SIMG-716
Linear Imaging Mathematics I, Handout 04

1 1-D REAL-VALUED SPECIAL FUNCTIONS

- 1-D special functions defined over real-valued domain with independent spatial variable x
- Many functions (e.g., $RECT[x]$) defined by specifying amplitude f at different coordinates by "rule" or mathematical relation.

1.1 Scaling the "width"

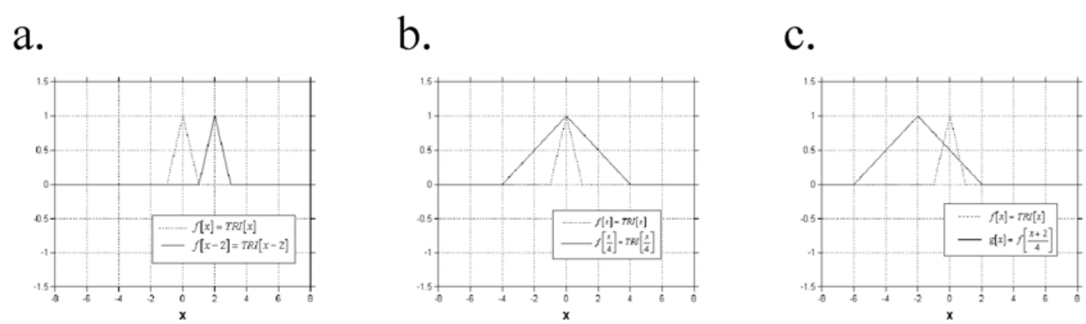
- "width" varied by scaling independent variable x by real-valued factor b to evaluate $f\left[\frac{x}{b}\right]$
- consider $f[x]$ with compact support such that $f[x] = 0$ for $|x| > 1$

1. scaled function $f\left[\frac{x}{b}\right] = 0$ for $|x| > b$
2. scaled function is "wider" if $b > 1$
3. "narrower" if $b < 1$
4. If $f[x]$ has finite support, then b is the scale factor for both that support interval and of the area of the function.
5. If $f[x]$ has infinite support, then b is the scale factor of separation distance between "features" of the function (such as the locations of zeros).

- A few functions are defined by a single criterion over entire infinite domain
  - Examples include sinusoids and Gaussian function $e^{-\pi x^2}$.
  - These functions may be scaled by complex-valued width parameters to produce functions with complex-valued amplitudes.

1.2 Translating the "center"

- recast independent variable into form $x - x_0$.
- amplitude originally at origin of coordinates is translated to $x = x_0$
- Both scaling and translation may be applied to a function at one time by applying the general argument $\frac{x-x_0}{b}$



Effects of parameters of argument of function $f[x]$: (a) shifting $f[x-2]$; (b) scaling $f\left[\frac{x}{4}\right]$; (c) combination $f\left[\frac{x+2}{4}\right]$.

1.3 1-D CONSTANT FUNCTIONS

- Amplitude $\propto x^0$

1.3.1 unit constant:

- infinite support
- infinite area.

$$1[x] \equiv 1 \text{ for all } x$$

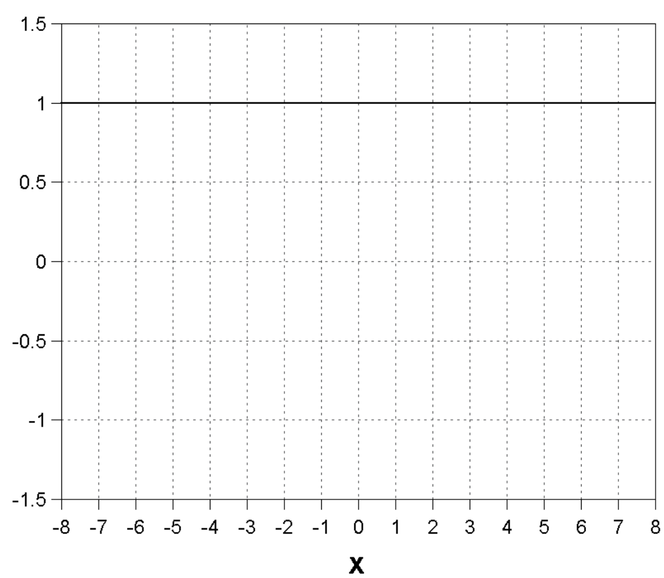
1.3.2 null constant:

- null support
- null area
- used as "place holder", e.g., $f[x] = 1[x] + i\,0[x]$ reminds us that function is complex with null imaginary part.

$$0[x] \equiv 0 \text{ for all } x$$

- A constant function with any desired complex-valued amplitude is generated trivially by multiplying 1[x] by that complex constant.
- Translation or scaling applied to argument of any constant function has no effect on amplitude at any coordinate:

$$1\left[\frac{x-x_0}{b}\right] = 1[x] = 1$$
$$0\left[\frac{x-x_0}{b}\right] = 0[x] = 0$$



Unit constant function $1[x]$

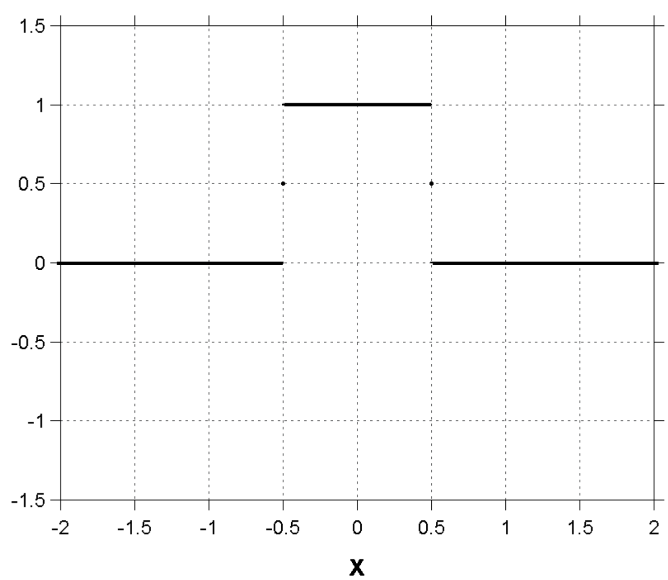
1.4 RECTANGLE FUNCTION

- Combination of the unit constant and null constant
- amplitude $\propto x^0$, except at discontinuous transitions
- Many uses in imaging
  - multiply (modulate), and thus truncate, a "wide" or infinite-support function
- Rectangle function has unit amplitude within its finite support
  - endpoint amplitudes are averages of neighboring amplitudes
  - endpoint amplitudes are VERY important, particularly when sampling

$$RECT[x] \equiv \begin{cases} 1 & \text{for} \quad |x| < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad |x| = \frac{1}{2} \\ 0 & \text{for} \quad |x| > \frac{1}{2} \end{cases}$$

