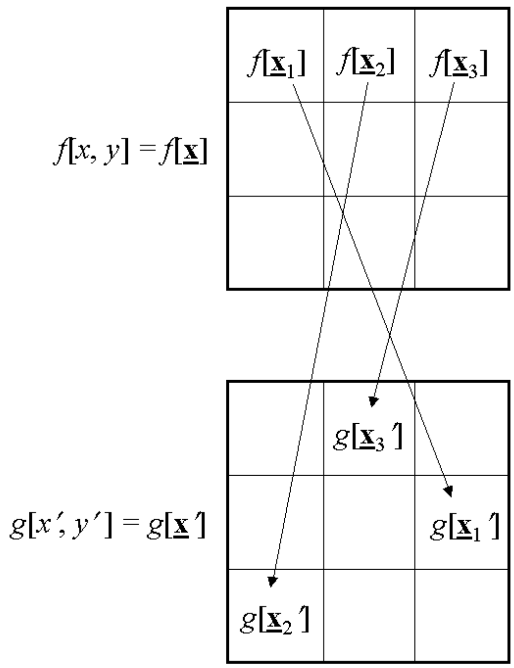
SIMG-716
Linear Imaging Mathematics I, 07

1 LINEAR OPERATIONS

- Mathematical operator $\mathcal{O}$ specifies action of system on input functions to generate outputs

$$\mathcal{O}\{f[\underline{\mathbf{x}}]\} = g[\underline{\mathbf{x}}'] .$$

  - $f[\underline{\mathbf{x}}]$ = input amplitude at location $\underline{\mathbf{x}}$ within the N-D continuous input domain
  - $g[\underline{\mathbf{x}}']$ = output amplitude at location $\underline{\mathbf{x}}'$ within the M-D continuous output domain.
  - Often $N > M$
    - $f[x, y, z, \lambda] \to g[x', y']$.
- $\mathcal{O}$ represents mathematical recipe for action of system on input function
- Presumably, output function differs from input in some way
  - "units" of respective input and output amplitudes f and g need not be (usually are not) identical.
  - Imaging systems convert one type of signal amplitude or energy to another form
    - energy "transducers"
    - example is photographic-film camera: converts electric-field amplitude (volts per meter) to dimensionless "film density".
- General system operator $\mathcal{O}$ may affect amplitude (brightness) f and/or "location" $\underline{\mathbf{x}}$ in arbitrary way
- Examples:
  1. System could amplify input amplitude f at each $\underline{\mathbf{x}}$ by deterministic numerical factor without "moving" or "spreading" to neighboring locations
  2. Amplify input amplitude at each location by numerical factors that vary randomly with position
  3. Affect the position of signal amplitude in deterministic way (e.g., translation or rotation about origin)
  4. Affect position in random way
  5. Mix any combination of such operations on amplitude and location.
- Action of operator may be conceptually simple to describe but difficult or impossible to write mathematically.
  - system #2 "moves" amplitude from 2-D input location $\underline{\mathbf{x}}$ to a new and unique $\underline{\mathbf{x}}'$ selected at random
  - output image is "scrambled" version of input function
  - $\mathcal{O}$ is random transformation of coordinates; $\underline{\mathbf{x}}'[\underline{\mathbf{x}}]$.



$$\mathcal{O}\{f[\underline{\mathbf{x}}]\} = g[\underline{\mathbf{x}}'] \Rightarrow \mathcal{O}\{f[\underline{\mathbf{x}}]\} = \underline{\mathbf{x}}'[\underline{\mathbf{x}}]$$

Schematic of the image "scrambling" operator $\underline{\mathbf{x}}'\,[\underline{\mathbf{x}}]$

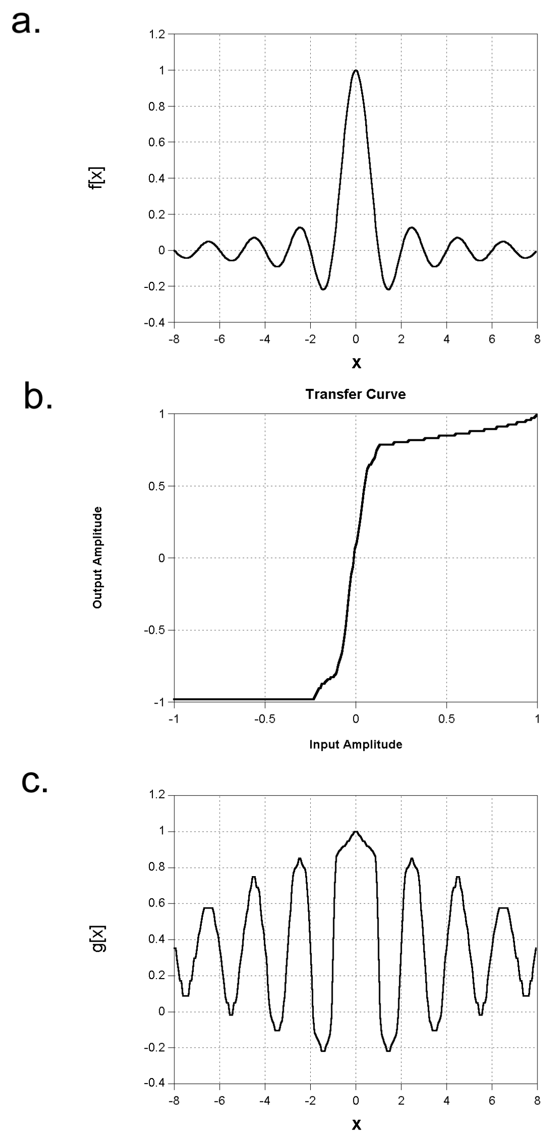
- Conditions for "inverse" imaging operator exists if each sampled input pixel $\underline{\mathbf{x}}_n$ is mapped to unique pixel output $\underline{\mathbf{x}}'_n$: $\underline{\mathbf{x}}_n[\underline{\mathbf{x}}'_n]$.
- Mathematically tractable operators exist only for systems whose actions are constrained from complete arbitrariness.
- Two most important such restrictions:
  1. linearity: restricts possible effect on amplitude f of input signal
  2. shift invariance: constrains action on coordinate x.

2 LINEARITY

- Example of a simple and yet useful operator: action depends upon input amplitude f ONLY.
  - specific location within domain has no influence upon output.
  - Domain of output image must include entire domain to ensure that unique output location exists for each input coordinate.
  - "primed" notation is superfluous.
  - action of system upon all amplitudes in input signal with same numerical value (say, f_0) must generate same output amplitude (e.g., g_0)
  - $\mathcal{O}$ described fully by single-valued functional relationship between amplitudes of input and output at same location:

$$\mathcal{O}\{f[\underline{\mathbf{x}}]\} = g\{\underline{\mathbf{x}}\} \Longrightarrow \mathcal{O}\{f\} = g \Longrightarrow \mathcal{O}\{f\} = g[f]$$

-
  - transfer characteristic" or "tone-transfer curve"
  - action is a "lookup table"
  - "point operation"



Effect of the "point" operator $g[f]$ on the 1-D input function $f[x]$ to construct the output function $g[x]$: (a) input $f[x] = SINC[x]$; (b) nonlinear "tone-transfer" operator $g[f]$; (c) output "image" $g[x]$.

- Classify by the shape of transfer characteristic function $g[f]$
- $\mathcal{O}\{f[x]\} \Longrightarrow g[f]$ is linear if $g \propto f \Longrightarrow g = kf$, k is numerical proportionality constant with appropriate dimensions.
- $g[f]$ is a straight line through origin
- action upon sum of multiple inputs guaranteed to be identical to the sum of individual outputs
- Criterion for Linearity:

$$\text{if } g[f_1] = g_1 = k\ f_1 \text{ and } g[f_2] = g_2 = k\ f_2, \text{ then: } g[f_1 + f_2] = k\ (f_1 + f_2) = g_1 + g_2.$$

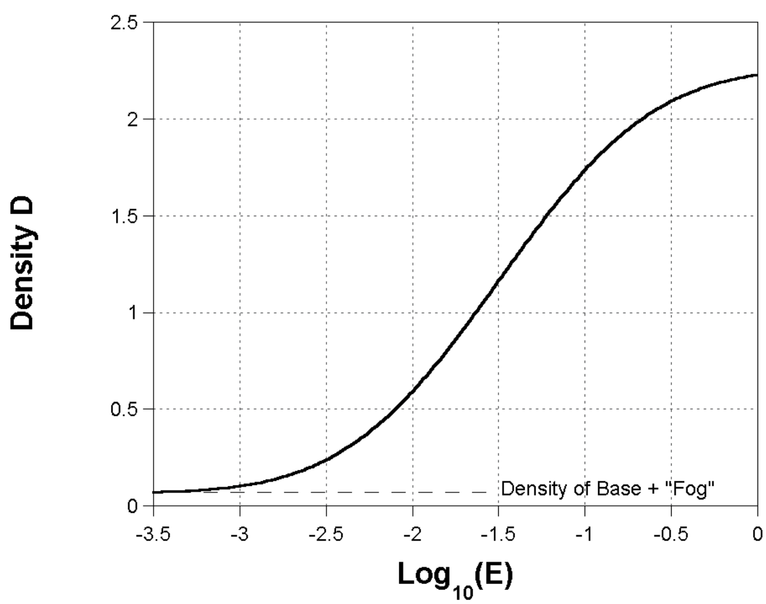
2.1 Formal Definition of Linear Operator

- Action of linear operator upon superposition (weighted sum or linear combination) of inputs is identically weighted sum of individual outputs:

$$\text{if } \mathcal{O}\{f_n[x]\} = g_n[x'], \text{ then } \mathcal{O}\left\{\sum_n \alpha_n\ f_n[x]\right\} = \sum_n \alpha_n\ \mathcal{O}\{f_n[x]\} = \sum_n \alpha_n\ g_n[x'].$$

  $\{\alpha_n\}$ are (generally complex-valued) weighting constants

- If input/output relationship is defined for one input amplitude, then relationship for *any* input amplitude is proportional scaling.
- Corollary that action of linear system on null input results in null output
- Linearity condition describes system operator $\mathcal{O}$ rather than input and/or output signals
- $\mathcal{O}$ must satisfy criterion over infinite range of possible input amplitudes
- Any operator that is not linear over infinite range of input amplitudes is "nonlinear"
- Realistic systems cannot possibly satisfy linearity condition over infinite range of input amplitudes.
  - Response of all realistic detectors "saturate"$\Longrightarrow$ addition of incremental input amplitude has no effect upon output
  - Familiar example of saturable (nonlinear) imaging system is photographic emulsion
    * converts intensity of incident electromagnetic radiation ("E") to spatial variation in recorded optical density "D".
    * "*H&D*" (Hurter and Driffield) or *characteristic curve*
    * graph of density *vs.* logarithm of exposure
    * sigmoidal shape for most photographic films
    * "flat" response at large exposures is evidence of detector saturation.



