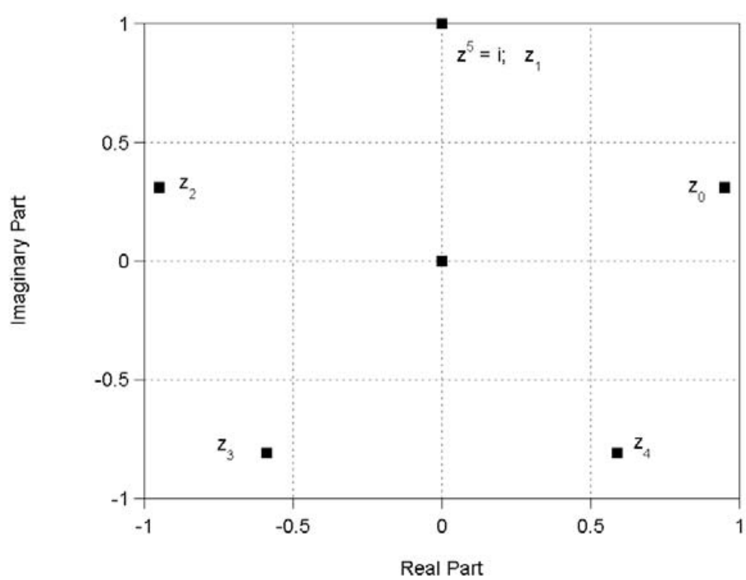
1051-716-20081 Solution Set #1

1. Calculate all roots of the following equations and express them as both real/imaginary parts and as magnitude/phase

   (a) $z^5 - i = 0$

$$
\begin{aligned}
z^5 - i &= 0 \implies z^5 = i = 1 \cdot e^{i\left(\frac{\pi}{2}+2\pi\ell\right)} = e^{i\pi\left(2\ell+\frac{1}{2}\right)},\ \textit{where}\ \ell = 0,1,2,\ldots(5-1) = 4 \\
\textit{magnitude} &: z^5 = \left(|z|\,e^{i\theta}\right)^5 = |z|^5\,e^{i(5\theta)} \implies |z|^5 = 1 \implies |z| = 1 \\
\textit{phase} &: 5\phi = \pi\left(2\ell+\frac{1}{2}\right) \implies \phi_\ell = \frac{\pi}{10} + \frac{2\pi\ell}{5}\ \textit{for}\ = 0,1,2,3,4 \\
\phi_0 &= \frac{\pi}{10} \implies z_0 = 1 \cdot e^{i\frac{\pi}{10}} \cong 0.951\,06 + 0.309\,02i \\
\phi_1 &= \frac{\pi}{10} + \frac{2\pi}{5}\cdot 1 = \frac{\pi}{2} \implies z_1 = 1 \cdot e^{i\frac{\pi}{2}} = 0 + 1 \cdot i \\
\phi_2 &= \frac{\pi}{10} + \frac{2\pi}{5}\cdot 2 = \frac{9}{10}\pi \implies z_2 = 1 \cdot e^{i\frac{9\pi}{10}} \cong -0.951\,06 + 0.309\,02 \cdot i \\
\phi_3 &= \frac{\pi}{10} + \frac{2\pi}{5}\cdot 3 = \frac{13}{10}\pi = -\frac{7}{10}\pi \implies z_3 = 1 \cdot e^{i\frac{-7\pi}{10}} \cong -0.587\,79 - 0.809\,02 \cdot i \\
\phi_4 &= \frac{\pi}{10} + \frac{2\pi}{5}\cdot 4 = \frac{17}{10}\pi = -\frac{3}{10}\pi \implies z_4 = 1 \cdot e^{i\frac{-3\pi}{10}} \cong 0.587\,79 - 0.809\,02 \cdot i
\end{aligned}
$$