- symmetric (even) with respect to the origin of coordinates: $RECT[-x] = RECT\left[\frac{x}{-1}\right] = RECT[x]$
- discontinuous range
- More general form of RECT includes parameters for location of center of symmetry x_0 and real-valued width b:

$$RECT\left[\frac{x-x_0}{b}\right] \equiv \begin{cases} 1 & \text{for} \quad |x-x_0| < \frac{|b|}{2} \\ \frac{1}{2} & \text{for} \quad |x-x_0| = \frac{|b|}{2} \\ 0 & \text{for} \quad |x-x_0| > \frac{|b|}{2} \end{cases}$$



Rectangle function $RECT[x]$

- Rectangle with negative amplitude is $f[x] = -RECT\left[\frac{x}{b}\right]$

1.5 TRIANGLE FUNCTION

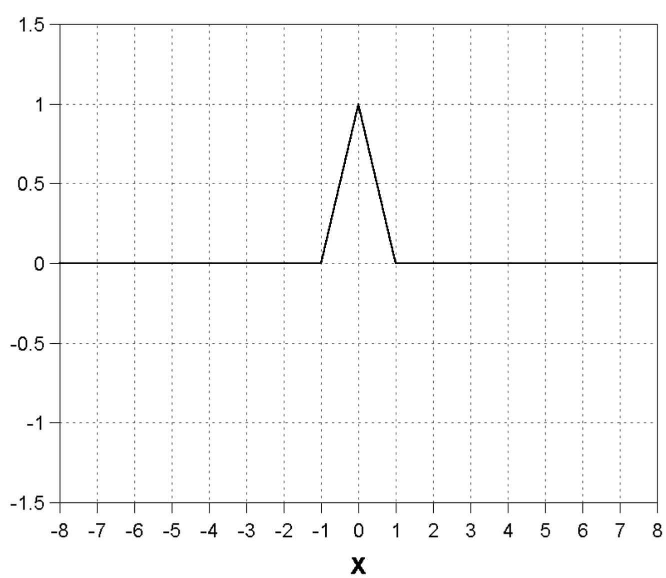
- Unit amplitude at origin
- Unit area
- Amplitudes vary as $\pm x^1$ within region of support

$$
\begin{aligned}
TRI[x] &\equiv \left\{ \begin{array}{ccc} 0 & \text{for} & x \le -1 \\ x+1 & \text{for} & -1 \le x \le 0 \\ 1-x & \text{for} & 0 \le x \le 1 \\ 0 & \text{for} & x \ge 1 \end{array} \right\} \\
&= \left\{ \begin{array}{ccc} 1-|x| & \text{for} & |x| < 1 \\ \\ 0 & \text{for} & |x| \ge 1 \end{array} \right\} \\
&= (1-|x|)\ RECT\left[\frac{x}{2}\right]
\end{aligned}
$$

- support is two units
- slope of sides are $+1$ for $-1 \le x \le 0$ and -1 for $0 \le x \le 1$
- More general expression for triangle centered at $x = x_0$ with real-valued scale parameter b:

$$
\begin{aligned}
TRI\left[\frac{x-x_0}{b}\right] &\equiv \left\{ \begin{array}{ccc} 1-\left|\frac{x}{b}\right| & \text{for} & |x-x_0| < |b| \\ \\ 0 & \text{for} & |x-x_0| \ge |b| \end{array} \right\} \\
&= \left(1-\frac{|x-x_0|}{b}\right)\ RECT\left[\frac{x-x_0}{2b}\right]
\end{aligned}
$$

- Support is $2\ |b|$
- Area is $|b|$
- often used to modulate other functions in imaging situations
  - particularly useful as apodizing (or "multiplicative weighting") function.



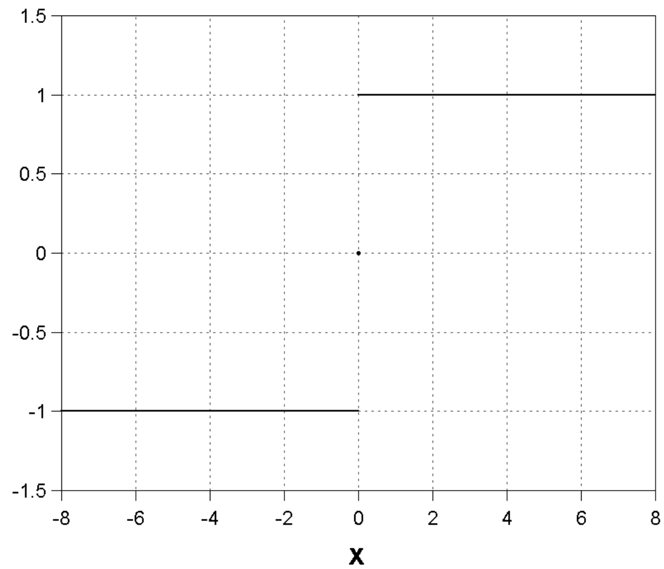
Triangle function $TRI[x]$

1.6 SIGNUM FUNCTION

The "signum" function (pronounced *sig'num)* assigns the numerical values 0 and ± 1 to the dependent variable based on the algebraic sign of the argument. Its name is intended to prevent any confusion of the homonyms "sign" and "sine". Our definition differs slightly from that of some authors:

$$SGN[x] \equiv \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

- Bracewell does not explicitly include null value at $x = 0$



"SIGNUM" function $SGN[x]$

- Location of abscissa where amplitude transition occurs may be translated via additive constant factor into argument:

$$SGN[x - x_0] \equiv \begin{cases} 1 & \text{for} \quad x - x_0 > 0 \Longrightarrow x > x_0 \\ 0 & \text{for} \quad x - x_0 = 0 \Longrightarrow x = x_0 \\ -1 & \text{for} \quad x - x_0 < 0 \Longrightarrow x < x_0 \end{cases}$$

- Scaling of argument of $SGN[x]$ by a real-valued factor b has no effect on amplitude at any location:

$$SGN\left[\frac{x - x_0}{b}\right] \equiv SGN[x - x_0]$$

1.7 STEP FUNCTION

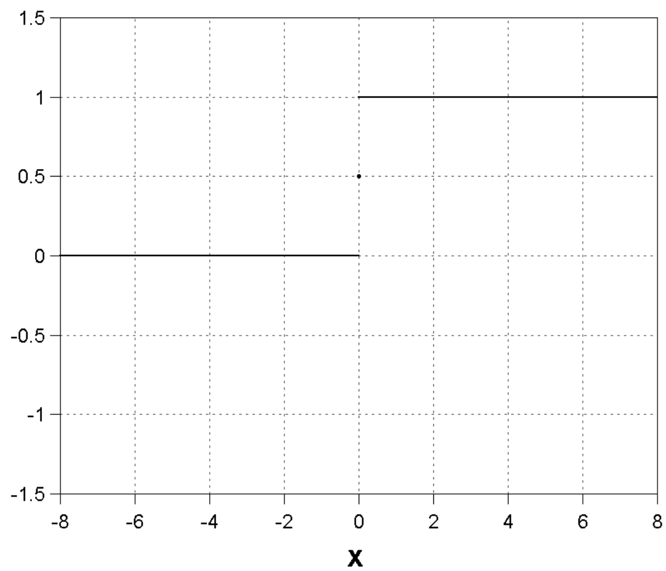
- resembles $SGN[x]$
- amplitude "switches" between two extrema (0 and +1) at origin:

$$STEP[x] \equiv \begin{cases} 1 & \text{for} \quad x > 0 \\ \frac{1}{2} & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