H&D, or characteristic, curve of a typical photographic emulsion. The "light exposure" is plotted on the x-axis, and the developed density on the y-axis. The nonlinear character of the curve is evident.

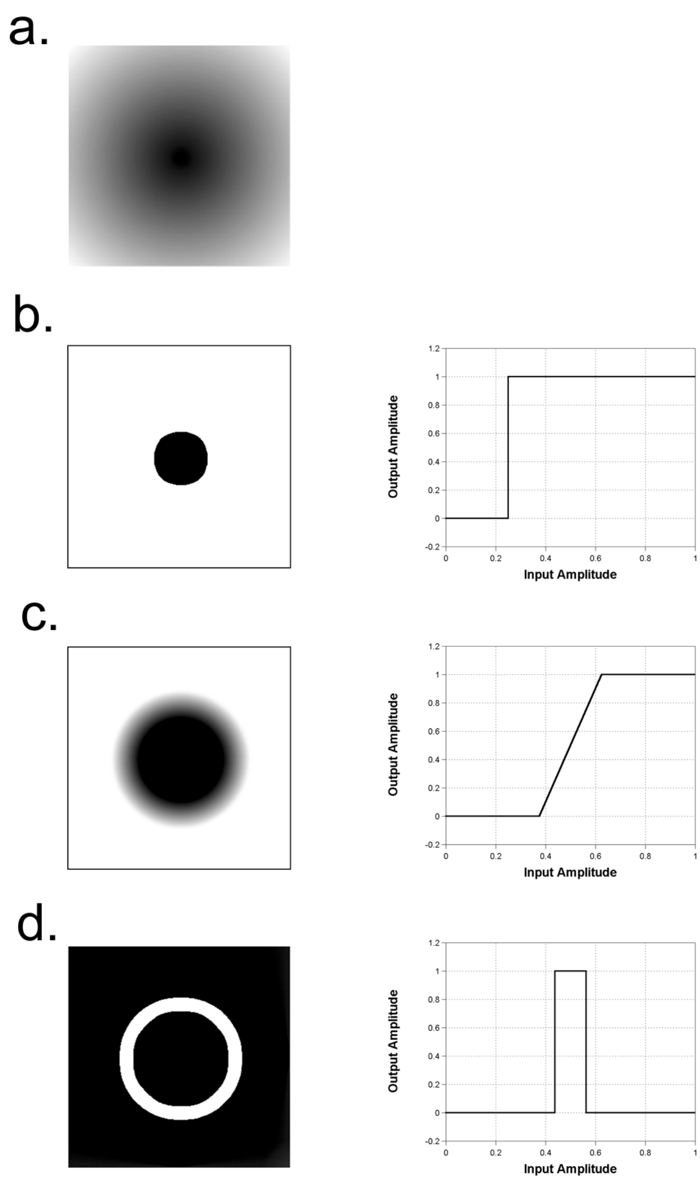
- realistic systems often obey linearity condition to good approximation over some finite range of input amplitudes
- Other common nonlinear operations include "thresholders" (also called "hard limiters) and "quantizers"
  - thresholding $\Longrightarrow$ output saturates "instantly"
  - quantizer is a collection of thresholders with different saturation levels
    - "greatest integer" operator = "truncation"
    - assigns integer output to each real-valued input by lopping off decimal

$$\mathcal{T}\{e\} = \mathcal{T}\{2.71828\cdots\} = 2$$

    - does not satisfy criterion for linearity, demonstrated by applying $\mathcal{T}$ to $2e$:

$$\begin{aligned} \mathcal{T}\{e+e\} &= \mathcal{T}\{2 \times 2.71828\} = \mathcal{T}\{5.436\cdots\} = 5 \\ &\neq \mathcal{T}\{e\} + \mathcal{T}\{e\} = 4 \end{aligned}$$

  - shape of transfer characteristic establishes immediately that truncation nonlinear



Examples of “point” operations common in digital imaging: (a) input object: the 2-D “ramp” function $f[x,y] = \sqrt{x^2+y^2} = RAMP(r)$; (b) image obtained via “thresholding” lookup table; (c) image resulting from “clipping” lookup table; and (d) image from “window threshold”.

- Linear regime determined by qualities of imaging sensor
- Sensors can be fabricated that measure complex amplitude f (magnitude and phase) of electromagnetic waves for small ν (RF waves)
- Sensors for electromagnetic waves at high frequencies respond to incident “power”:
  - time-averaged squared magnitude of amplitude
- Examples of linear operators:

1. numerical scale factor to the input:

$$\mathcal{O}_1\{f[x]\} \equiv g[x] = k\ f[x].$$

The condition that must be satisfied for the operator $\mathcal{O}_1$ to be linear is:

$$\mathcal{O}_1\{\alpha_1\ f_1[x] + \alpha_2\ f_2[x]\} \stackrel{?}{=} \alpha_1\ \mathcal{O}_1\{f_1[x]\} + \alpha_2\ \mathcal{O}_1\{f_2[x]\}$$

$$k\ \{\alpha_1\ f_1[x] + \alpha_2\ f_2[x]\} \stackrel{?}{=} \alpha_1 \cdot k\ f_1[x] + \alpha_2 \cdot k\ f_2[x]$$

These obviously are equal, and thus the operation consisting of multiplication of the input amplitude by a constant is linear. This should be no surprise, as this operation defines the property of linearity.

2. differentiation with respect to x.

$$\mathcal{O}_2\{f[x]\} = g[x] = \frac{\partial^n}{\partial x^n} f[x]$$

$$\begin{aligned}
\mathcal{O}_2\{\alpha_1\ f_1[x] + \alpha_2\ f_2[x]\} &= \frac{\partial^n}{\partial x^n}(\alpha_1\ f_1[x] + \alpha_2\ f_2[x]) \\
&= \alpha_1\ \frac{\partial^n}{\partial x^n}(f_1[x]) + \alpha_2\ \frac{\partial^n}{\partial x^n}(f_2[x]) \\
&= \alpha_1\ \mathcal{O}_2\{f_1[x]\} + \alpha_2\ \mathcal{O}_2\{f_2[x]\} \\
&= \alpha_1\ g_1[x] + \alpha_2\ g_2[x]
\end{aligned}$$

3. integration over fixed interval. The expression for the operator is:

$$\mathcal{O}_3\{f[x]\} = g_3[x] = \int_a^b f[u]\ du$$

$$\begin{aligned}
\mathcal{O}_3\{\alpha_1\ f_1[x] + \alpha_2\ f_2[x]\} &= \int_a^b (\alpha_1\ f_1[u] + \alpha_2\ f_2[u])\,du \\
&= \alpha_1 \int_a^b f_1[u]\ du + \beta \int_a^b f_2[u]\ du \\
&= \alpha_1 \mathcal{O}_3\{f_1[x]\} + \alpha_2 \mathcal{O}_3\{f_2[x]\}
\end{aligned}$$

Result is valid even if the case where either limit (or both) is $\pm\infty$.

- Null input must generate a null output strict-sense linearity:

$$\text{if } f[x] = 0[x]\,, \text{then } g[x] = 0[x]$$

- any operator that adds constant "bias" to input must be a strict-sense nonlinear

$$\mathcal{O}\{f[x]\} = f[x] + b \equiv g[x]$$

$$\mathcal{O}\{\alpha_1 f_1[x] + \alpha_2 f_2[x]\} = (\alpha_1\ f_1[x] + \alpha_2\ f_2[x]) + b$$

- Weighted sum of individual outputs includes "extra" bias term:

$$\begin{aligned}
\alpha_1 \mathcal{O}\{f_1[x]\} + \alpha_2 \mathcal{O}\{f_2[x]\} &= \alpha_1(f_1[x] + b) + \alpha_2(f_2[x] + b) \\
&= \alpha_1 f_1[x] + \alpha_2 f_2[x] + (\alpha_1 + \alpha_2) \cdot b
\end{aligned}$$

- Subsequent operation may create cascaded linear process.

$$\begin{aligned}
g_2[x] &= \mathcal{O}_2\{g_1[x]\} = \mathcal{O}_2\{\mathcal{O}_1\{f[x]\}\} \\
&= \exp\{\log[f[x]]\} = \exp\{g_1[x]\} = f[x].
\end{aligned}$$

- The cascaded process $\mathcal{O} \equiv \mathcal{O}_2\ \mathcal{O}_1$ is identical to the identity operator, which is linear.