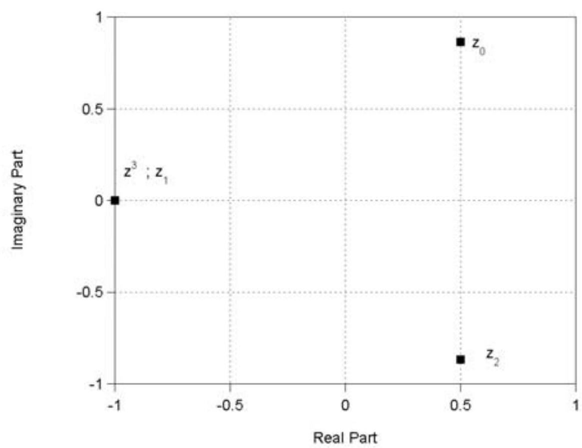


Check answers:

$$
\begin{aligned}
z_0^5 &\cong (0.951\,06 + 0.309\,02i)^5 \cong i \\
z_1^5 &= (0+i)^5 = i \\
z_2^5 &\cong (-0.951\,06 + 0.309\,02 \cdot i)^5 \cong i \\
z_3^5 &\cong (-0.587\,79 - 0.809\,02 \cdot i)^5 \cong i \\
z_4^5 &\cong (0.587\,79 - 0.809\,02 \cdot i)^5 \cong i
\end{aligned}
$$

(b) $z^3 + 1 = 0$

$$\begin{aligned}
z^3 &= -1 = 1 \cdot e^{i\pi} = e^{i\pi} \cdot e^{2\pi i \ell} = e^{i\pi(2\ell+1)} \;\; where \;\; \ell = 0, 1, 2 \\
magnitude &: \;\; z^3 \, |z|^3 \, e^{i(3\theta)} \to |z|^3 = 1 \implies |z| = 1 \\
phase &: \;\; 3\phi = \pi\,(2\ell + 1) \implies \phi_\ell = \frac{\pi}{3} + \frac{2\pi\ell}{3} \;\; for \;\; = 0, 1, 2 \\
\phi_0 &= \frac{\pi}{3} \implies z_0 = 1 \cdot e^{i\frac{\pi}{3}} \cong 0.5 + 0.866\,03 \cdot i \\
\phi_1 &= \frac{\pi}{3} + \frac{2\pi}{3} = \pi \implies z_1 = 1 \cdot e^{i\pi} \cong -1 + 0 \cdot i \\
\phi_2 &= \frac{\pi}{3} + \frac{4\pi}{3} = \frac{5\pi}{3} \implies z_2 = e^{i\frac{5\pi}{3}} = \frac{1}{2} - \frac{1}{2} i\sqrt{3} \cong 0.5 - 0.866\,03 i = z_0^*
\end{aligned}$$

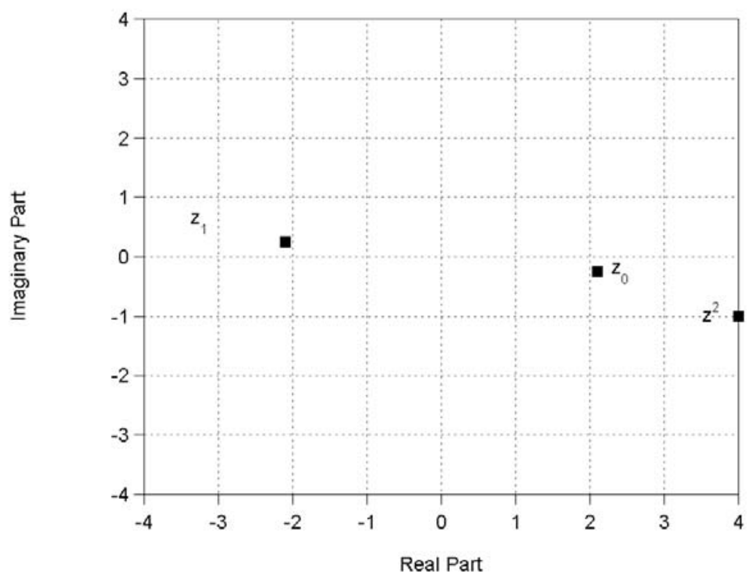


Check answers:

$$\begin{aligned}
z_0^3 &= (0.5 + 0.866\,03 \cdot i)^3 = -1 \\
z_1^3 &= (-1)^3 = -1 \\
z_2^3 &= (0.5 - 0.866\,03 \cdot i)^3 = -1
\end{aligned}$$

