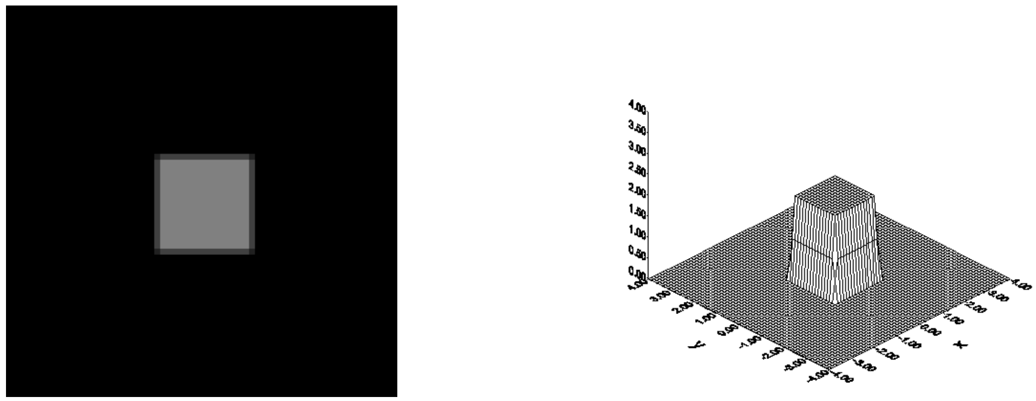
$$\begin{gathered}COR[x,y] * COR[x,y] = (\delta[x+1] + 2\ \delta[x] + \delta[x-1])\,TRI[y] \\ +TRI[x]\,(\delta[y+1] + 2\ \delta[y] + \delta[y-1]) \\ +2\ RECT\left[x+\frac{1}{2}\right]\ \ RECT\left[y+\frac{1}{2}\right] + 2\ RECT\left[x-\frac{1}{2}\right]\ \ RECT\left[y-\frac{1}{2}\right] \\ +2\ RECT\left[x+\frac{1}{2}\right]\ \ RECT\left[y-\frac{1}{2}\right] + 2\ RECT\left[x-\frac{1}{2}\right]\ \ RECT\left[y+\frac{1}{2}\right]\end{gathered}$$

$$= \boxed{(\delta[x+1] + 2\ \delta[x] + \delta[x-1])\,TRI[y] + TRI[x]\,(\delta[y+1] + 2\ \delta[y] + \delta[y-1]) + 2\ RECT\left[\tfrac{x}{2}, \tfrac{y}{2}\right]}$$

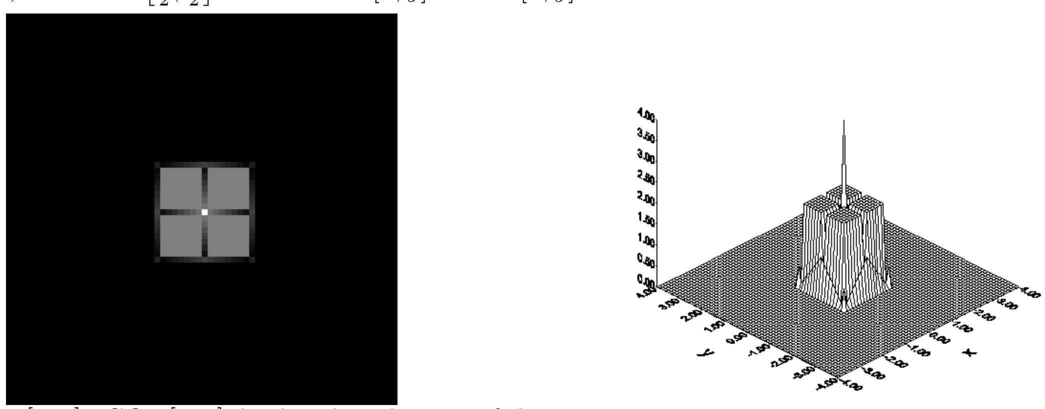
The sum results in 6 "line-triangle" functions and a rectangle function with amplitude 2. The sum of the "line-triangle" functions at the origin yields a 2-D Dirac delta with volume 4 added to the rectangle. The maximum amplitude is located at the origin, as it should be since the autoconvolution of a real-valued function is the autocorrelation, which has its maximum amplitude (equal to the volume of its square) at the origin. The results are shown in pieces and summed together below:



"Line Triangles" composed of weighted line delta functions that appear as part of the convolution $COR\,[x] * COR\,[x]$



Rectangular part, $2 \cdot RECT\left[\frac{x}{2}, \frac{y}{2}\right]$ within $COR\,[x, y] * COR\,[x, y]$



End Result: $COR\,[x, y] * COR\,[x, y]$ displayed as the sum of the two terms before with the weighted line delta functions "on the axis".

7. The convolution of two scaled Gaussian functions and of two $SINC$ functions are $GAUS\left[\frac{x}{3}\right] * GAUS\left[\frac{x}{4}\right]$ and $SINC[3x] * SINC[2x]$.

Evaluate these integrals either rigorously (by direct integration) or approximately by graphical means (HINT: the rigorous solution is possible using complex integration and thus requires background that most of you probably do not have. A graphical approximation is reasonable.).

(a) $GAUS\left[\frac{x}{3}\right] * GAUS\left[\frac{x}{4}\right] = \int_{-\infty}^{+\infty} \exp\left[-\pi\left(\frac{\alpha}{3}\right)^2\right]\exp\left[-\pi\left(\frac{x-\alpha}{4}\right)^2\right]\,d\alpha$

$$\begin{aligned}
GAUS\left[\frac{x}{3}\right] * GAUS\left[\frac{x}{4}\right] &= \int_{-\infty}^{+\infty} \exp\left[-\pi\left(\frac{\alpha}{3}\right)^2\right]\exp\left[-\pi\left(\frac{x-\alpha}{4}\right)^2\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\left(\frac{\alpha}{3}\right)^2\right]\exp\left[-\pi\left(\left(\frac{x}{4}\right)^2+\left(\frac{\alpha}{4}\right)^2-\frac{x\alpha}{8}\right)\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\left(\left(\frac{\alpha}{3}\right)^2+\left(\frac{\alpha}{4}\right)^2\right)\right]\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\exp\left[+\pi\frac{x\alpha}{8}\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\left(\frac{5\alpha}{12}\right)^2\right]\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\exp\left[+\pi\frac{x\alpha}{8}\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\frac{25}{144}\left(\alpha^2-\frac{144}{25\cdot 8}\alpha x+\frac{144}{25}\cdot\frac{1}{16}x^2\right)\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\frac{25}{144}\left(\alpha^2-2\cdot\frac{9}{25}\alpha x+\left(\frac{9}{25}x\right)^2\right)\right]\,d\alpha
\end{aligned}$$

complete the square:

$$\begin{aligned}
\left(\alpha-\frac{9}{25}x\right)^2 &= \alpha^2-\frac{6}{5}x\alpha+\frac{9}{25}x^2 \\
&\Longrightarrow \left(\alpha^2-\frac{18}{25}\alpha x+\frac{81}{625}x^2\right)
\end{aligned}$$

$$\left(\alpha^2-\frac{18}{25}\alpha x+\left(\frac{9}{25}x\right)^2\right)=\left(\alpha-\frac{9}{25}x\right)^2+\frac{144}{625}x^2$$