- translated to arbitrary coordinate x_0:

$$STEP[x - x_0] \equiv \begin{cases} 1 & \text{for} \quad x - x_0 > 0 \Longrightarrow x > x_0 \\ \frac{1}{2} & \text{for} \quad x - x_0 = 0 \Longrightarrow x = x_0 \\ 0 & \text{for} \quad x - x_0 < 0 \Longrightarrow x < x_0 \end{cases}$$

- Other authors use other notations.
  - Bracewell ignores amplitude of $STEP[0]$
  - uses $H[x]$ as notation for "Heaviside unit step"



Step function $STEP[x]$

- semi-infinite support
- infinite area
- conveniently expressed in terms of signum function:

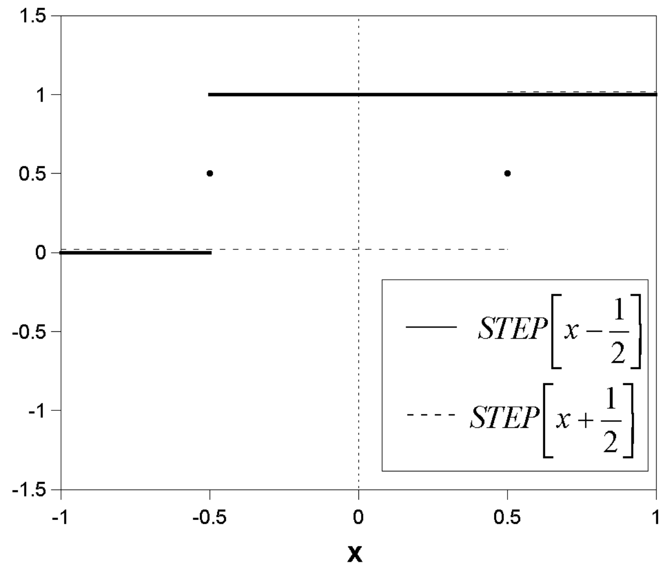
$$STEP[x] = \frac{1}{2}(1 + SGN[x])$$

- scale factor in argument has no effect on form of function

- Sometimes convenient to represent $RECT[x]$ as difference of two $STEP$ functions

$$RECT[x] = STEP\left[x - \left(-\frac{1}{2}\right)\right] - STEP\left[x - \left(+\frac{1}{2}\right)\right] = STEP\left[x + \frac{1}{2}\right] - STEP\left[x - \frac{1}{2}\right]$$

$$RECT\left[\frac{x}{b}\right] = STEP\left[x + \frac{b}{2}\right] - STEP\left[x - \frac{b}{2}\right]$$



$RECT[x]$ constructed from the difference of two translated $STEP$ functions, $RECT[x] = STEP\left[x + \frac{1}{2}\right] - STEP\left[x - \frac{1}{2}\right]$.

1.8 EXPONENTIAL FUNCTION

- Most frequently occuring function in physical and engineering problems (other than sinusoid)

$$f_1[x] = e^{-x}$$

- $f_1[0] = 1$
- $f_1[1] = e^{-1} \simeq 0.3679$
- $f_1[x \to +\infty] = 0$
- $f_1[x \to -\infty] = +\infty$
- area is infinite:

$$\int_{-\infty}^{+\infty} e^{-x} dx = +\infty$$

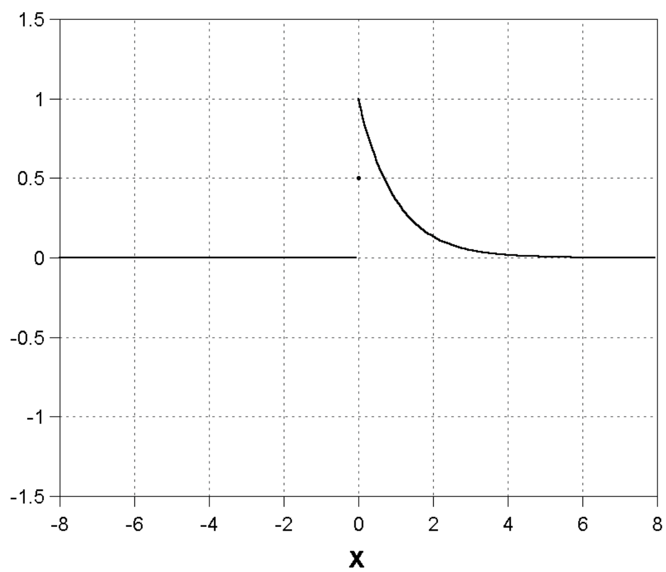
- Multiply by $STEP[x]$:

$$f_2[x] = e^{-x}\ STEP[x]$$

- Unit area

$$\int_{-\infty}^{+\infty} e^{-x}\, STEP[x]\ dx = \int_{0}^{+\infty} e^{-x}\ dx = -\ e^{-x}\Big|_{x=0}^{x=+\infty} = 1$$

- $f_2[0] = \frac{1}{2}$
- useful when modeling dynamic systems that lose energy or amplitude due to losses (e.g., friction)



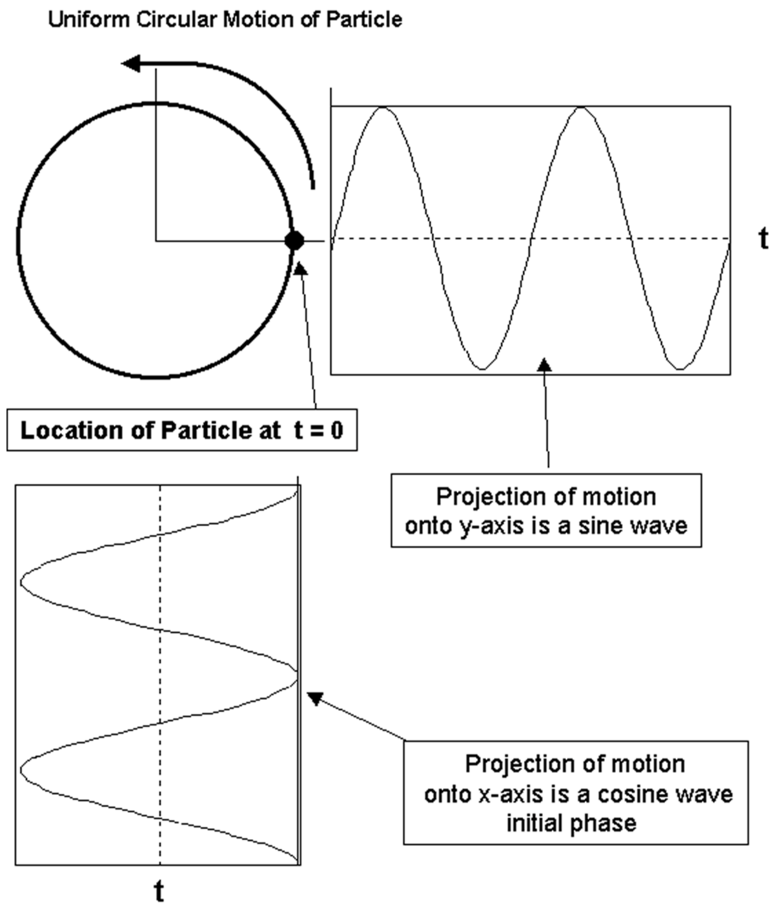
Decaying exponential function $f[x] = e^{-x} STEP[x]$