(c) $z^2 + i = 4$

$$
\begin{aligned}
z^2 &= 4 - i = \sqrt{(4^2+1^2)} \cdot e^{i\theta} = \sqrt{17} \cdot e^{i\theta} \implies |z| = \sqrt{17} \cong 4.1231 \\
\theta &= \tan^{-1}\left[\frac{-1}{4}\right] \cong -0.244\,98 \ \textit{radians} \cong -0.0779\pi \ \textit{radians} \cong -14.04^\circ \\
\frac{\theta_0}{2} &= \frac{1}{2} \cdot \tan^{-1}\left[\frac{-1}{4}\right] \cong -0.122\,49 \ \textit{radians} \cong -7.02^\circ \\
z &= \left(\sqrt{17}\right)^{\frac{1}{2}} \exp\left[+i \cdot \left(\frac{1}{2} \cdot \tan^{-1}\left[\frac{-1}{4}\right] + \frac{\ell}{2} \cdot 2\pi\right)\right] \ \ \ell = 0, 1 \\
z_0 &= (17)^{\frac{1}{4}} \exp\left[+i \cdot \frac{1}{2} \tan^{-1}\left[\frac{-1}{4}\right]\right] \cong 2.015\,3 - 0.248\,10 \cdot i, \ \theta_0 \cong -0.122\,49 \ \textit{radians} \\
z_1 &= \left(\sqrt{17}\right)^{\frac{1}{2}} \exp\left[+i \cdot \left(\tan^{-1}\left[\frac{-1}{4}\right] + \frac{1}{2} \cdot 2\pi\right)\right] \cong -2.015\,3 + 0.248\,10 \cdot i, = -z_0 \\
\theta_0 &= \frac{1}{2} \tan^{-1}\left[\frac{-1}{4}\right] + \pi \cong 3.02 \ \textit{radians} \cong +172.98^\circ
\end{aligned}
$$



Check answers:

$$
\begin{aligned}
z_0^2 &= (2.015\,3 - 0.248\,10 \cdot i)^2 = 4 - i \\
z_1^2 &= \left(-(2.015\,3 - 0.248\,10 \cdot i)^2 = 4 - i\right)
\end{aligned}
$$

2. Determine the requirement that must be satisfied for the three complex numbers z_1, z_2, and z_3 to lie on a straight line in the complex plane:

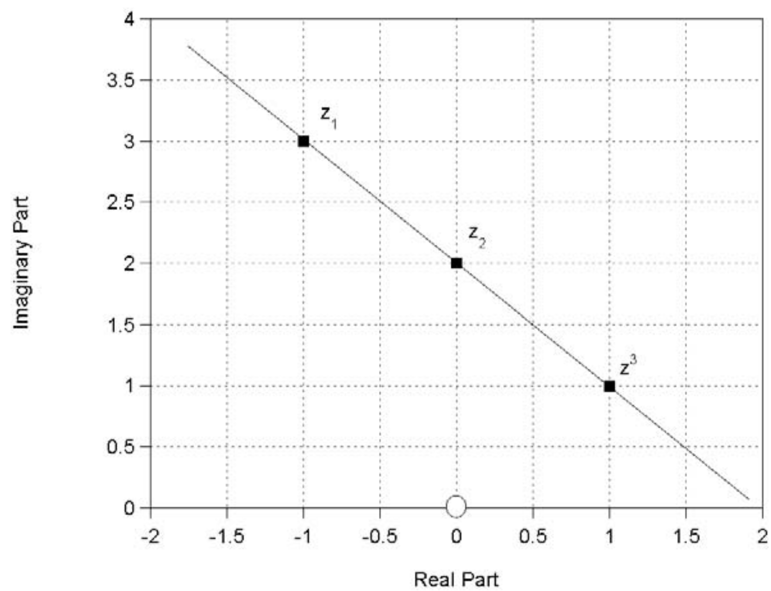
The real and imaginary parts must satisfy the constraints for a straight line. The slopes of the lines connecting z_1 *and* z_2 and z_1 *and* z_3 *must be identical. Recall the "two-point" equation for a straight line:*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $[x, y] \to x + iy = z$*, so that* $x = \text{Re}\{z\}$ *and* $y = \text{Im}\{z\}$*. In this situation, the same equation must be satisfied for both* z_2 *and* z_3 *measured from* z_1*, which means that:*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$$

This also means that the ratios $\frac{z_1 - z_3}{z_2 - z_3}$ *must be real. Put another way, the lines connecting pairs of points must have the same slopes AND the same intercepts of the two axes.*