$$\begin{aligned}
GAUS\left[\frac{x}{3}\right] * GAUS\left[\frac{x}{4}\right] &= \int_{-\infty}^{+\infty} \exp\left[-\pi\frac{25}{144}\left(\left(\alpha-\frac{9}{25}x\right)^2+\frac{144}{625}x^2\right)\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} \exp\left[-\pi\frac{25}{144}\left(\alpha-\frac{9}{25}x\right)^2\right]\cdot\exp\left[-\pi\cdot\frac{25}{144}\cdot\frac{144}{625}x^2\right]\,d\alpha \\
&= \cdot\exp\left[-\pi\cdot\left(\frac{x}{5}\right)^2\right]\cdot\int_{-\infty}^{+\infty}\exp\left[-\pi\frac{25}{144}\left(\alpha-\frac{9}{25}x\right)^2\right]\,d\alpha
\end{aligned}$$

$$\begin{aligned}
\textit{Solve the integral by substituting } u &= \frac{5}{12}\left(\alpha-\frac{9}{25}x\right)\Longrightarrow d\alpha=\frac{12}{5}\,du \\
\alpha &= \pm\infty\Longrightarrow u=\pm\infty
\end{aligned}$$

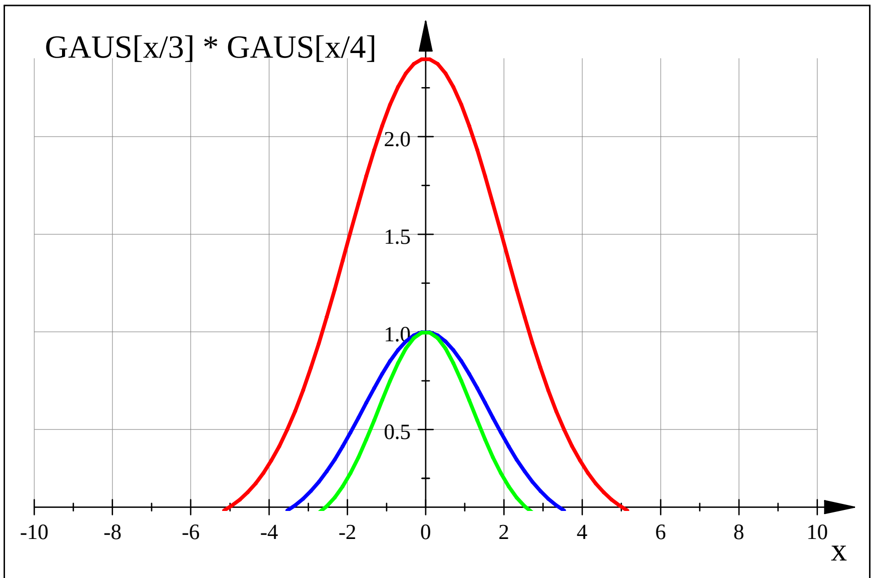
$$\Longrightarrow \quad \exp\left[-\pi\left(\frac{x}{5}\right)^2\right] \int_{u=+\infty}^{-\infty} \exp\left[-\pi u^2\right] \quad \left(+\frac{12}{5}\right) du$$

$$= \quad +\frac{12}{5} \exp\left[-\pi\left(\frac{x}{5}\right)^2\right] \int_{u=-\infty}^{+\infty} \exp\left[-\pi u^2\right] \; du$$

$$\int_{u=-\infty}^{+\infty} \exp\left[-\pi u^2\right] \; du \quad = \quad 1 \Longrightarrow$$

$$GAUS\left[\frac{x}{3}\right] * GAUS\left[\frac{x}{4}\right] \quad = \quad +\frac{12}{5} \exp\left[-\pi\left(\frac{x}{5}\right)^2\right] \cdot 1 = \frac{12}{5} e^{-\pi\left(\frac{x}{5}\right)^2} = \frac{12}{5} GAUS\left[\frac{x}{5}\right]$$

Result is a "taller" and "wider" Gaussian



$\exp\left[-\pi\left(\frac{x}{3}\right)^2\right]$ *(green)*, $\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]$ *(blue), and their convolution (red)*