1.9 SINUSOID

- most pervasive function in physical science
- form derived in any of several ways
  - solution to second-order linear differential equation:

$$\frac{d^2}{dx^2}(f[x]) + \alpha^2 f[x] = 0$$

- sinusoid is projection of endpoint of vector that rotates about origin at uniform rate



Projection of uniform circular motion onto orthogonal axes generates two sinusoidal functions in quadrature.

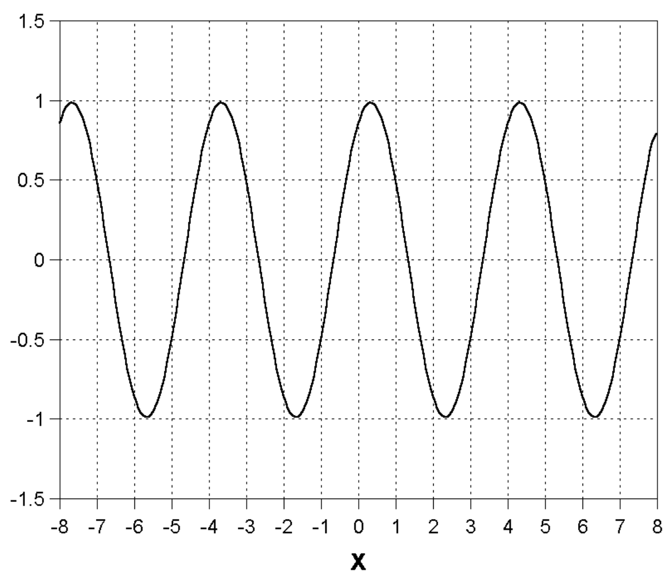
- distance from tip of vector to origin projected onto line perpendicular to distance of closest approach is sinusoidal function of time
- Often more convenient to define form of general sinusoid in terms of symmetric cosine
- General sinusoidal function in the space domain is:

$$A_0 \cos[\Phi[x]] = A_0 \cos\left[\frac{2\pi x}{X_0} + \phi_0\right] = A_0 \cos[2\pi\xi_0 x + \phi_0]$$

- sinusoidal functions may be rewritten in terms of complex exponential function

$$A_0 \cos[2\pi\xi_0 x + \phi_0] = \frac{1}{2}\left(e^{+i(2\pi\xi_0 x+\phi_0)} + e^{-i(2\pi\xi_0 x+\phi_0)}\right)$$

- The proportionality constant is $\frac{2\pi}{X_0}$
  - describes number of radians traversed by argument per unit distance
  - "angular spatial frequency"
  - Reciprocal of period X_0 is "spatial frequency" ξ_0
  - describes number of cycles traversed by sinusoid in unit distance.
- "cycle" describes a single period of any periodic function
  - assigned to sinusoidal functions only
  - analogous terms square waves is "line pairs per mm"
- Other parameter is amplitude A_0 measured in appropriate units
- cosine may be computed for any real-valued argument, including higher-order functions of x
- sinusoid is symmetric cosine when initial phase ϕ_0 is integer multiple of π radians.



Sinusoidal function $\cos[2\pi\xi_0 x + \phi_0]$ with $\xi_0 = \frac{1}{4}$ cycle per unit length and $\phi_0 = -\frac{\pi}{6}$ radians.

$$\xi = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} = \frac{1}{2\pi}\frac{\partial}{\partial x}[2\pi\xi_0 x + \phi_0] = \left(\frac{1}{2\pi}\right)2\pi\xi_0 = \xi_0$$

- Positive and negative lobes of sinusoid have equal areas of opposite sign
- Area of harmonic sinusoid is zero:

$$\int_{-\infty}^{+\infty} \cos[2\pi\xi_0 x + \phi_0]\; dx = 0 \text{ if } \xi_0 \neq 0$$

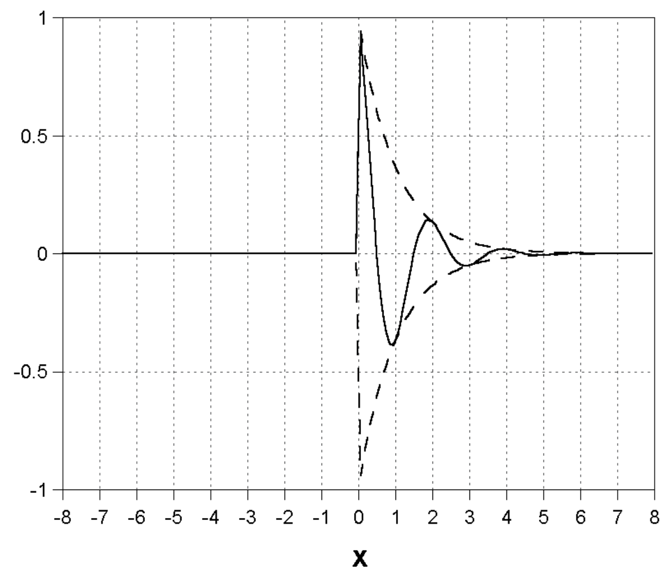
- Applied as modulations (multipliers) of other functions with compact support

– particularly common in electrical engineering to describe temporal signals.

– Example:

$$f[x] = \cos\left[2\pi\xi_0 x - \frac{\pi}{2}\right]\ e^{-x}\ STEP[x] = \sin[2\pi\xi_0 x]\ e^{-x}\ STEP[x]$$

– Odd sinusoid is modulating function because $\sin[0] = 0$, increases "slowly" with increasing x



Sinusoid modulating a decaying exponential: $f[x] = COS[2\pi\xi_0 x]\ \left(e^{-x}\ STEP[x]\right)$

- squared magnitude is *spatial power* or *intensity*
  - guaranteed to be nonnegative
  - recast into sum of constant and sinusoidal parta by squaring the complex-exponential expression for cosine

$$\cos^2[2\pi\xi_0 x] = \left(\frac{e^{2\pi i\xi_0 x} + e^{-2\pi i\xi_0 x}}{2}\right)^2 = \frac{1}{4}\left(2 + e^{4\pi i\xi_0 x} + e^{-4\pi i\xi_0 x}\right)$$
$$= \frac{1}{2} + \frac{1}{2}\cos[4\pi\xi_0 x] = \frac{1}{2}\left(1 + \cos[2\pi(2\xi_0)x]\right)$$

  - $\cos^2[2\pi\xi_0 x]$ is equivalent to sum of half-unit additive constant and cosine function at doubled spatial frequency
  - additive constant is the *bias* that ensures that amplitude is nonnegative.