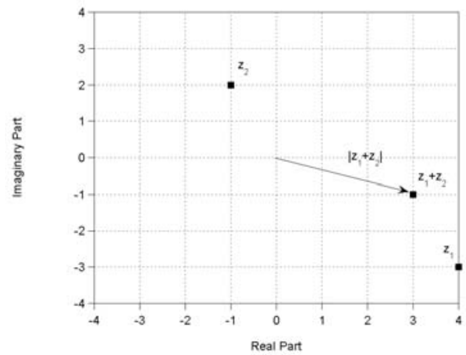


3. If $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$, find the analytic and graphical solutions to:

(a) $|z_1 + z_2|$

$$|z_1 + z_2| = |(4 - 3i) + (-1 + 2i)| = |3 - i| = \sqrt{3^2 + 1^2} = \sqrt{10} \cong 3.162\,3$$

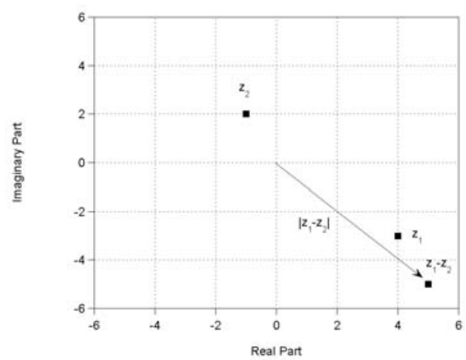
$$\textit{magnitude is real and nonnegative} \implies \textit{phase } \phi = 0$$



(b) $|z_1 - z_2|$

$$|z_1 - z_2| = |(4 - 3i) - (-1 + 2i)| = |5 - 5i| = \sqrt{5^2 + 5^2} = 5\sqrt{2} \cong 7.071\,1$$

$$\textit{again, phase } \phi = 0 \textit{ because magnitude is real and nonnegative}$$

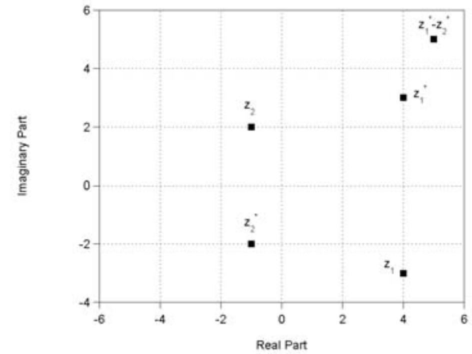


(c) $z_1^* - z_2^*$

$$z_1^* - z_2^* = (4 - 3i)^* - (-1 + 2i)^* = (4 + 3i) - (-1 - 2i) = 5 + 5i$$

$$|z_1^* - z_2^*| = |5 + 5i| = 5 \cdot |1 + i| = 5 \cdot \sqrt{2}$$

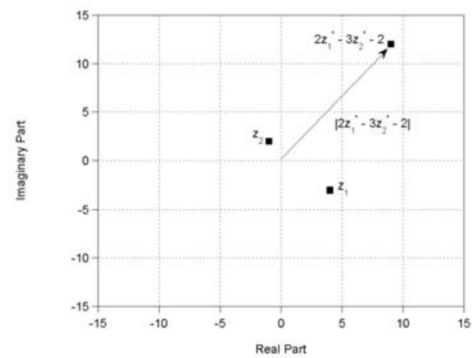
$$\textit{angle is } \tan^{-1}\left[\frac{\mathrm{Im}\{z_1^* - z_2^*\}}{\mathrm{Re}\{z_1^* - z_2^*\}}\right] = \tan^{-1}\left[\frac{+5}{+5}\right] = +\frac{\pi}{4}$$



(d) $|2z_1^* - 3z_2^* - 2|$

$$\begin{aligned} |2z_1^* - 3z_2^* - 2| &= |2 \cdot (4+3i) - 3 \cdot (-1-2i) - 2| = |8 + 6i + 3 + 6i - 2| \\ &= |9 + 12i| = \sqrt{(9^2 + 12^2)} = 15 \\ & \textit{angle is 0} \end{aligned}$$