(b) $SINC[3x] * SINC[2x] = \int_{-\infty}^{+\infty} \left(\frac{\sin[3\pi\alpha]}{3\pi\alpha}\right)\left(\frac{\sin[2\pi(x-\alpha)]}{2\pi(x-\alpha)}\right) d\alpha$

Solution of integral (b) is more difficult, enough so that you are EXCUSED from having to do it – this is just to show the difficulty of doing convolutions by direct integration:

$$SINC[3x] * SINC[2x] = \int_{-\infty}^{+\infty} \left(\frac{\sin[3\pi\alpha]}{3\pi\alpha}\right)\left(\frac{\sin[2\pi(x-\alpha)]}{2\pi(x-\alpha)}\right) d\alpha$$

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{+3\pi i\alpha} - e^{-3\pi i\alpha}}{(3\pi\alpha)(2i)}\right)\left(\frac{e^{+3\pi i(x-\alpha)} - e^{-3\pi i(x-\alpha)}}{2\pi(x-\alpha)(2i)}\right) d\alpha$$

$$= -\frac{1}{24\pi^2}\int_{-\infty}^{+\infty} \frac{\left(e^{+3\pi i\alpha} - e^{-3\pi i\alpha}\right)\left(e^{+3\pi i(x-\alpha)} - e^{-3\pi i(x-\alpha)}\right)}{\alpha(x-\alpha)} d\alpha$$

$$= -\frac{1}{24\pi^2}\int_{-\infty}^{+\infty} \left(\frac{e^{+i\pi(\alpha+2x)} + e^{-i\pi(\alpha+2x)} - e^{-i\pi(5\alpha-2x)} - e^{+i\pi(5\alpha-2x)}}{\alpha(x-\alpha)}\right) d\alpha$$

Now, break up integral into sum of terms with positive and negative exponents

$$= -\frac{1}{24\pi^2}\left(\int_{-\infty}^{+\infty}\left(\frac{e^{+i\pi(\alpha+2x)} - e^{+i\pi(5\alpha-2x)}}{\alpha(x-\alpha)}\right) d\alpha + \int_{-\infty}^{+\infty}\left(\frac{e^{-i\pi(\alpha+2x)} - e^{-i\pi(5\alpha-2x)}}{\alpha(x-\alpha)}\right) d\alpha\right)$$

Solve these integrals by contour integration in the complex plane (not that I expect you to know or recall how to do this....)

close contour of the first integral (with positive exponents) in upper half plane

so that $e^{+i\pi(iy)} \to 0$ *as* $y \to +\infty$

$$\int_{-\infty}^{+\infty}\left(\frac{e^{+i\pi(\alpha+2x)} - e^{+i\pi(5\alpha-2x)}}{\alpha(x-\alpha)}\right) d\alpha = \oint\left(\frac{e^{+i\pi(z+2x)} - e^{+i\pi(5z-2z)}}{z(x-z)}\right) dz$$

The contour integral may be solved via the residue theorem

evaluate the residues at $z = 0$ *and* $z = x$

at $z = 0$*, residue is:*

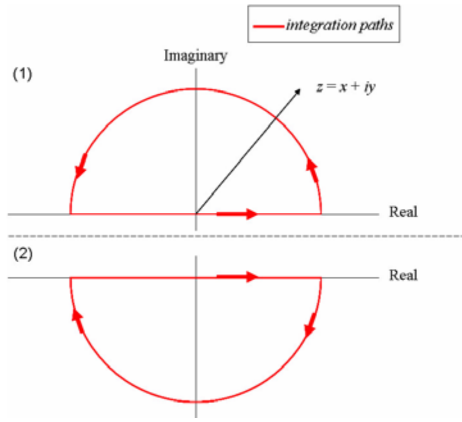
$$-\frac{1}{24\pi^2}\left(\left(\frac{e^{+i\pi(z+2x)} - e^{+i\pi(5z-2x)}}{z(x-z)}\right) z\Big|_{z=0} + \left(\frac{e^{+i\pi(z+2x)} - e^{+i\pi(5z-2x)}}{z(x-z)}\right)(z-x)\Big|_{z=x}\right)$$