3 SHIFT-INVARIANT OPERATORS

- Describes action of system upon location.
- Example: (admittedly unphysical) defined by:

$$g[x] = \mathcal{O}\{f[x]\} = \left\{ \begin{array}{ll} f[x+2] & \text{for} \quad INT[x] \text{ is even} \\ \\ f[x-2] & \text{for} \quad INT[x] \text{ is odd} \end{array} \right\}$$

  - translates amplitude leftwards or rightwards by two units depending on evenness or oddness of integer part of x
  - "response" depends upon location within input signal $f[x]$,not at all on amplitude.
- System whose response varies with and/or is determined by location is *space-variant* or *shift-variant*.
- System whose action does not depend on the specific position is easier to describe mathematically
  - *space-invariant* or *shift-invariant*.
- Mathematical criterion of shift invariance expressed in terms of action on translated signal:

$$\text{If } \mathcal{O}\{f[x]\} = g[x]\,, \text{ then } \mathcal{O}\{f[x-x_0]\} = g[x-x_0]\,.$$

- quality of "shift invariance" ensures that effect of translating 1-D input by some distance x_0 is identical translation of original output.
- 2-D shift invariance:

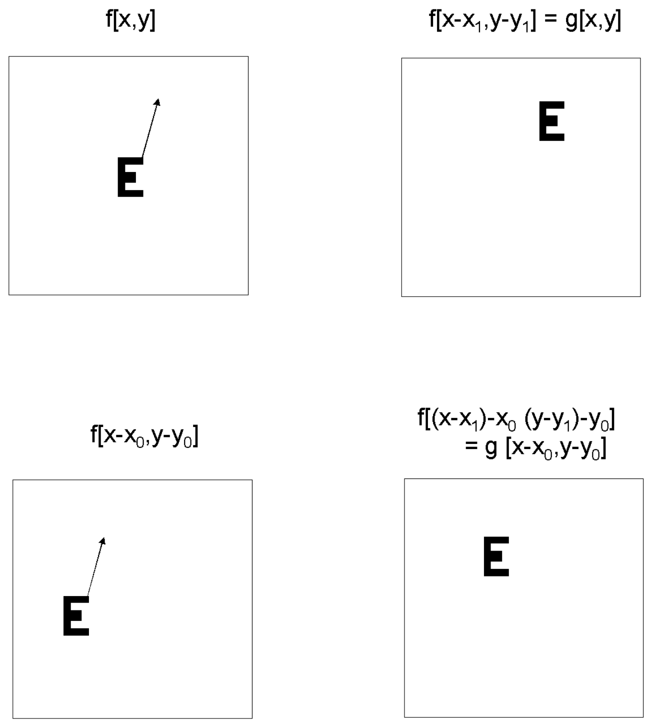
$$\text{If } \mathcal{O}\{f[x,y]\} = g[x,y]\,, \text{ then } \mathcal{O}\{f[x-x_0,y-y_0]\} = g[x-x_0,y-y_0]\,.$$

- Shift variance tested by measuring system output before and after translating input.
  1. operator that translates input through x_1:

$$\mathcal{O}_4\{f[x,y]\} = g_4[x,y] = f[x-x_1,y-y_1]$$

     Condition is:

$$\begin{aligned} \mathcal{O}_4\{f[x-x_0,y-y_0]\} &= f[(x-x_0)-x_1,(y-y_0)-y_1] \\ &= f[(x-x_1)-x_0,(y-y_1)-y_0] \\ &= g_4[x-x_1,y-y_1] \end{aligned}$$



Test of the properties of the translation operator. Since the same result is obtained for the translated input, the translation operator is shift invariant.

2. operator that "magnifies" input by factor of α without changing amplitude

$$\mathcal{O}_5\left\{f\left[x\right]\right\} = g_5\left[x\right] = f\left[\frac{x}{\alpha}\right]$$

$\alpha > 1 \Longrightarrow$output is "larger" than input (magnified image). Test of shift-invariance condition yields two distinct expressions:

$$\mathcal{O}_5\left\{f[x-x_0]\right\} = f\left[\frac{x-x_0}{\alpha}\right]$$

Image magnification is not shift-invariant operation.

3. Rotation of 2-D function $f[x,y]$ by θ radians:

$$\begin{aligned}\mathcal{O}_6\left\{f\left[x,y\right]\right\} &= g_6\left[x,y\right] = f\left[x',y'\right] \\ &= f\left[x\cos\left[\theta\right] + y\sin\left[\theta\right], -x\sin\left[\theta\right] + y\cos\left[\theta\right]\right] \\ &= f\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}, \underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\right]\end{aligned}$$

Test for shift invariance by computing result of cascade of translation followed by rotation:

$$\mathcal{O}_6\left\{f\left[x-x_0, y-y_0\right]\right\} = f\left[x'', y''\right]$$

The coordinates $[x'', y'']$ are obtained by substituting $x - x_0$ and $y - y_0$ for x and y:

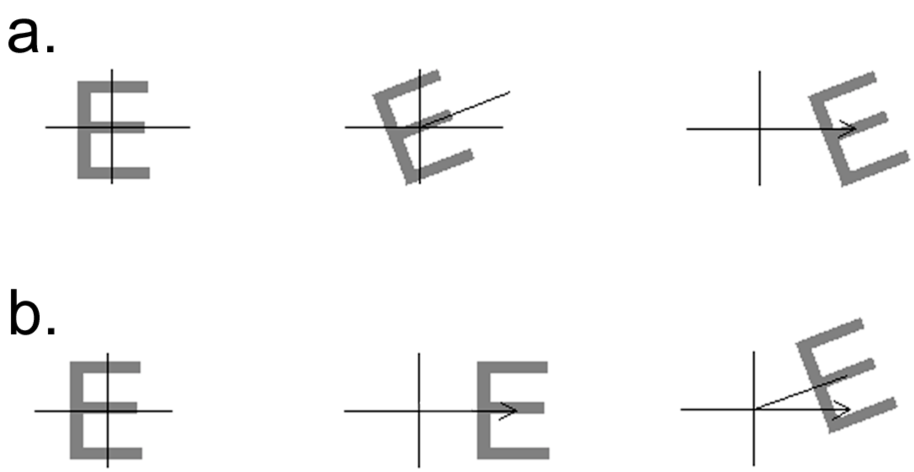
$$\begin{aligned}x'' &= (x-x_0)\ \cos\left[\theta\right] + (y-y_0)\ \sin\left[\theta\right] \\ &= (x\ \cos\left[\theta\right] + y\ \sin\left[\theta\right]) - (x_0\ \cos\left[\theta\right] + y_0\ \sin\left[\theta\right]) \\ &= \underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}} - \underline{\mathbf{r}}_0 \bullet \underline{\widehat{\mathbf{p}}} = (\underline{\mathbf{r}} - \underline{\mathbf{r}}_0) \bullet \underline{\widehat{\mathbf{p}}} \\ y'' &= -(x-x_0)\ \sin\left[\theta\right] + (y-y_0)\ \cos\left[\theta\right] \\ &= \underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp} - \underline{\mathbf{r}}_0 \bullet \underline{\widehat{\mathbf{p}}}^{\perp} = (\underline{\mathbf{r}} - \underline{\mathbf{r}}_0) \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\end{aligned}$$

If $\mathcal{O}_5$ is shift invariant, then new coordinates are identical to those obtained by translating rotated coordinates x' and y':

$$\begin{aligned} x' - x_0 &= x \cos[\theta] - y \sin[\theta] - x_0 \neq x'' \\ y' - y_0 &= x \sin[\theta] + y \cos[\theta] - y_0 \neq y'' \end{aligned}$$

Rotation is be shift variant.

- Magnification and rotation are shift variant because the amplitude at origin of coordinates is unchanged while amplitudes at other locations are "moved" by different increments



Demonstration that "rotation" about the origin of coordinates is a shift-variant operator: (a) $f[x, y]$ is rotated and then translated; (b) $f[x.y]$ is translated and then rotated. Because the output images differ, the rotation operation is shift variant.

4 LINEAR, SHIFT-INVARIANT OPERATORS

- Combination of two constraints ensures that mathematical description of system has form particularly applicable to imaging systems.
- From definition of linearity, can calculate output amplitude by scaling output from identical input function normalized to unity.
- Property of shift invariance ensures that output function could be translated to any location.
- Output of system that satisfies both constraints may be determined by translating, scaling, and summing output generated from "convenient" input function centered at origin
- In other words, combined constraints of linearity and shift invariance allows calculation of output function by decomposing input function into superposition of outputs from a set of "basis" functions
- Most common convenient decomposition of input function uses the sifting property
- Output from each of scaled and translated Dirac delta functions is scaled and translated replica of the output due to a unit-area Dirac delta function located at the origin:

$$\mathcal{O}\{\delta[x-0]\} = \mathcal{O}\{\delta[x]\} \equiv h[x]$$

  notation $h[x]$ is the usual name for the *1-D impulse response*.

- Since $\mathcal{O}$ is shift invariant, translation of input results in identical translation of impulse response:

$$\mathcal{O}\{\delta[x-x_0]\} = h[x-x_0]$$

- Linearity of $\mathcal{O}$ ensures that multiplicative weighting of input results in identical weighting applied to individual output:

$$\begin{aligned}\mathcal{O}\{\alpha\ \delta[x-x_0]\} &= \alpha\ \mathcal{O}\{\delta[x-x_0]\}\\ &= \alpha\ h[x-x_0].\end{aligned}$$

  - Note that response of LSI system due to pair of shifted and scaled impulses is superposition of pair of shifted and scaled impulse responses:

$$\mathcal{O}\{\alpha_0\ \delta[x-x_0] + \alpha_1\ \delta[x-x_1]\} = \alpha_1\ h[x-x_0] + \alpha_2\ h[x-x_1].$$

- General expression for output of LSI operator by decomposing input function into constituent set of weighted Dirac delta functions via sifting property:

$$f[x] = \int_{-\infty}^{+\infty} f[\alpha]\ \delta[x-\alpha]\ d\alpha$$

- Response of system $\mathcal{O}$ to this input obtained by applying:

$$g[x] = \mathcal{O}\{f[x]\} = \mathcal{O}\left\{\int_{-\infty}^{+\infty} f[\alpha]\ \delta[x-\alpha]\ d\alpha\right\}$$

- Linearity property of $\mathcal{O}$ allows application either after summing inputs (i.e., outside integral) or applied to constituent Dirac delta functions before integrating:

$$\mathcal{O}\left\{\int_{-\infty}^{+\infty} f[\alpha]\ \delta[x-\alpha]\, d\alpha\right\} = \int_{-\infty}^{+\infty} \{\mathcal{O}\{f[\alpha]\ \delta[x-\alpha]\}\}\, d\alpha$$

- Because $\mathcal{O}$ acts on functions of x, weighting term $f[\alpha]$ "seen" as a multiplicative constant by $\mathcal{O}$.
  - Linearity criterion allows weighting to be applied after operator is applied to $\delta[x-\alpha]$.

$$\int_{-\infty}^{+\infty} \mathcal{O}\{f[\alpha]\ \delta[x-\alpha]\}\, d\alpha = \int_{-\infty}^{+\infty} f[\alpha]\ \mathcal{O}\{\delta[x-\alpha]\}\, d\alpha$$