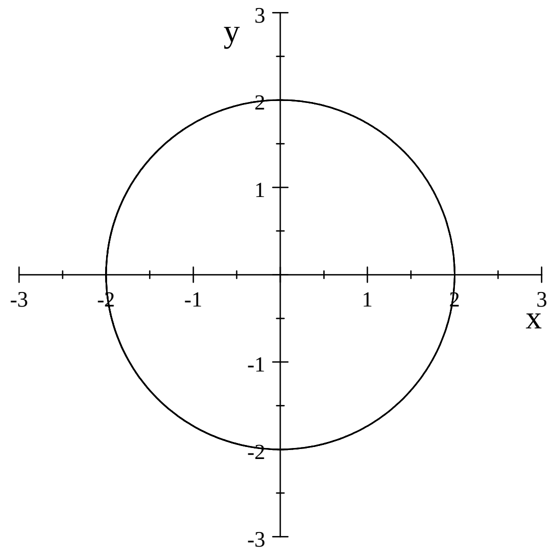
:



4. Describe the set of z that satisfy the constraint $z \cdot z^* = 4$

$z = x + iy \rightarrow zz^* = x^2 + y^2 = 4 \implies$ *Circle centered at origin of complex plane with radius* $r = 2$.

$$x^2 + y^2 = 4$$



5. Find the complex numbers z that are complex conjugates of the values of $z^{-\frac{1}{2}}$.

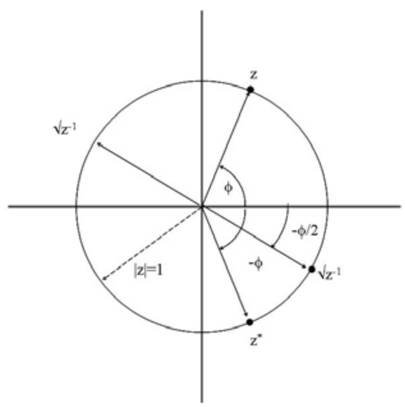
$$\begin{aligned}
z &= \left(z^{-\frac{1}{2}}\right)^* \implies z^* = z^{-\frac{1}{2}} \\
z &= |z| \exp\left[+i\phi\right] \implies z^* = |z| \exp\left[-i\phi\right] \\
z^{-\frac{1}{2}} &= z_\ell = |z|^{-\frac{1}{2}} \exp\left[-i\left(\frac{\phi}{2}\right) + \frac{2\pi\ell}{2}\right], \; \ell = \textit{integer} \\
\textit{equate magnitudes} &: \quad |z| = \frac{1}{\sqrt{|z|}} \implies |z|^{\frac{3}{2}} = 1 \implies |z| = 1 \\
\textit{equate phases} &: \quad -\phi = -\frac{\phi}{2} + \ell\pi \implies -\frac{\phi}{2} + \ell\pi \implies \phi = 2\pi\ell \\
\ell &= 0 \implies \phi = 0 \implies z_0 = 1 \cdot e^{i \cdot 0} = 1 \\
\ell &= 1 \implies \phi = 2\pi \implies z_1 = 1 \cdot e^{i \cdot 2\pi} = 1 = z_0
\end{aligned}$$

$$\boxed{z = 1 + i \cdot 0 = 1 \exp\left[+i \cdot 2\pi\ell\right] \textit{ are solutions}}$$

check:

$$\begin{aligned}
z_0^2 &= 1^2 = 1^* = z_0^* \\
z_1^{-\frac{1}{2}} &= |1|^{-\frac{1}{2}} \left(\exp\left[-i \cdot \left(\frac{\phi}{2} + \ell\pi\right)\right]\right)^2 \\
z_1^* &= |1| \exp\left[-i \cdot \phi\right] \\
&\implies \frac{\phi}{2} + \ell\pi = \phi \implies \phi + 2\pi\ell = 2\phi \implies \phi = 2\pi\ell
\end{aligned}$$

More easily seen in complex plane:



If you believe that $|z| = 1$, select an arbitrary complex number z with phase angle ϕ. The complex conjugate AND the reciprocal are both located at angle $-\phi$. The square roots of the reciprocal are located at angles $-\frac{\phi}{2}$ and $-\frac{\phi}{2} \pm \pi$. The only possible solution for $-\phi = \frac{\phi}{2}$ is $\phi = 0$.