$$= -\frac{1}{24\pi^2}\left(\frac{e^{+i\pi(2x)} - e^{+i\pi(-2x)}}{x} - \frac{e^{+i\pi(x+2x)} - e^{+i\pi(5x-2x)}}{x}\right)$$

$$= -\frac{1}{24\pi^2}\left(\frac{e^{+i\pi(2x)} - e^{+i\pi(-2x)} - e^{+i\pi(3x)} + e^{+i\pi(3x)}}{x}\right) = -\frac{1}{24\pi^2}\frac{e^{+i\pi(2x)} - e^{+i\pi(-2x)}}{x}$$

$$= -\frac{1}{24\pi^2}\cdot\frac{2i}{2i}\cdot\frac{e^{+i\pi(2x)} - e^{+i\pi(-2x)}}{x} = -\frac{1}{24\pi^2}\cdot\frac{2i}{x}\cdot\left(\frac{e^{+2\pi ix} - e^{-2\pi ix}}{2i}\right)$$

$$= -\frac{1}{24\pi^2}\cdot\frac{2i}{x}\sin[2\pi x]$$



close contour of the 2nd integral (with negative exponents) in lower half plane

so that $e^{-i\pi(iy)} \to 0$ *as* $y \to -\infty$

$$\int_{-\infty}^{+\infty} \left(\frac{e^{-i\pi(\alpha+2x)} - e^{-i\pi(5\alpha-2x)}}{\alpha (x-\alpha)} \right) d\alpha = \oint \left(\frac{e^{-i\pi(z+2x)} - e^{-i\pi(5z-2z)}}{z (x-z)} \right) dz$$

no poles enclosed by this contour, so

residue at $z = x$ *evaluates to 0*

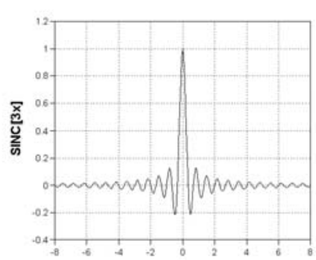
$$\textit{integral} = 2\pi i \cdot \textit{ sum of residues}$$

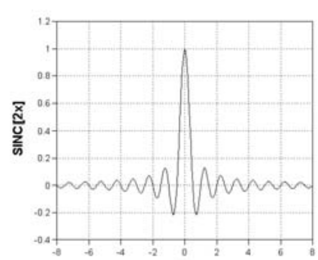
$$= 2\pi i \cdot \left(-\frac{1}{24\pi^2} \right) \frac{2i}{x} \sin[2\pi x] = 2\pi i \cdot \left(-\frac{1}{24\pi^2} \right) \frac{2i}{x} \frac{2\pi}{2\pi} \sin[2\pi x]$$

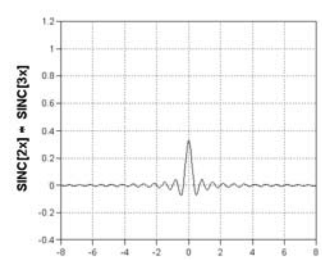
$$= 2\pi i \cdot 4\pi i \cdot \left(-\frac{1}{24\pi^2} \right) \frac{\sin[2\pi x]}{2\pi x}$$

$$= \frac{1}{3} SINC[2x]$$

$$\boxed{SINC[3x] * SINC[2x] = \frac{1}{3} SINC[2x]}$$







(c) Comment on the difficulty of the evaluation.

We will revisit these problems after proving the so-called "filter theorem" of the 1-D Fourier transform, where the solution becomes "trivial".

Actually the first one is not too difficult, just tedious. The second is both difficult and tedious. As we shall see shortly, solution of either problem by the filter theorem is "trivial".