- Shift invariance of $\mathcal{O}$ substitutes shifted replicas of impulse response:

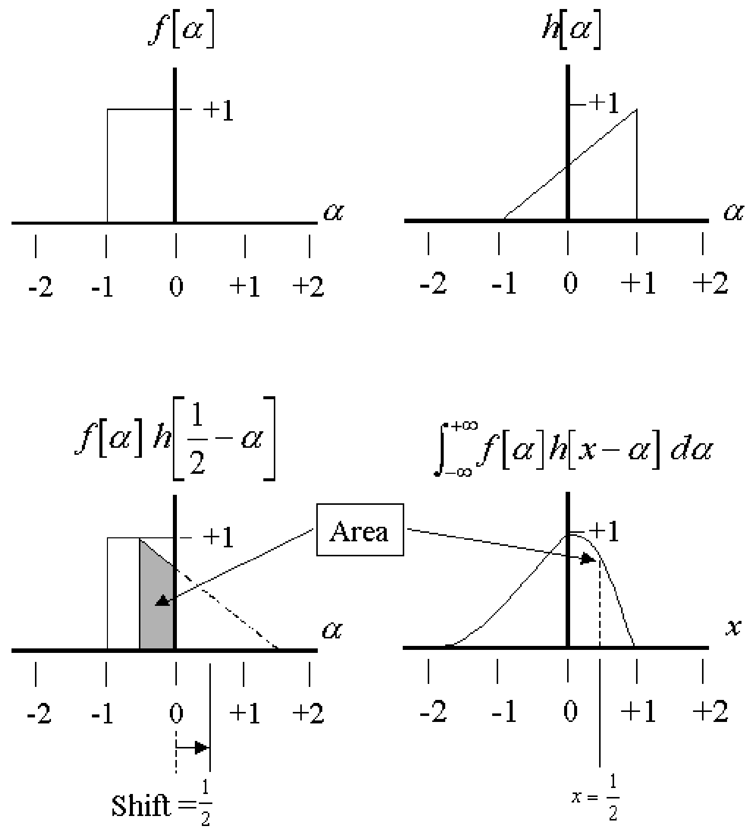
$$\int_{-\infty}^{+\infty} f[\alpha]\ \mathcal{O}\{\delta[x-\alpha]\}\ d\alpha = \int_{-\infty}^{+\infty} f[\alpha]\ h[x-\alpha]\ d\alpha$$

- the *convolution* integral of input $f[x]$ and impulse response $h[x]$

$$\begin{aligned} g[x] &= \int_{-\infty}^{+\infty} f[\alpha]\ h[x-\alpha]\ d\alpha \\ &\equiv f[x] * h[x] \end{aligned}$$

- Sifting property of Dirac delta function is just convolution of $f[x]$ with $\delta[x]$!
- Output of system from impulsive input at origin $f[x] = \delta[x-0]$ is important descriptor of the system
  - Mathematicians call it the *kernel*
  - Physicists $\Longrightarrow$ *Green's function*
  - Electrical engineers $\Longrightarrow$ *impulse response*
  - Optical scientists $\Longrightarrow$ *point-spread function* (or *psf*).
- Convolution called *linear filtering* of input signal by system
- Output of convolution computed at single coordinate (such as $g[x_0]$) obtained by summing products obtained by point-by-point multiplication of the input $f[\alpha]$ and copy of $h[\alpha]$ that has been "reverted" (or "flipped" or "folded") about origin, and then translated to place central ordinate at $\alpha = x_0$.
- If $f[x]$ and $h[x]$ have infinite support, then amplitude g at a single coordinate x_0 depends on amplitudes of constituent functions at all coordinates
- For this reason, mathematicians refer to output $g[x]$ as "functional" of $f[x]$ and $h[x]$
- The individual steps of the computation of the convolution integral are:

1. copy $f[x]$ and $h[x]$ to the α-domain to create $f[\alpha]$ and $h[\alpha]$,
2. "fold" the impulse response about the origin to obtain $h[-\alpha]$,
3. translate $h[-\alpha]$ by the distance x_0 to generate $h[x_0-\alpha]$,
4. multiply the two functions $h[x_0-\alpha]$ by $f[\alpha]$ at each coordinate α to generate the product function,
5. compute the area by integrating over the α-domain,
6. assign this area to the output amplitude g at coordinate x_0, and
7. loop to 3 for the next output coordinate x_1.



Demonstration of convolution: the convolution $f[x] * h[x]$ is evaluated at $x = \frac{1}{2}$.

- "Reversal" (or "folding" or "flipping") operation in step 2 is source of German name of *Faltung* (folding) integral for convolution.
- Less common names are *composition product* and *running mean*.
- The product of component functions (step 5) often misconstrued by students because of special properties of functions used in common examples
  - Either input $f[x]$ or impulse response $h[x]$ (or both) generally is a unit-amplitude rectangle function
  - Specific case where impulse response $h[x] = RECT\left[\frac{x}{b}\right]$; sole effect of $h[x]$ in multiplication (step 4) truncates $f[\alpha]$ to region between $x_0 - \frac{b}{2}$ and $x_0 + \frac{b}{2}$
  - Convolution evaluated at x_0 is merely area of $f[\alpha]$ within rectangle's "window"
  - Potential for confusion in minds of students may be reduced by using more general case to illustrate
  - Consider action of system with impulse response $h[x] = STEP[x]\ e^{-x}$ acting on input $f[x] = TRI[x]$. Convolution is evaluated at two particular locations $x_0 = +1$ and $x_1 = -0.5$ in figure. Note the differences in amplitude of the component functions and of the product function.
- Results may be summarized in single sentence: *"the output of any linear, shift-invariant system is the convolution of the input with the impulse response of the system."*
- Statement encapsulates reasons why simplification of arbitrary system operator to LSI is desirable

- Action of any N-D LSI system may be specified completely by single N-D function – the impulse response.
- "Character" of impulse response determines much about action of imaging system
- Expend significant effort to classify action of systems based upon "shape" of impulse response function
- Describe general properties of "differencing" and "averaging" filters.
- Identity Operator

$$g[x] = f[x] * \delta[x] = \int_{-\infty}^{+\infty} f[\alpha]\ \delta[x-\alpha]\ d\alpha = f[x]$$

- Result used to prove several identities of Dirac delta function:

$$\begin{gathered}\delta[x] * \delta[x] = \delta[x] \\ \delta[x-x_0] * \delta[x] = \delta[x] * \delta[x-x_0] = \delta[x-x_0] \\ \delta[x-x_0] * \delta[x-x_1] = \delta[x-x_0-x_1] = \delta[x-(x_0+x_1)]\end{gathered}$$

- Local Differencing Operator
  - Also easy to use convolution integral to derive sifting property for derivative of Dirac delta function by simply setting $h[x] = \frac{d\delta[x]}{dx} \equiv \delta'[x]$ in the convolution integral:

$$\begin{aligned} g[x] &= f[x] * \delta'[x] = \int_{-\infty}^{+\infty} f[\alpha]\ \frac{d}{dx}(\delta[x-\alpha])\ d\alpha \\ &= \int_{-\infty}^{+\infty} \frac{d}{dx}(f[\alpha]\ \delta[x-\alpha])\ d\alpha \text{ by linearity} \\ &= \frac{d}{dx}\int_{-\infty}^{+\infty} (f[\alpha]\ \delta[x-\alpha])\, d\alpha \text{ by linearity} \\ &= \frac{d}{dx} f[x] = f'[x] \text{ by sifting} \end{aligned}$$

  - In words, derivative of input function $f[x]$ is obtained by convolving with $\delta'[x]$
  - Generalized to compute n^{th} derivative of $f[x]$ via convolution with n^{th} derivative of $\delta[x]$:

$$\begin{aligned} \frac{d^n}{dx^n}(f[x]) &= \int_{-\infty}^{+\infty} f[\alpha]\ \delta^{(n)}[x-\alpha]\ d\alpha \\ &= f[x] * \delta^{(n)}[x] \end{aligned}$$

  - Convolution of $f[x]$ with $\delta'[x]$ computes differences in amplitude of $f[x]$ at "adjacent" coordinates
  - "local differencing operation".
- Local Averaging Operator: impulse response is unit-area rectangle function with support b:

$$h[x] = \frac{1}{|b|} RECT\left[\frac{x}{b}\right]$$

- Convolution $g[x]$ is:

$$
\begin{aligned}
g[x] &= f[x] * h[x] = \int_{-\infty}^{+\infty} \frac{1}{|b|} RECT\left[\frac{x-\alpha}{b}\right] \; f[\alpha] \; d\alpha \\
&= \frac{1}{|b|} \int_{x-\frac{b}{2}}^{x+\frac{b}{2}} f[\alpha] \; d\alpha
\end{aligned}
$$

  uniformly weighted average of input function $f[x]$ over domain of width b.

- "local averaging operation" tends to "smooth out" variation in amplitude of input over distances less than b

- Any impulse response with unit area, finite support, and with nonnull amplitudes of one sign (positive or negative) acts as local averager.