6. Use complex analysis to demonstrate that:

   (a) $\cos[3\theta] = 4\cos^3[\theta] - 3\cos[\theta]$

   (b) $\sin[3\theta] = 3\sin[\theta] - 4\sin^3[\theta]$

Euler relation: $e^{i\theta} = \cos[\theta] + i\sin[\theta]$

(a): $\left(e^{i\theta}\right)^3 = e^{i3\theta} = \cos[3\theta] + i\sin[3\theta]$

(b):

$$\begin{aligned}(\cos[\theta] + i\sin[\theta])^3 &= \cos^3[\theta] + i\left(3\cos^2[\theta]\sin[\theta]\right) + 3i^2\cos[\theta]\sin^2[\theta] + \left(i^3\sin^3[\theta]\right)\\ &= \cos^3[\theta] + i\left(3\cos^2[\theta]\sin[\theta]\right) - 3\cos[\theta]\sin^2[\theta] - \left(i\sin^3[\theta]\right)\\ &= \left(\cos^3[\theta] - 3\cos[\theta]\sin^2[\theta]\right) + i\cdot\left(3\cos^2[\theta]\sin[\theta] - \sin^3[\theta]\right)\end{aligned}$$

Real part may be rewritten as:

$$\begin{aligned}\cos^3[\theta] - 3\cos[\theta]\sin^2[\theta] &= \cos[\theta]\cdot\left(\cos^2[\theta] - 3\sin^2[\theta]\right)\\ &= \cos[\theta]\cdot\left(\cos^2[\theta] - 3\cdot\left(1 - \cos^2[\theta]\right)\right)\\ &= \cos[\theta]\cdot\left(4\cos^2[\theta] - 3\right)\\ &= 4\cos^3[\theta] - 3\cos[\theta]\end{aligned}$$

Imaginary part may be rewritten as:

$$\begin{aligned}3\cos^2[\theta]\sin[\theta] - \sin^3[\theta] &= 3\cdot\left(1 - \sin^2[\theta]\right)\cdot\sin[\theta] - \sin^3[\theta]\\ &= 3\sin[\theta] - 3\sin^3[\theta] - \sin^3[\theta]\\ &= 3\sin[\theta] - 4\sin^3[\theta]\end{aligned}$$

Equate real and imaginary parts:

$$\boxed{\cos[3\theta] = \cos^3[\theta] - 3\cos[\theta]\sin^2[\theta]}$$

$$\boxed{\sin[3\theta] = 3\cos^2[\theta]\sin[\theta] - \sin^3[\theta]}$$

7. Evaluate the integral

$$\int_{-\infty}^{+\infty} (A_0 + A_1 \sin[2\pi\xi_0\beta]) \cdot RECT[x-\beta] \; d\beta$$

Since this is integrating over the "space" variable β, the answer is a function of x, A_0, A_1, and ξ_0.

First, note that

$$\sin[2\pi\xi_0\beta] = \cos\left[2\pi\xi_0\beta - \frac{\pi}{2}\right]$$

I will solve this for the general case:

$$\int_{-\infty}^{+\infty} (A_0 + A_1 \cos[2\pi\xi_0\beta + \phi_0]) \cdot RECT[x-\beta] \; d\beta$$

and then substitute in $\phi_0 = -\frac{\pi}{2}$ *to get the specific result (you did not have to do this).*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (A_0 + A_1 \cos[2\pi\xi_0\beta + \phi_0]) \cdot RECT[x-\beta] \; d\beta \\
&= A_0 \cdot \int_{-\infty}^{+\infty} RECT[x-\beta] \; d\beta + A_1 \cdot \int_{-\infty}^{+\infty} \cos[2\pi\xi_0\beta + \phi_0] \cdot RECT[x-\beta] \; d\beta \\
&= A_0 \cdot \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} 1 \; d\beta + A_1 \cdot \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos[2\pi\xi_0\beta + \phi_0] \; d\beta \\
&= A_0 \cdot \beta\Big|_{x-\frac{1}{2}}^{x+\frac{1}{2}} + A_1 \cdot \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos[2\pi\xi_0\beta + \phi_0] \; d\beta \\
&= A_0 \cdot \left(\left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)\right) + A_1 \cdot \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos[2\pi\xi_0\beta + \phi_0] \; d\beta \\
&= A_0 + A_1 \cdot \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \cos[2\pi\xi_0\beta + \phi_0] \; d\beta
\end{aligned}$$

now do the integral of the cosine:

$$\textit{recall} \;\; \cos[A \pm B] = \cos[A]\cos[B] \mp \sin[A]\sin[B]$$

$$\sin[A \pm B] = \sin[A]\cos[B] \pm \cos[A]\sin[B]$$

$$\implies \int \cos[2\pi\xi_0\beta + \phi_0] \; d\beta = \cos[\phi_0] \int \cos[2\pi\xi_0\beta] \; d\beta - \sin[\phi_0] \int \sin[2\pi\xi_0\beta] \; d\beta$$