1.9.1 Modulation of Sinusoid

- Relative sizes of maxima and minima of nonnegative sinusoid give a metric used in optics and image processing

- For $f[x]$ a biased nonnegative sinusoid with maximum and minimum amplitudes are called $f_{\max}$ and $f_{\min}$,

$$m_f \equiv \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}$$

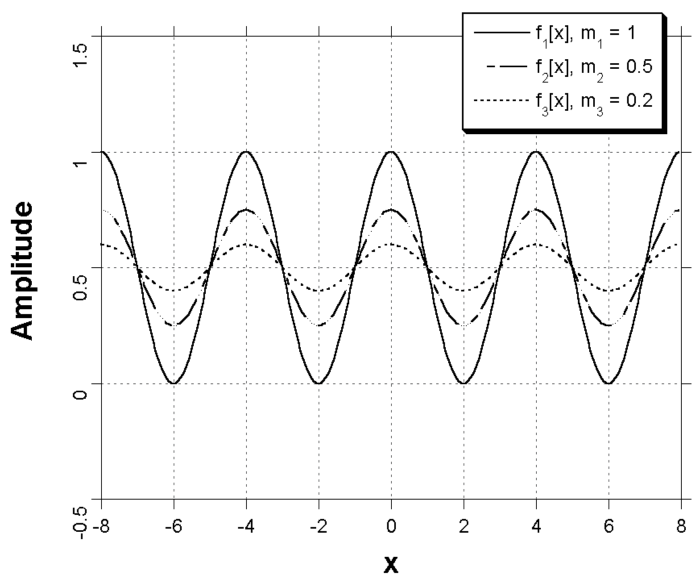
- The range of m_f for nonnegative sinusoids is $0 \leq m_f \leq 1$.

- $m_f = 1$ for all functions with $f_{min} = 0$
- $m_f < 1$ if $f_{min} > 0$
- $m_f = 0$ if $f_{max} = f_{min}$ (amplitude of sinusoid is zero).
- modulation of $\cos^2[2\pi\xi_0 x]$ is unity
- modulation of

$$f[x] = A_0 + A_1 \cos[2\pi\xi_0 x + \phi_0] \quad \text{where } A_0 \geq A_1$$

$$m_f = \frac{(A_0 + A_1) - (A_0 - A_1)}{(A_0 + A_1) + (A_0 - A_1)} = \frac{A_1}{A_0}$$

- modulation of nonnegative sinusoid is ratio of amplitude to bias
- provides measure of relative "brightnesses" of maxima and minima.



Nonnegative sinusoidal functions with modulation factors of 1, 0.5, and 0.2.

- Note that we have introduced two different definitions of "modulation"
  1. multiplication of two functions
  2. quality factor of sinusoid
- Other names for similar factors:
  - Michelson: m_f is "visibility" of sinusoid
  - "contrast" (we use to refer to nonnegative square-wave signals rather than sinusoids)
- *modulation transfer function* (MTF) for sine waves plots m vs. ξ
- *contrast transfer function* (CTF).for square waves

1.10 SINC FUNCTION

- product of odd sinusoid and $(\pi x)^{-1}$, which is not continuous at $x = 0$
- amplitude of $(\pi x)^{-1} \to 0$ as $x \to \pm\infty$:

$$SINC[x] \equiv \frac{\sin[\pi x]}{\pi x} = \frac{\cos\left[\frac{2\pi x}{2} - \frac{\pi}{2}\right]}{\pi x}$$

- Some authors define $SINC[x]$ without factors of π in arguments.
- In our definition, $SINC[x] = 0$ for $x = \pm n$, $(n = 1, 2, 3, \cdots)$.
- Both numerator and denominator are zero at the origin $\Longrightarrow$ the amplitude of $SINC[0]$ determined via L'Hôspital's rule:

$$SINC[0] = \frac{\lim_{x \to 0}\left\{\frac{d}{dx}\sin[\pi x]\right\}}{\lim_{x \to 0}\left\{\frac{d}{dx}[\pi x]\right\}} = \frac{\pi \cos[0]}{\pi} = 1$$

- Between zeros, magnitude increases to local extremum in vicinity of (though not coincident with) half-integer values of x.
- First few local extrema of the magnitudes are approximately:

$$SINC[\pm 1.428] \simeq -0.2172$$

$$SINC[\pm 2.459] \simeq +0.1284$$

$$SINC[\pm 3.471] \simeq -0.09132$$

- Amplitudes at half-integer arguments are:

$$SINC\left[\tfrac{1}{2}\right] = \tfrac{2}{\pi} \simeq 0.6366$$

$$SINC\left[\tfrac{3}{2}\right] = -\tfrac{2}{3\pi} \simeq -0.2122$$

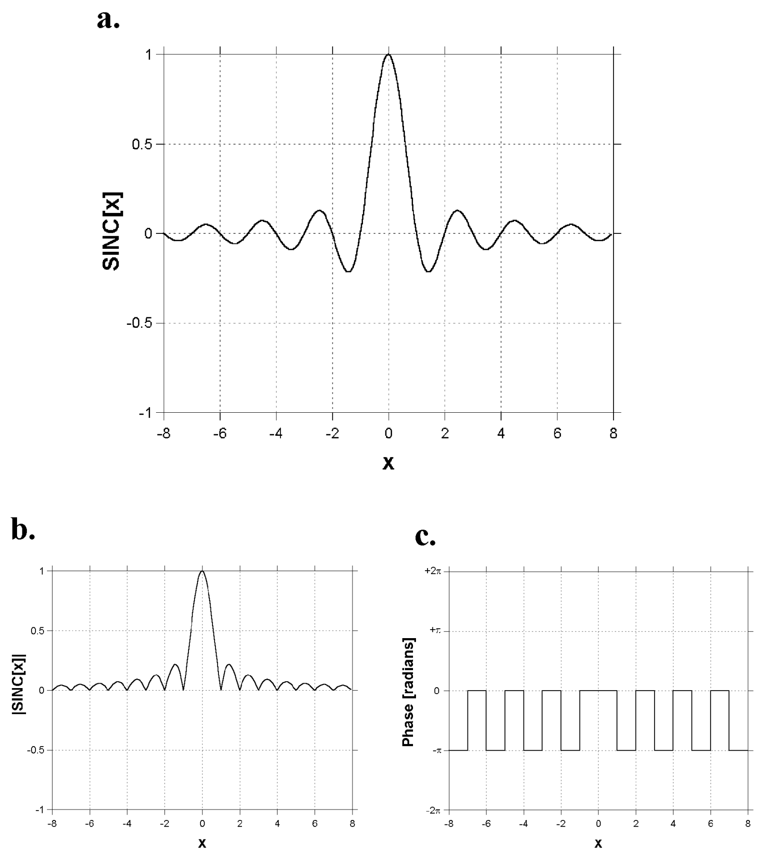
$$SINC\left[\tfrac{5}{2}\right] = +\tfrac{2}{5\pi} \simeq 0.1273$$

$$SINC\left[\tfrac{7}{2}\right] = -\tfrac{2}{7\pi} \simeq -0.09095$$

$$\vdots$$

$$SINC\left[\tfrac{m}{2}\right] = (-1)^m \tfrac{2}{(2m+1)\pi}$$

- Phase of real-valued $SINC$ is 0 or $-\pi$ radians



Representations of $SINC[x]$: (a) real part (imaginary part is $0[x]$); (b) magnitude; (c) phase.