  - Consider convolution of unit-area finite-support rectangle with input function composed of sinusoid with amplitude A_1 and additive bias A_0.
  - Bias satisfies condition $A_0 \geq A_1$ to ensure that input function is nonnegative
  - Modulation of input sinusoid is $m_f = \frac{A_1}{A_0}$.
  - Convolution is straightforward to evaluate:

$$
\begin{aligned}
g[x] &= (A_0 + A_1 \cos[2\pi\xi_0\alpha + \phi_0] * \frac{1}{|b|} RECT\left[\frac{x}{b}\right] \\
&= \int_{-\infty}^{+\infty} \left|\frac{1}{b}\right| RECT[x-\alpha] \, (A_0 + A_1 \cos[2\pi\xi_0\alpha + \phi_0] \; d\alpha \\
&= \frac{1}{|b|} \int_{x-\frac{b}{2}}^{x+\frac{b}{2}} (A_0 + A_1 \cos[2\pi\xi_0\alpha + \phi_0] \; d\alpha \\
&= \frac{A_0}{|b|} \int_{x-\frac{b}{2}}^{x+\frac{b}{2}} 1[\alpha] \, d\alpha + \frac{A_1}{|b|} \int_{x-\frac{b}{2}}^{x+\frac{b}{2}} \cos[2\pi\xi_0\alpha + \phi_0] \; d\alpha \\
&= \frac{A_0}{|b|} \; |b| + \frac{A_1}{|b|} \; \frac{\sin[2\pi\xi_0\alpha + \phi_0]}{2\pi\xi_0}\Bigg|_{\alpha=x-\frac{b}{2}}^{\alpha=x+\frac{b}{2}} \\
&= A_0 + A_1 \, SINC[\xi_0 \; b] \;\; \cos[2\pi\xi_0 x + \phi_0]
\end{aligned}
$$

  - Because convolution is linear, same result obtained by separately convolving constant and oscillating parts of input function and adding results
  - Convolution of constant (DC component) is identical to input constant: average value of a constant is the constant
  - Convolution of oscillating part differs from input by scale factor $SINC[\xi_0 \; b]$.
  - Constant A_0 interpreted as symmetric sinusoid with amplitude A_0 and spatial frequency $\xi_0 = 0$
  - Expression for scale factor of oscillating part of $f[x]$ valid for constant part, i.e., $SINC[0 \times b] = 1$.

- Modulation of $g[x]$:

$$
\begin{aligned}
m_g &= \left(\frac{g_{max} - g_{min}}{g_{max} + g_{min}}\right) \\
&= \frac{A_1 SINC[\xi_0 \; b]}{A_0} \\
&= m_f \; SINC[\xi_0 \; b]
\end{aligned}
$$

- Rearrange to obtain a description of effect of LSI system on input sinusoid:

$$\frac{m_g}{m_f} = SINC\left[\xi_0 \; b\right]$$

- *modulation transfer function* (or MTF)

$$\text{MT}\left[\xi\right] = \frac{m_g\left[\xi\right]}{m_f\left[\xi\right]} = SINC\left[\xi \; b\right]$$

- modulation $m_g < m_f$ unless $\xi_0 = 0$ and/or $b = 0$.
  - First condition implies that original function $f\left[x\right]$ is constant of amplitude A_0, $m_f = 0$
  - Second condition $b = 0$ implies that uniform averager has infinitesimal support and finite area, i.e., $h\left[x\right] = \delta\left[x\right]$

4.1 GENERALIZATION to LINEAR SPACE-VARIANT OPERATIONS

- Generalized to construct general mathematical expression for linear and shift-variant (LSV) operations
- Linearity ensures that mathematical operation is superposition expressed as integral
- Shift variance modeled as superposition with different impulse response for each location in output domain
- Each location in 1-D output domain specified by x has associated 1-D impulse response $h[\alpha]$
- Space-variant analogue of 1-D impulse response in LSV system is 2-D function $h[x, \alpha]$:

$$\mathcal{O}\left\{f\left[x\right]\right\} \equiv g\left[x\right] = \int_{-\infty}^{+\infty} f[\alpha] \; h[x, \alpha] \; d\alpha$$

- 1-D *superposition integral*

5 CALCULATING CONVOLUTIONS

- Complicated form of convolution integral ensures that analytic expressions exist for few pairs of functions
- If two constituent functions have finite support, convolution is null when evaluated at all translations that exceed sum of their widths

$$\begin{aligned} g[x_0] &= \int_{-\infty}^{+\infty} RECT[\alpha]\; RECT[x_0-\alpha]\; d\alpha \\ &= \begin{cases} \int_{-\frac{1}{2}}^{x_0+\frac{1}{2}} 1\; d\alpha = \left(x_0+\frac{1}{2}\right)+\frac{1}{2} = 1+x_0 & \text{for} \quad -1 < x_0 < 0 \\ \int_{x_0-\frac{1}{2}}^{\frac{1}{2}} 1\; d\alpha = \frac{1}{2}-\left(x_0-\frac{1}{2}\right) = 1-x_0 & \text{for} \quad 0 < x_0 < +1 \end{cases} \\ &= (1-|x|)\; RECT\left[\frac{x}{2}\right] = TRI[x] \end{aligned}$$

- In words, convolution of $RECT[x]$ with itself (the *autoconvolution*) yields $TRI[x]$

5.1 EXAMPLES of CONVOLUTIONS

1. Autoconvolution of decaying exponential

$$\begin{aligned} g_1[x] &= e^{-x} STEP[x] * e^{-x} STEP[x] \\ &= \int_{-\infty}^{+\infty} \left(e^{-\alpha}\; STEP[\alpha]\right)\; \left(e^{-(x-\alpha)}\; STEP[x-\alpha]\right)\; d\alpha \\ &= \int_{-\infty}^{+\infty} STEP[\alpha]\; STEP[x-\alpha]\, e^{-x} e^{+\alpha} e^{-\alpha} d\alpha \\ &= e^{-x} \int_{-\infty}^{+\infty} STEP[\alpha]\; STEP[x-\alpha]\; d\alpha \\ &= e^{-x} \int_{0}^{+\infty} STEP[x-\alpha]\; d\alpha \end{aligned}$$

$$\begin{aligned} g_1[x] &= e^{-x} STEP[x] * e^{-x} STEP[x] \\ &= e^{-x} \int_0^x d\alpha \\ &= \begin{cases} x\; e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases} \end{aligned}$$

Final result expressed in concise form by reapplying definition of $STEP[x]$ to "window" result to positive values of x

$$\begin{aligned} g_1[x] &= e^{-x}\; STEP[x] * e^{-x}\; STEP[x] \\ &= x\; STEP[x]\; e^{-x} \end{aligned}$$

2. Convolution of two quadratic-phase functions of opposite sign

$$\begin{aligned}
g_2[x] &= e^{-i\pi x^2} * e^{+i\pi x^2} \\
&= \int_{-\infty}^{+\infty} e^{-i\pi\alpha^2} e^{+i\pi(x\,-\,\alpha)^2} d\alpha \\
&= \int_{-\infty}^{+\infty} e^{-i\pi\alpha^2} e^{+i\pi x^2} e^{+i\pi\alpha^2} e^{-2\pi i\alpha x} d\alpha \\
&= e^{+i\pi x^2} \int_{-\infty}^{+\infty} e^{-2\pi i\alpha x} d\alpha \\
&= e^{+i\pi x^2}\ \delta[x] = e^{+i\pi x^2}\ \delta[x-0] \\
&= e^{+i\pi 0^2}\ \delta[x-0] = \delta[x]
\end{aligned}$$

Both constituent functions have infinite support, but support of convolution is infinitesimal.

6 PROPERTIES of CONVOLUTIONS

1. From definition of convolution of two continuous functions $f[x]$ and $h[x]$, an equivalent formulation derived by changing integration variable to $u = x - \alpha$, so that $\alpha = x - u$ and $du = -d\alpha$:

$$\begin{aligned} f[x] * h[x] &= \int_{u=+\infty}^{u=-\infty} f[x-u]\ h[u]\ (-du) \\ &= \int_{-\infty}^{+\infty} f[x-u]\ h[u]\ (+du) \\ &= \int_{-\infty}^{+\infty} h[\alpha]\ f[x-\alpha]\ (+d\alpha) = h[x] * f[x] \end{aligned}$$

   - Convolution is commutative
   - roles of input and impulse response are interchangeable

2. Convolution with specific impulse response $h[x]$ is linear

$$\begin{aligned} f[x] * h[x] &= g[x] \\ &\Longrightarrow \alpha\ f[x] * h[x] \\ &= f[x] * \alpha h[x] = \alpha g[x]\,. \end{aligned}$$

3. Distributive with respect to addition:

$$\{\alpha_1 f_1[x] + \alpha_2 f_2[x]\} * \{\beta_1 h_1[x] + \beta_2 h_2[x]\}$$
$$= \alpha_1\beta_1(f_1[x] * h_1[x]) + \alpha_2\beta_1(f_2[x] * h_1[x]) + \alpha_1\beta_2(f_1[x] * h_2[x]) + \alpha_2\beta_2(f_2[x] * h_2[x])$$

4. Convolution is shift-invariant:

$$\begin{aligned} \mathcal{O}\{f[x]\} &\equiv f[x] * h[x] = g[x] \\ &\Longrightarrow \mathcal{O}\{f[x-x_0]\} = f[x-x_0] * h[x] = g[x-x_0]\,. \end{aligned}$$

5. Convolution is associative:

$$(f_1[x] * f_2[x]) * f_3[x] = f_1[x] * (f_2[x] * f_3[x]).$$

   - Cascades of convolutions may be computed in any order:

$$(f_1[x] * f_2[x]) * f_3[x] = f_3[x] * (f_2[x] * f_1[x]).$$

6.1 REGION of SUPPORT of CONVOLUTIONS

- When both functions have compact support, support of convolution is sum of supports of constituent functions
  - confirmed by autoconvolution of $RECT[x]$:

$$RECT[x] * RECT[x] = TRI[x]$$

  - If $f[x]$ and $h[x]$ have support b_f and b_h support of $g[x]$ is $b_g = b_f + b_h$
- Simple relation fails if support of either (or both) of constituent functions is infinite or infinitesimal

– Consider convolution of $f[x] = 1[x]$ and $h[x] = \delta'[x]$, $b_f = \infty$ and b_h is infinitesimal

$$\begin{aligned} g[x] &= 1[x] * \delta'[x] \\ &= \frac{d}{dx}(1[x]) \\ &= 0[x] \end{aligned}$$

– $b_g = 0$;

$$f[x] * h[x] = e^{+i\pi x^2} * e^{-i\pi x^2} = \delta[x],\ b_f = b_h = \infty,\ b_g = 0$$

6.2 AREA of a CONVOLUTION

$$\begin{aligned} \int_{-\infty}^{+\infty} g[x]\,dx &= \int_{-\infty}^{+\infty} f[x] * h[x]\ dx \\ &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f[\alpha]\, h[x-\alpha]\ d\alpha \right) dx \\ &= \int_{-\infty}^{+\infty} f[\alpha] \left(\int_{-\infty}^{+\infty} h[x-\alpha]\ dx \right) d\alpha \\ &= \left(\int_{-\infty}^{+\infty} f[\alpha]\ d\alpha \right) \left(\int_{-\infty}^{+\infty} h[x]\ dx \right) \end{aligned}$$