$$\textit{recall:} \int \cos[\alpha_0\beta] \; d\beta = +\frac{1}{\alpha_0} \cdot \sin[\alpha_0\beta] \;\; \textit{and} \;\; \int \sin[\alpha_0\beta] \; d\beta = -\frac{1}{\alpha_0} \cdot \cos[\alpha_0\beta]$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (A_0 + A_1 \cos[2\pi\xi_0\beta + \phi_0]) \cdot RECT[x-\beta] \; d\beta \\
&\qquad = A_0 + \cos[\phi_0] \cdot A_1 \cdot \frac{\sin[2\pi\xi_0\beta]}{2\pi\xi_0}\Bigg|_{x-\frac{1}{2}}^{x+\frac{1}{2}} - \left(-\sin[\phi_0] \cdot A_1 \cdot \frac{\cos[2\pi\xi_0\beta]}{2\pi\xi_0}\Bigg|_{x-\frac{1}{2}}^{x+\frac{1}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
= A_0 &+ \frac{A_1}{2\pi\xi_0}\left(\cos[\phi_0] \cdot \left(\sin\left[2\pi\xi_0\left(x + \frac{1}{2}\right)\right] - \sin\left[2\pi\xi_0\left(x - \frac{1}{2}\right)\right]\right)\right) \\
&+ \frac{A_1}{2\pi\xi_0}\left(\sin[\phi_0] \cdot \left(\cos\left[2\pi\xi_0\left(x + \frac{1}{2}\right)\right] - \cos\left[2\pi\xi_0\left(x - \frac{1}{2}\right)\right]\right)\right)
\end{aligned}$$

Second term:

$$\begin{aligned}&\sin\left[2\pi\xi_0\left(x+\frac{1}{2}\right)\right]-\sin\left[2\pi\xi_0\left(x-\frac{1}{2}\right)\right]\\&\quad=(\sin[2\pi\xi_0 x]\cos[\pi\xi_0]+\cos[2\pi\xi_0 x]\sin[\pi\xi_0])-(\sin[2\pi\xi_0 x]\cos[\pi\xi_0]-\cos[2\pi\xi_0 x]\sin[\pi\xi_0])\\&\quad=2\sin[\pi\xi_0]\cos[2\pi x\xi_0]\end{aligned}$$

Third term:

$$\begin{aligned}&\cos\left[2\pi\xi_0\left(x+\frac{1}{2}\right)\right]-\cos\left[2\pi\xi_0\left(x-\frac{1}{2}\right)\right]\\&\quad=(\cos[2\pi\xi_0 x]\cos[\pi\xi_0]-\sin[2\pi\xi_0 x]\sin[\pi\xi_0])-(\cos[2\pi\xi_0 x]\cos[\pi\xi_0]+\sin[2\pi\xi_0 x]\sin[\pi\xi_0])\\&\quad=-2\sin[\pi\xi_0]\sin[2\pi x\xi_0]\end{aligned}$$

Collect terms:

$$\begin{aligned}&\int_{-\infty}^{+\infty}(A_0+A_1\cos[2\pi\xi_0\beta+\phi_0])\cdot RECT[x-\beta]\;d\beta\\=\;&A_0+A_1\cos[\phi_0]\cdot\frac{2\sin[\pi\xi_0]\cos[2\pi x\xi_0]}{2\pi\xi_0}-A_1\sin[\phi_0]\cdot\frac{2\sin[\pi\xi_0]\sin[2\pi x\xi_0]}{2\pi\xi_0}\\=\;&A_0+A_1\cos[\phi_0]\cdot\frac{\sin[\pi\xi_0]}{\pi\xi_0}\cdot\cos[2\pi x\xi_0]-A_1\sin[\phi_0]\cdot\frac{\sin[\pi\xi_0]}{\pi\xi_0}\cdot\sin[2\pi x\xi_0]\\\equiv\;&A_0+A_1 SINC[\xi_0]\cdot(\cos[\phi_0]\cdot\cos[2\pi x\xi_0]-\sin[\phi_0]\cdot\sin[2\pi x\xi_0])\end{aligned}$$