1.10.1 Power-Series Representation of $SINC[x]$:

$$\begin{aligned}
\frac{\sin[\pi x]}{\pi x} &= \frac{1}{\pi x}\left(\frac{(\pi x)^1}{1!} - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots\right) \\
&= \left(1 - \left(\frac{\pi^2}{6}\right)x^2 + \left(\frac{\pi^4}{120}\right)x^4 - \left(\frac{\pi^6}{5040}\right)x^6 + \cdots\right) \\
&= \sum_{n=0}^{+\infty} (-1)^n \frac{(\pi x)^{2n}}{(2n+1)!}
\end{aligned}$$

- demonstrates that $SINC[0]$ must be unity
- Note denominators of sequence increase rapidly with order

1.10.2 Area of $SINC[x]$

- determined rigorously by evaluating appropriate contour integral in the complex plane
- simpler method uses *central-ordinate theorem* of Fourier transform.
  - state without proof that the area of the unscaled SINC function is unity:

$$\int_{-\infty}^{+\infty} SINC[x]\ dx = 1$$

1.10.3 Translated and Scaled $SINC$

$$SINC\left[\frac{x-x_0}{b}\right] = \frac{\sin\left[\pi\left(\frac{x-x_0}{b}\right)\right]}{\left[\pi\left(\frac{x-x_0}{b}\right)\right]}$$

- amplitude vanishes at $x = n\,|b| + x_0$ $(n = \pm 1, \pm 2, \ldots)$
- area is $|b|$ regardless of the translation x_0
- support of $SINC\left[\frac{x}{b}\right]$ is infinite
- magnitude $\left|SINC\left[\frac{x-x_0}{b}\right]\right| < |x|^{-1}$

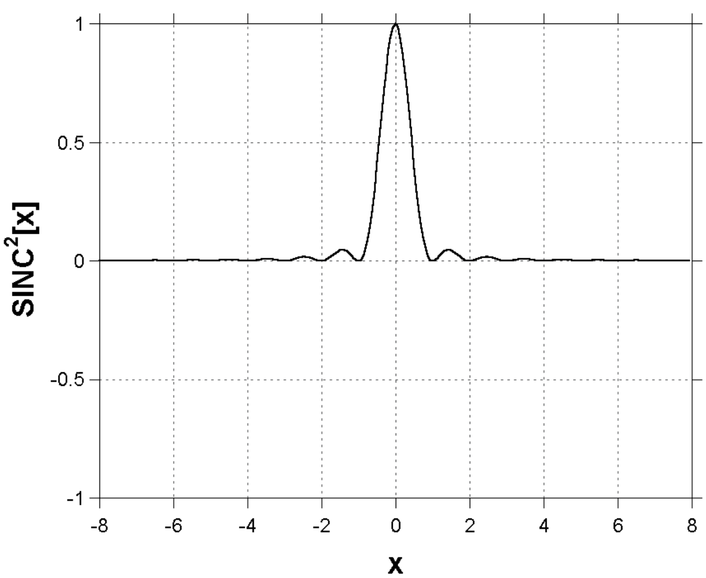
1.11 $SINC^2$ FUNCTION

- Appears in many imaging contexts
  - description of optical imaging systems with rectangular cross-sections used in natural (incoherent) light
- zeros of $SINC^2[x]$ and $SINC[x]$ occur at the same locations ($x = \pm n$ for $n \neq 0$)
- $SINC^2[x]$ varies "smoothly" in vicinity of zeros (novices often visualize cusps in $SINC^2[x]$ similar to those of $|SINC[x]|$)
- $SINC^2[x] \geq 0$ everywhere $\Longrightarrow$ phase = 0 radians for all x.
- area of $SINC^2[x]$ evaluated via appropriate integral in the complex plane or by central-ordinate theorem

$$\int_{-\infty}^{+\infty} SINC^2[x]\ dx = \int_{-\infty}^{+\infty} SINC[x]\ dx = 1$$

- Areas of $SINC^2\left[\frac{x}{b}\right]$ and $SINC\left[\frac{x}{b}\right]$ are identical:

$$\int_{-\infty}^{+\infty} SINC^2\left[\frac{x}{b}\right]\ dx = \int_{-\infty}^{+\infty} SINC\left[\frac{x}{b}\right]\ dx = |b|$$



$SINC^2[x]$

1.12 GAMMA FUNCTION $\Gamma[x]$

- Different "animal"
- $\Gamma[x]$ does not represent useful signal or descriptive function of useful imaging system
- Computational tool for solving imaging problems that involve other special functions
  - avenue for deriving the areas and Fourier transforms of quadratic-phase sinusoid and Gaussian functions
- Gamma function evaluated at positive argument x_0 is area of decaying exponential $e^{-\alpha}\, STEP[\alpha]$ modulated by α^{x_0-1}:

$$\Gamma[x] = \int_0^{+\infty} e^{-\alpha}\ \alpha^{x-1}\ d\alpha = \int_{-\infty}^{+\infty} STEP[\alpha]\ e^{-\alpha}\ \alpha^{x-1}\ d\alpha, \quad x > 0$$

- Argument x appears only in exponential term α^{x-1}.
  - $\Gamma[1]$ is area of $STEP[\alpha]\ e^{-\alpha}$, which was shown to be unity

$$\Gamma[1] = 1.$$

  - For a fixed positive finite value of x, $e^{-\alpha}$ decreases with increasing α while α^{x-1} increases rapidly
  - Combined conflicting behaviors ensure that area remains finite for finite and positive values of x
  - Area of the product function increases rapidly with x for $x > 1$.
- Changing variable of integration from α to $e^{-\alpha}$

$$d\left(e^{-\alpha}\right) = -e^{-\alpha}\ d\alpha$$

$$\Gamma[x] = \int_0^{+\infty} \left(-\alpha^{x-1}\right)\ d\left(e^{-\alpha}\right)$$

$$\begin{aligned} u &= \alpha^{x-1} \\ du &= (x-1)\ \alpha^{x-2}\ d\alpha \\ v &= -e^{-\alpha} \\ dv &= +e^{-\alpha}\ d\alpha \end{aligned}$$