6.3 CONVOLUTION of FUNCTIONS after SCALING

- Not-so-intuitive property demonstrated by changing scale of both $f[x]$ and $h[x]$ by same factor b.
- Given that $f[x] * h[x] = g[x]$, then:

$$\begin{aligned} f\left[\frac{x}{b}\right] * h\left[\frac{x}{b}\right] &= \int_{-\infty}^{+\infty} f\left[\frac{\alpha}{b}\right]\ h\left[\frac{x-\alpha}{b}\right] d\alpha \\ &= |b| \int_{-\infty}^{+\infty} f[\beta]\ h\left[\frac{x}{b} - \beta\right] dx \\ &= |b|\ g\left[\frac{x}{b}\right] \end{aligned}$$

- Equal changes of scale in both input and kernel change both scale and amplitude of output function by same factor.
- Perhaps more clearly stated as:

$$f\left[\frac{x}{b}\right] * h\left[\frac{x}{b}\right] = |b|\ (f[u] * h[u])\big|_{u=\frac{x}{b}}$$

$$f_1\left[\frac{x}{b}\right] * f_2\left[\frac{x}{b}\right] * \cdots * f_N\left[\frac{x}{b}\right] = |b|^{N-1}\ g\left[\frac{x}{b}\right]$$

7 AUTOCORRELATION

- Define 1-D operator that resembles convolution
- Differs in significant details that often cause confusion.
- Operator is very useful in certain signal-processing applications
  - Finding position of known signal in input object.
- Operator introduced as special case and then generalized.
- Inner product of N-D discrete vector with itself :

$$|\underline{\mathbf{x}}|^2 = \underline{\mathbf{x}}^* \bullet \underline{\mathbf{x}} = \sum_{n=1}^{N} |x_n|^2$$

- Inner product of $\underline{\mathbf{x}}$ with itself is square of length of vector and guaranteed to be real valued
- Analogous construction for complex-valued continuous function $f[x]$ is integral of squared magnitude:

$$\begin{aligned}\int_{-\infty}^{+\infty} (f[\alpha])^* f[\alpha]\, d\alpha &= \int_{-\infty}^{+\infty} f[\alpha] (f[\alpha])^* \, d\alpha \\ &= \int_{-\infty}^{+\infty} |f[\alpha]|^2 \, d\alpha\end{aligned}$$

  - Also real valued.
  - Operation yields numerical value proportional to *projection* of $f[x]$ onto itself
  - Metric of analogue to *length* of function.
- Integral computes projection of input function $f[x]$ onto "shifted replica" of $f[x]$:

$$\int_{-\infty}^{+\infty} f[\alpha] (f[\alpha - x_0])^* \, d\alpha = \int_{-\infty}^{+\infty} f[\alpha] \ f^*[\alpha - x_0] \ d\alpha$$

- Projection computed for all possible values of translation parameter x
- Operation recast into convolution integral by algebraic manipulation:

$$\begin{aligned}\int_{-\infty}^{+\infty} f[\alpha] \ f^*[\alpha - x] \ d\alpha &= \int_{-\infty}^{+\infty} f[\alpha] \ f^*[-(x-\alpha)] \ d\alpha \\ &= f[x] * f^*[-x]\end{aligned}$$

- Measures "similarity" of $f[x]$ to its translated replica
- All functions are most similar to untranslated replicas, amplitude must be largest at $x = 0$.
- *Autocorrelation* of $f[x]$
- signified by "pentagram" (five-pointed star):

$$f[x] * f^*[-x] \equiv f[x] \bigstar f[x]$$

- Definition explicitly includes complex conjugation of "reference" function
  - Some authors (e.g., Gaskill) do not explicitly complex conjugate

- Autocorrelation integral is not defined in many cases where $f[x]$ has infinite support or infinite integrated power.
  - Trivial example is $f[x] = 1[x]$
  - Autocorrelation does not exist for finite-amplitude periodic functions
- Autocorrelation of infinite-support $f[x]$ redefined to compute "average" area of product function over finite but indefinite region of support, extrapolated to infinite limits:

$$(f[x] \bigstar f[x])_{\text{infinite support}} \simeq \lim_{B\to\infty} \left\{ \frac{1}{2B} \int_{-B}^{+B} f[\alpha]\;\; f^*[\alpha - x]\;\; d\alpha \right\}$$

- Assumes expression converges in limit $B \to \infty$
- Variant of autocorrelation for periodic $f[x]$ defined over single period only:

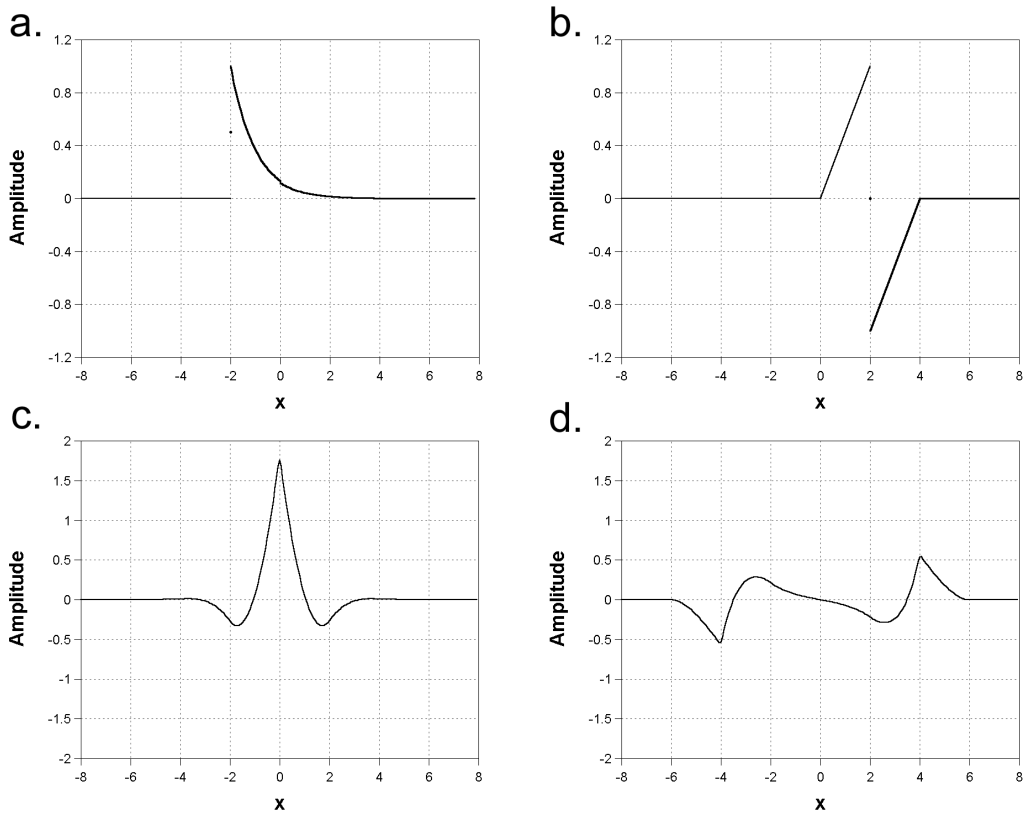
$$(f[x] \bigstar f[x])_{\text{periodic}} \simeq \frac{1}{X} \int_{-\infty}^{+\infty} \left(f[\alpha]\; RECT\left[\frac{\alpha}{X}\right] \right) \left(f^*[\alpha - x]\; RECT\left[\frac{\alpha - x}{X}\right] \right) d\alpha$$

  where X is period of $f[x]$.

- Maximum amplitude of autocorrelation always occurs at origin
- Comparisons of autocorrelations of different functions facilitated by normalizing by integrated power
- Unit amplitude at origin.
- Common notation for normalized autocorrelation is γ_{ff},
- "ff" signifies that the same function is used as "input" and "reference":

$$\gamma_{ff}[x] \equiv \frac{\int_{-\infty}^{+\infty} f[\alpha]\;\; f^*[\alpha - x]\;\; dx}{\int_{-\infty}^{+\infty} |f[\alpha]|^2\;\; dx} = \frac{(f[x] \bigstar f[x])}{(f[x] \bigstar f[x])|_{x=0}}$$

- Qualities of symmetry of autocorrelation function determined by character of $f[x]$
  - If $f[x]$ is real OR imaginary , then complex conjugation applied to "reference" ensures that $\gamma_{ff}[x]$ is real valued and symmetric
  - $\gamma_{ff}[x] = \gamma_{ff}^*[x] = \gamma_{ff}[-x]$ for $f[x]$ real OR imaginary.
  - Projection of complex function onto itself at zero shift yields a real-valued "length"$\Longrightarrow$ $\gamma_{ff}[0]$ is real
  - Complex-valued $f[x] \Longrightarrow$"cross terms" from product of real part of $f[x]$ and imaginary part of $f^*[x]$ generate imaginary part $\gamma_{ff}[x]$ that must be zero at $x = 0$ and form an odd function.
  - Autocorrelation of complex-valued $f[x]$ is Hermitian (even real part, odd imaginary part).



The autocorrelation of a complex-valued function $f[x]$ is Hermitian: (a) $\Re\{f[x]\} = e^{-(x+2)} STEP[x+2]$, (b) $\Im\{f[x]\} = TRI[x-2]\, SGN[x-2]$, (c) $\Re\{f[x] \bigstar f[x]\}$ is even, (d) $\Im\{f[x] \bigstar f[x]\}$ is odd.

- Shape of autocorrelation measures spatial "variability" of function
  - If $f[x]$ varies "slowly" with x, area of product function also varies "slowly" with translation parameter $\Longrightarrow$ broad autocorrelation "peak"
  - Conversely, function whose amplitude varies over short distances will have narrow autocorrelation peak
  - Autocorrelation and Fourier transform intimately related via Wiener-Khintchin theorem.