Now select $\phi_0=-\frac{\pi}{2}$:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(A_0+A_1\cos\left[2\pi\xi_0\beta-\frac{\pi}{2}\right]\right)\cdot RECT[x-\beta]\;d\beta\\=\;&\int_{-\infty}^{+\infty}(A_0+A_1\sin[2\pi\xi_0\beta])\cdot RECT[x-\beta]\;d\beta\\=\;&A_0+A_1 SINC[\xi_0]\cdot\left(-\sin\left[-\frac{\pi}{2}\right]\cdot\sin[2\pi x\xi_0]\right)\\=\;&A_0+A_1 SINC[\xi_0]\cdot\sin[2\pi x\xi_0]\end{aligned}$$

The constant part is unaffected by the integration over a region of unit length, but the sinusoid is scaled by the SINC function, which evaluates to unity only for $\xi_0=0$ (which would mean that the sinusoid has infinite period, i.e., it is a constant too). The SINC function evaluates to zero at $\xi_0=1$, which would occur if the sinusoid had unit period, which is the same as the width of the rectangle. In other words, the integral of a cosine over a single period is zero.

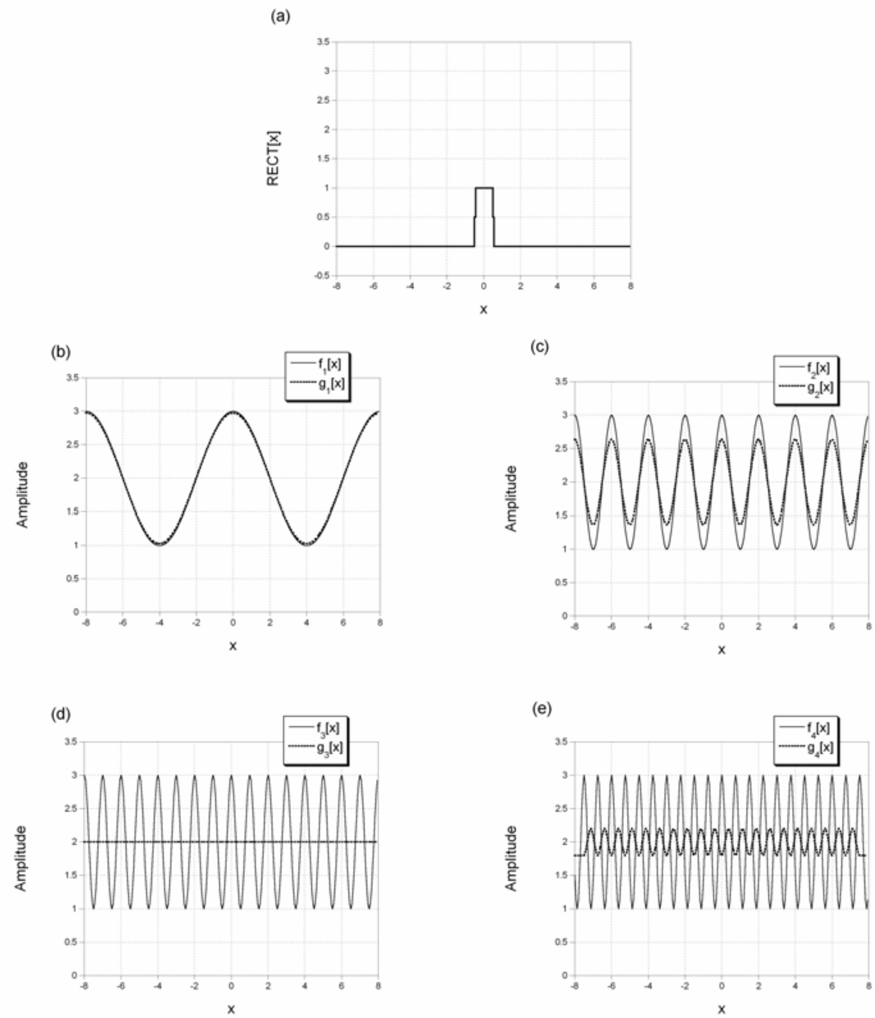


Illustration of amplitude reduction by uniform averaging: (a) the rectangle function; (b) a low-frequency sinusoid before and after averaging, showing small reduction in amplitude; (c) higher-frequency sinusoid shows more reduction; (d) sinusoid with period equal to width of rectangle is averaged to the constant; (e) sinusoid with shorter period has its amplitude "inverted" (contrast reversal), so that what were maxima are now minima.