  - Substitute into integration-by-parts formula to obtain a recursion relation for $\Gamma[x]$ for positive values of x:

$$\begin{aligned} \Gamma[x] &= \int_0^{+\infty} u\, dv = uv|_{u=0}^{u=+\infty} - \int_{u=0}^{u=+\infty} v\ du \\ &= \alpha^{x-1} e^{-\alpha}|_{\alpha=0}^{\alpha=+\infty} + (x-1) \int_0^{+\infty} \alpha^{x-2}\ e^{-\alpha}\ d\alpha = 0 + (x-1)\ \Gamma[x-1] \\ \Gamma[x] &= (x-1)\ \Gamma[x-1] \end{aligned}$$

  - Repeat to demonstrate:

$$\begin{aligned} \Gamma[x] &= (x-1)\ \Gamma[x-1] \\ &= (x-1)\left((x-2)\ \Gamma[x-2]\right) \\ &= (x-1)(x-2)(x-3)\cdots(x - INT[x])\ \Gamma[x - INT[x]] \end{aligned}$$

  where $INT[x]$ is integer part of positive real number x

– Relationship valid because $x - INT[x] > 0$.

– If $x = n$, simplifies to:

$$\Gamma[n] = (n-1)(n-2)(n-3)\cdots(1)\ \Gamma[1] = (n-1)!$$

– *Gamma function = factorial function*

$$\Gamma[1] = \Gamma[2] = 1$$

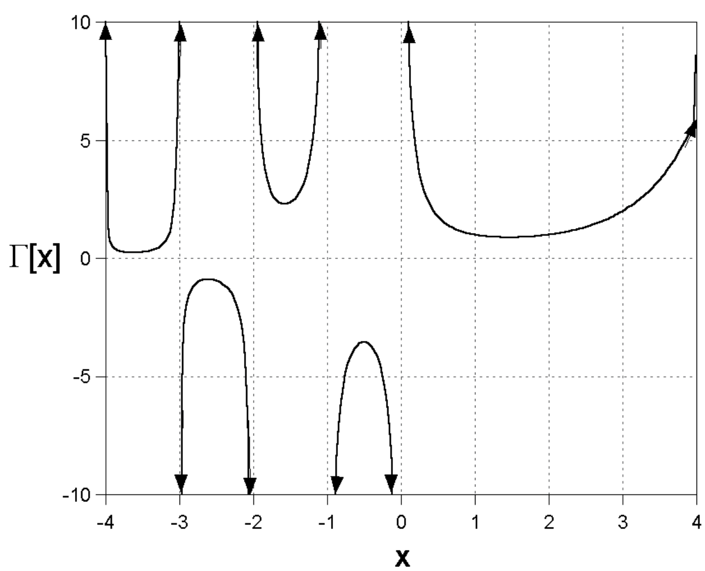
$$0! = 1$$

– Generalize definition of "factorial"

$$x! \equiv \Gamma[x+1] = \int_0^{+\infty} \alpha^x\ e^{-\alpha}\ d\alpha$$

- Extend gamma function to negative arguments

$$\Gamma\left[-\frac{1}{2}\right] = -2\ \Gamma\left[+\frac{1}{2}\right]$$



Graphical representation of $\Gamma[x]$, showing that the amplitude is undefined at all integer values of $x \leq 0$.

- $\Gamma[x]$ has no zeros $\Longrightarrow (\Gamma[x])^{-1}$ has no singularities

  – expressed as Taylor series valid for all x

$$(\Gamma[x])^{-1} = 1 + 0.57721\ x - 0.65587\ x^2 - 0.04200\ x^3 + 0.16654\ x^4 + \cdots$$

- Other expansions converge more efficiently to correct value

1.12.1 Gamma function for half-integer arguments:

- $\left(x = \frac{1}{2}, \frac{3}{2}, \text{etc.}\right)$ are obtained by applying recursion relation to $\Gamma\left[\frac{1}{2}\right]$
  - Evaluate $\Gamma\left[\frac{1}{2}\right]$ by recasting into form of easily evaluated "error function"

$$\Gamma\left[\frac{1}{2}\right] = \int_0^{+\infty} \alpha^{\left(\frac{1}{2}-1\right)}\, e^{-\alpha}\, d\alpha = \int_0^{+\infty} \alpha^{-\frac{1}{2}}\, e^{-\alpha}\, d\alpha$$

1. Change the variable of integration to $\beta = \sqrt{\alpha}$ to obtain an integral that is defined as I:

$$\begin{aligned}\Gamma\left[\frac{1}{2}\right] &= \int_{\beta=0}^{\beta=+\infty} \beta^{-1}\, e^{-\beta^2}\, 2\beta\, d\beta = 2\int_{\beta=0}^{\beta=+\infty} e^{-\beta^2}\, d\beta \\ &= \int_{\beta=-\infty}^{\beta=+\infty} e^{-\beta^2} d\beta \equiv I\end{aligned}$$

2. Evaluate I by constructing square as product of two integrals with independent variables, convert to polar coordinates:

$$\begin{aligned}I^2 &= \left(\int_{\beta=-\infty}^{\beta=+\infty} e^{-\beta^2} d\beta\right)\left(\int_{\gamma=-\infty}^{\gamma=+\infty} e^{-\gamma^2} d\gamma\right) \\ &= \int_{\beta=-\infty}^{\beta=+\infty}\int_{\gamma=-\infty}^{\gamma=+\infty} e^{-\left(\beta^2+\gamma^2\right)} d\beta\, d\gamma \\ &= \int_{\theta=-\pi}^{\theta=+\pi}\int_{\rho=0}^{\rho=+\infty} e^{-\rho^2}\rho\, d\rho\, d\theta \\ &= 2\pi\int_{\rho=0}^{\rho=+\infty} e^{-\rho^2}\, \rho\, d\rho\end{aligned}$$

3. Change integration variable again to $u = e^{-\rho^2}$

$$du = -2\rho\, e^{-\rho^2}\, d\rho$$

$$I^2 = 2\pi\int_{u=1}^{u=0}\left(-\frac{1}{2}\right)\, du = 2\pi\left(\frac{1}{2}\right) = \pi$$

Required result is:

$$\Gamma\left[\frac{1}{2}\right] = \sqrt{I^2} = \sqrt{\pi} \simeq 1.7725$$

Apply recursion relation:

$$\begin{aligned}\Gamma\left[\frac{5}{2}\right] &= \frac{3}{2}\,\Gamma\left[\frac{3}{2}\right] = \frac{3\sqrt{\pi}}{4} \simeq 1.3293 \\ \Gamma\left[\frac{3}{2}\right] &= \frac{1}{2}\,\Gamma\left[\frac{1}{2}\right] = \frac{\sqrt{\pi}}{2} \simeq 0.8862 \\ \Gamma\left[-\frac{1}{2}\right] &= -2\,\Gamma\left[\frac{1}{2}\right] = -2\sqrt{\pi} \simeq -3.5449 \\ \Gamma\left[-\frac{3}{2}\right] &= -\frac{2}{3}\,\Gamma\left[-\frac{1}{2}\right] = -\frac{4}{3}\sqrt{\pi} \simeq +2.3633\end{aligned}$$

Note growth in $\Gamma[x]$ for increasing values of $x > 1$.