7.1 AUTOCORRELATION of STOCHASTIC FUNCTIONS (NOISE)

- Useful application of autocorrelation is to describe characteristics of stochastic functions
- If noise function is truly random, no systematic relationship between amplitudes at "adjacent" coordinates is expected
- Central peak of resulting autocorrelation function must be "narrow
- If amplitudes of noise function at adjacent coordinates are "correlated", then autocorrelation of noise function will have "wider" peak.
- If stochastic noise function is nonnegative everywhere, average amplitude is itself guaranteed to be nonnegative
- Autocorrelation of such a noise function exhibits large peak at zero shift due to overlap of positive values
- Variation called the "autocovariance" often is convenient.

7.2 AUTOCOVARIANCE of STOCHASTIC FUNCTIONS (NOISE)

- *Autocovariance* is autocorrelation of bipolar function created by subtracting mean amplitude

$$
\begin{aligned}
C_{ff}[x] &= \langle f[x] - \langle f \rangle \rangle \bigstar \langle f[x] - \langle f \rangle \rangle = \int_{-\infty}^{+\infty} (f[\alpha] - \langle f \rangle)(f[\alpha - x] - \langle f \rangle)^* \; d\alpha \\
&= \lim_{X \to \infty} \left\{ \frac{1}{2X} \int_{-X}^{+X} \left(f[\alpha] f^*[\alpha - x] - f[\alpha] \langle f \rangle^* - \langle f \rangle f^*[\alpha - x] + |\langle f \rangle|^2 \right) d\alpha \right\} \\
&= \lim_{X \to \infty} \left\{ \frac{1}{2X} \int_{-X}^{+X} (f[\alpha] \; f^*[\alpha - x]) \, d\alpha \right\} - \langle f \rangle^* \lim_{X \to \infty} \left\{ \frac{1}{2X} \int_{-X}^{+X} f[\alpha] \, d\alpha \right\} \\
&\quad - \langle f \rangle \lim_{X \to \infty} \left\{ \frac{1}{2X} \int_{-X}^{+X} f^*[\alpha - x] \; d\alpha \right\} + |\langle f \rangle|^2 \\
&= f[x] \bigstar f[x] - \langle f \rangle^* \langle f \rangle - \langle f \rangle \langle f \rangle^* + |\langle f \rangle|^2 \\
&= f[x] \bigstar f[x] - |\langle f \rangle|^2
\end{aligned}
$$

- Has advantage of removing "bias" from autocorrelation that may hide subtle variations.

8 CROSSCORRELATION

- Generalize concept of autocorrelation to compute measure of "similarity" between $f[x]$ and arbitrary "reference" function $m[x]$

  - Projection of $f[x]$ onto $m[x]$ at "zero" shift is a (generally complex) number g_1 :

$$g_1 = \int_{-\infty}^{+\infty} f[\alpha] \; m^*[\alpha] \; d\alpha$$

  - g_1 is large when $f[x]$ and $m[x]$ have similar "shapes"
  - Central ordinate of reference function $m^*[\alpha]$ translated by adding factor x_0 to the argument to measure the projection of $f[x]$ onto $m[x - x_0]$:

$$\int_{-\infty}^{+\infty} f[\alpha] \; m^*[\alpha - x_0] \; d\alpha = g_1[x_0]$$

  - Measures similarity between $f[x]$ and translated reference $m[x - x_0]$
  - Different x_0 generally gives different values of inner product
  - Evaluate inner product for all possible shift parameters to construct function of x that measures similarity between $f[x]$ and complete set of translated replicas of $m[x - x_0]$
  - "Crosscorrelation" of input function $f[x]$ and "reference" function $m[x]$

$$\begin{aligned} g_1[x] &= \int_{-\infty}^{+\infty} f[\alpha] \; m^*[\alpha - x] \; d\alpha \\ &\equiv f[x] \bigstar m[x] \end{aligned}$$

  - Area must be finite at all translations x for the crosscorrelation integral to exist
  - Largest values of inner product occur at abscissa(s) x_n where input function $f[x]$ is "most similar" to translated reference function $m[x - x]$

- Note both similarity and two significant differences between crosscorrelation and convolution

  - Both measure area of product of two functions after one has been translated by some distance

    * Crosscorrelation: input function multiplied by shifted replicas of complex conjugate of reference function $m[x]$
    * Convolution: analogue of "reference function" is "impulse response" $h[x]$, which is "reversed" prior to translation, no complex conjugation.

  - Crosscorrelation: Argument of reference function m is negative of argument of impulse response h in convolution

- Some authors define crosscorrelation with the "negative" of shift parameter so that argument of m^* has positive sign:

$$g_2[x] \equiv \int_{-\infty}^{+\infty} f[\alpha] \; m^*[\alpha + x] \; d\alpha$$

  - Different definitions of crosscorrelation g_1 and g_2 produce mutually "reversed" functions: $g_1[x] = g_2[-x]$.
  - We use former definition with "minus sign" in argument to maintain similarity to definition of convolution

- Gaskill defines variant of crosscorrelation that does not include complex conjugate of "reference" function $m[x]$:

$$g_3[x] \equiv \int_{-\infty}^{+\infty} f[\alpha]\ \ m[\alpha - x]\ \ d\alpha$$

- Explicit definition of crosscorrelation:

$$\boxed{g_1[x] = f[x] \bigstar m[x] \equiv \int_{-\infty}^{+\infty} f[\alpha]\ \ m^*[\alpha - x]\ \ d\alpha}$$

- Definition of autocorrelation may be generalized to construct relation between crosscorrelation and convolution:

$$\begin{aligned} f[x] \bigstar m^*[x] &= f[x] * m[-x] \\ f[x] \bigstar m[x] &= f[x] * m^*[-x] \end{aligned}$$

8.1 Properties of Crosscorrelation: Linearity and Shift Invariance

- Linear: Crosscorrelation is a linear operation because it is defined by an integral over fixed limits
- Behavior under translation:

$$\begin{aligned} f[x-x_0] \bigstar m[x] &= \int_{-\infty}^{+\infty} f[\alpha - x_0]\ m^*[\alpha - x]\ d\alpha \\ &= \int_{-\infty}^{+\infty} f[\beta]\ m^*[\beta + x_0 - x]\ d\beta,\ \text{where } \beta \equiv \alpha - x_0 \\ &= \int_{-\infty}^{+\infty} f[\beta]\ m^*[\beta - (x - x_0)]\ d\beta \\ &= (f[x] \bigstar m[x])|_{x=x-x_0} \end{aligned}$$

  $\Longrightarrow$ crosscorrelation with $m[x]$ is shift invariant.

- Commutativity: change integration variable to $\beta = \alpha - x$ to show tghat correlation of "input function" $m[x]$ and "reference function" $f[x]$ is:

$$\begin{aligned} m[x] \bigstar f[x] &= \int_{-\infty}^{+\infty} m[\beta + x]\ f^*[\beta]\ d\beta \\ &\neq \int_{-\infty}^{+\infty} f[\alpha]\ m^*[x-\alpha]\ d\alpha = f[x] \bigstar m[x] \end{aligned}$$

  $\Longrightarrow$ crosscorrelation does not commute; roles of input and reference function are not so easily interchanged

- Amplitude of crosscorrelation may be negative for some shifts; functions are said to be "anti-correlated" because input function "resembles" negative of reference function
- Normalized crosscorrelation:

$$\gamma_{fm}[x] \equiv \frac{\int\limits_{-\infty}^{+\infty} f[\alpha]\ m^*[\alpha - x]\ dx}{\int\limits_{-\infty}^{+\infty} |m[\alpha]|^2\ dx} = \frac{f[x] \bigstar m[x]}{m[x] \bigstar m[x]|_{x=0}}$$

- Crosscorrelation is a generalized inner product evaluated for different translations of the reference.

9 LINEAR SHIFT-INVARIANT OPERATIONS on 2-D FUNCTIONS

- Sifting property of 2-D Dirac delta function may be applied to evaluate 2-D convolution integral

$$\begin{aligned}\mathcal{O}\{f[x,y]\} &= \iint_{-\infty}^{+\infty} f[\alpha,\beta]\ h[x-\alpha,y-\beta]\ d\alpha\ d\beta \\ &\equiv f[x,y]*h[x,y]\end{aligned}$$

- 2-D kernel $h[x,y]$ determines action of system
  - still called impulse response or point spread function
  - kernel $h[x,y]$ is "reflected" through the origin (or equivalently rotated about origin by π radians)
  - "volume" of product function is determined by integrating over 2-D domain
- If input and kernel both separable, the 2-D convolution integral is the product of two 1-D convolutions:

$$\begin{aligned}f[x,y]*h[x,y] &= (f_1[x]\cdot f_2[y])*(h_1[x]\cdot h_2[y]) \\ &= \iint_{-\infty}^{+\infty} f_1[\alpha]\ \ f_2[\beta]\,h_1[x-\alpha]\,h_2[y-\beta]\ d\alpha\ d\beta \\ &= \int_{-\infty}^{+\infty} f_1[\alpha]\ h_1[x-\alpha]\,d\alpha \int_{-\infty}^{+\infty} f_2[\beta]\ h_2[y-\beta]\,d\beta \\ &= (f_1[x]*h_1[x])\ (f_2[x]*h_2[x])\end{aligned}$$

- Example: 2-D convolution of $f[x,y]=RECT[x,y]$ with itself:

$$\begin{aligned}RECT[x,y]*RECT[x,y] &= (RECT[x]*RECT[x])\ (RECT[y]*RECT[y]) \\ &= TRI[x]\cdot TRI[y]\equiv TRI[x,y]\end{aligned}$$

- Some authors use "paired" asterisks to denote convolution of 2-D functions.
  - Rationale: distinguishes between 1-D and 2-D convolution operations when performed in same system.
  - Notation may be used to describe convolution of 2-D function with either 1-D or 2-D impulse responses
  - Divide asterisks among two 1-D convolutions for 2-D separable functions: "conservation of asterisks"
  - Example:convolve 2-D function $f[x,y]$ with two systems:
    1. 1-D convolution of 2-D function with 1-D Dirac delta function $\delta[x-x_0]$

$$\begin{aligned}f[x,y]*\delta[x-x_0] &= \int_{-\infty}^{+\infty} f[\alpha,y]\ \ \delta[x-x_0-\alpha]\ \ d\alpha \\ &= f[x-x_0,y]\end{aligned}$$

I try to avoid confusion by rewriting this as the 2-D convolution with $\delta[x-x_0,y]$

$$\begin{aligned}f[x,y]*\delta[x-x_0,y] &= \int_{-\infty}^{+\infty} f[\alpha,\beta]\ \ \delta[x-x_0-\alpha]\ \ \delta[y-\beta]\ \ d\alpha\ d\beta \\ &= \int_{-\infty}^{+\infty} f[\alpha,y]\ \ \delta[x-x_0-\alpha]\ \ d\alpha \\ &= f[x-x_0,y]\end{aligned}$$

2. 2-D convolution of 2-D function with 2-D Dirac delta function (line delta function):

$$
\begin{aligned}
f[x,y] ** \delta[x-x_0] &= \iint_{-\infty}^{+\infty} f[\alpha,\beta]\ \delta[x-x_0-\alpha]\ 1[\beta]\ d\alpha\ d\beta \\
&= \int_{-\infty}^{+\infty} f[x-x_0,\beta]\ d\beta
\end{aligned}
$$

This is 2-D convolution with the line delta function $\delta[x-x_0]\,1[y]$

$$
\begin{aligned}
f[x,y] * \delta[x-x_0]\ 1[y] &= \iint_{-\infty}^{+\infty} f[\alpha,\beta]\ \delta[x-x_0-\alpha]\ 1[y-\beta]\ d\alpha\ d\beta \\
&= \int_{-\infty}^{+\infty} f[x-x_0,y]\ 1[y]\ dy \\
&= \int_{-\infty}^{+\infty} f[x-x_0,y]\ dy
\end{aligned}
$$

which is a 2-D function that is constant along the y-direction and the value at each coordinate x is the translated "line-integral projection" of $f[x,y]$

- In short, I always use one asterisk for convolution and require that the two component functions have the same dimensionality.

9.1 LINE SPREAD and EDGE SPREAD FUNCTIONS

- Useful metrics for system

9.1.1 Line Spread Function

- impulse response is $h[x,y]$is output of system when input is a 2-D Dirac delta function:

$$\begin{aligned}\mathcal{O}\{\delta[x,y]\} &= \mathcal{O}\{\delta[x]\ \delta[y]\} \\ &= \iint_{-\infty}^{+\infty} \delta[x-\alpha, y-\beta]\ h[\alpha,\beta]\ d\alpha\, d\beta \\ &\equiv h[x,y]\end{aligned}$$

- Line spread function is output of system when input is 2-D "line delta function" $\delta[x]\ 1[y]$
  - Result is a 2-D function that varies in x-direction and constant along lines parallel to y-axis:

$$\begin{aligned}\mathcal{O}\{\delta[x]\ 1[y]\} &= \iint_{-\infty}^{+\infty} (\delta[x-\alpha]\ 1[y-\beta])\ h[\alpha,\beta]\ d\alpha\ d\beta \\ &= \int_{-\infty}^{+\infty} h[x,\beta]\ d\beta \\ &\equiv \ell[x]\end{aligned}$$

  - Description of the system from $\ell[x]$ is "incomplete"
    - No information about variation of $h[x,y]$ along the y-axis
    - $h[x,y]$ cannot be determined from $\ell[x]$ alone, unless $h[x,y]$ is circularly symmetric.

9.1.2 Edge Spread Function

- Response of system to "edge" input $STEP[x]\ 1[y]$

$$\begin{aligned}\mathcal{O}\{STEP[x]\ 1[y]\} &= \iint_{-\infty}^{+\infty} STEP[x-\alpha]\ 1[y-\beta]\, h[\alpha,\beta]\ d\alpha\ d\beta \\ &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{x} h[\alpha,\beta]\, d\alpha\right)\ 1[y-\beta]\ d\beta \\ &\equiv e[x]\end{aligned}$$

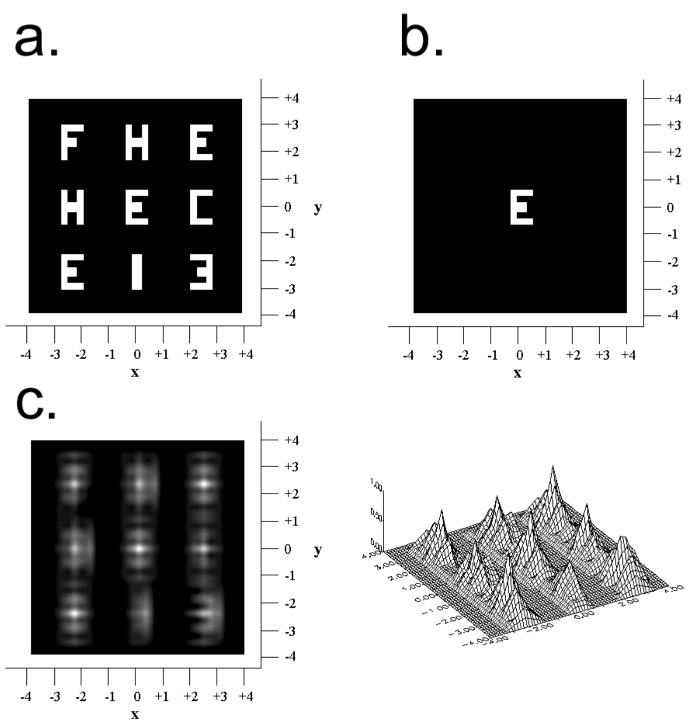
- Varies along x-axis and constant along lines parallel to the y-axis

10 CROSSCORRELATION of 2-D FUNCTIONS

- Translate complex conjugate of 2-D reference function $m[x,y]$ relative to input function $f[x,y]$
- Evaluating overlap volume
- Repeat for all possible translations to get 2-D function

$$
\begin{aligned}
f[x,y] \bigstar m[x,y] &\equiv \iint_{-\infty}^{+\infty} f[\alpha,\beta]\;\; h^*[\alpha-x, y-\beta]\;\; d\alpha\; d\beta \\
&= f[x,y] * m^*[-x,-y]
\end{aligned}
$$

- Example:
  - Input function is bitonal nine alphabetic characters
  - Reference function is one character "E"
  - Largest amplitudes in crosscorrelation occur at locations of replicas of $m[x,y]$
  - Basis for matched filtering



Crosscorrelation of the input "object" $f[x,y]$ with the "reference" function $m[x,y]$: (a) $f[x,y]$, (b) $m[x,y]$, (c) normalized crosscorrelation $\gamma_{fm}[x,y] = \frac{f[x,y]\bigstar m[x,y]}{\int\int_{-\infty}^{+\infty}|m[x,y]|^2 dxdy}$ as image and as 3-D surface.

11 AUTOCORRELATION of 2-D FUNCTIONS

$$f[x,y] \bigstar f[x,y] \equiv \iint_{-\infty}^{+\infty} f[\alpha,\beta] \;\; f^*[\alpha - x, y - \beta] \;\, d\alpha \; d\beta$$

- guaranteed to be Hermitian
- Amplitude at origin is area of squared magnitude of function:

$$\begin{aligned} (f[x,y] \bigstar f[x,y])|_{[x,y]=[0,0]} &= \iint_{-\infty}^{+\infty} f[\alpha,\beta] \;\; f^*[\alpha - 0, y - \beta] \;\, d\alpha \; d\beta \\ &= \iint_{-\infty}^{+\infty} |f[\alpha,\beta]|^2 \;\, d\alpha \; d\beta \end{aligned}$$

which is real valued

- "Normalized autocorrelation" is:

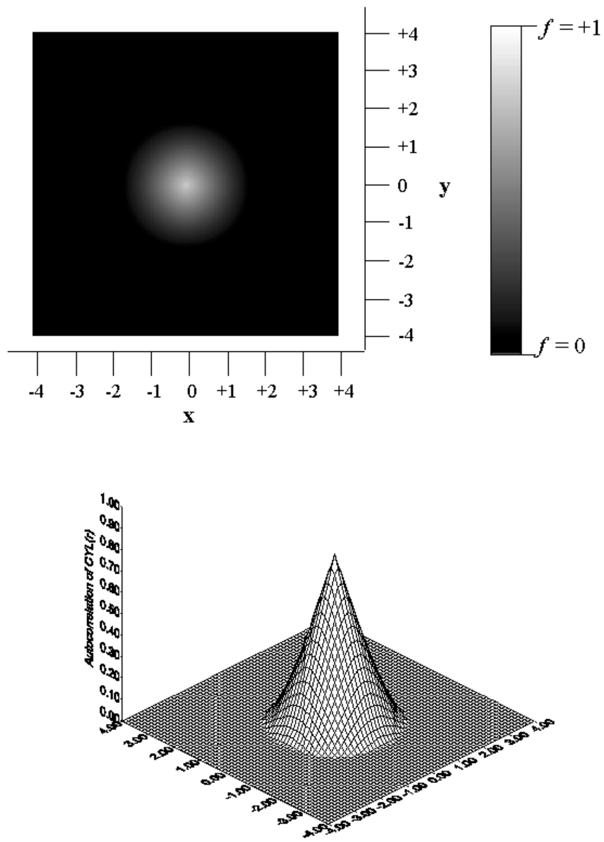
$$\begin{aligned} \gamma_{ff}[x,y] &\equiv \frac{\int\limits_{-\infty}^{+\infty} f[\alpha,\beta] \;\; f^*[\alpha - x, \beta - y] \;\, d\alpha \; d\beta}{\int\limits_{-\infty}^{+\infty} |f[\alpha,\beta]|^2 \;\, d\alpha \; d\beta} \\ &= \frac{f[x,y] \bigstar f[x,y]}{f[x,y] \bigstar f[x,y]|_{x=0,y=0}} \end{aligned}$$

11.1 AUTOCORRELATION of CYLINDER FUNCTION

- Without Proof:

$$CYL(r)\bigstar CYL(r) = \frac{2}{\pi}\left(\cos^{-1}(r) - r\,\sqrt{1-r^2}\right) CYL\left(\frac{r}{2}\right)$$

- circularly symmetric
- sides are not "straight"



Autocorrelation of $CYL(r)$ as "image" and as 3-D function. Note that the amplitude at $[x, y] = [0, 0]$ is $\frac{\pi}{4}$.