1.12.2 Gamma Function for "Reciprocal Integer" Arguments

- $\left(x = \frac{1}{n}, \text{ where } n = 1, 2, 3, \cdots\right)$evaluated via series expansion
- First 5 examples are:

$$\Gamma\left[\frac{1}{1}\right] = 0! = 1$$
$$\Gamma\left[\frac{1}{2}\right] = \sqrt{\pi} \simeq 1.7725$$
$$\Gamma\left[\frac{1}{3}\right] \simeq 2.6789$$
$$\Gamma\left[\frac{1}{4}\right] \simeq 3.6256$$
$$\Gamma\left[\frac{1}{5}\right] \simeq 4.5908$$

$\Gamma\left[\frac{1}{n}\right]$ increases as n increases $(\frac{1}{n} \to 0_+)$

- $\Gamma\left[\frac{1}{n}\right]$ in limit of large n evaluated from recursion relation:

$$\lim_{n\to\infty}\left\{\frac{1}{n}\,\Gamma\left[\frac{1}{n}\right]\right\} = \lim_{n\to\infty}\left\{\Gamma\left[1+\frac{1}{n}\right]\right\} \simeq \Gamma[1] = 1 \Longrightarrow \lim_{n\to\infty}\left\{\Gamma\left[\frac{1}{n}\right]\right\} = n$$

1.13 QUADRATIC-PHASE SINUSOID – "CHIRP" FUNCTION

- Phase is "dimensionless" or "unitless" (measured in radians) ensures that argument x^2 includes factors with aggregate dimensions of length^{-2}.

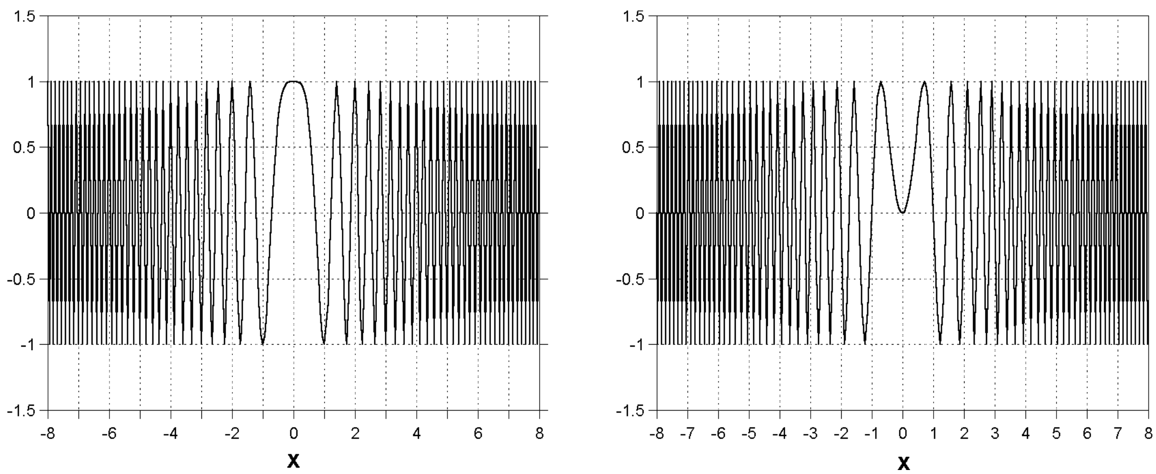
$$f[x] = A\cos[\Phi[x]] = A\ \cos\left[\frac{\pi x^2}{\alpha^2} + \phi_0\right]$$

  - scale parameter α has dimensions of length
  - α specifies "closest" coordinates ($x = \pm a$) where phase differs from ϕ_0 (at origin) by π radians.

- Amplitude symmetric with respect to origin regardless of ϕ_0
  - only term involving x in argument of sinusoid appears as x^2 $\Longrightarrow$symmetric ("even")

- center of symmetry may be translated by adding a constant to the argument:

$$f[x] = A\ \cos\left[\pi\frac{(x - x_0)^2}{\alpha^2} + \phi_0\right]$$

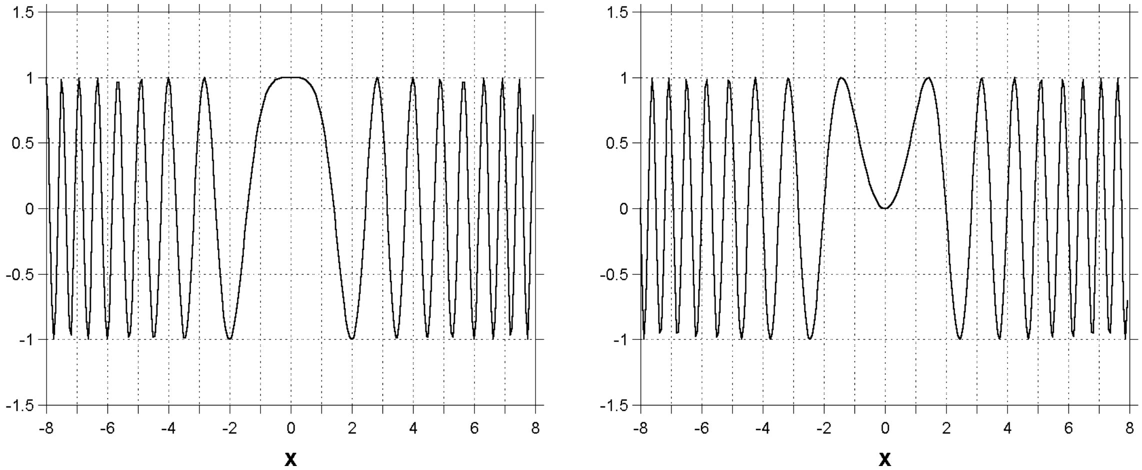


Quadratic-phase sinusoidal functions: (a) $\cos\left[\pi x^2\right]$, (b) $\cos\left[\pi x^2 - \frac{\pi}{2}\right] = \sin\left[\pi x^2\right]$

- spatial frequency is:

$$\xi[x] = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} = \frac{1}{2\pi}\left[\frac{2\pi x}{\alpha^2}\right] = +\frac{x}{\alpha^2} \Longrightarrow \xi[x<0]<0 \text{ and } \xi[x>0]>0$$

- spatial frequency ξ depends on x, $\xi[x]$ is the *instantaneous spatial frequency* of quadratic-phase sinusoid
- linear dependence of ξ on x leads to names "linear frequency modulation" or "linear FM signal"
- "chirp"
  - α is "chirp rate", smaller α $\Longrightarrow$more rapid change in spatial frequency



Effect of scale factor on quadratic-phase sinusoids: (a) $\cos\left[\pi\left(\frac{x}{2}\right)^2\right]$, (b) $\sin\left[\pi\left(\frac{x}{2}\right)^2\right]$

- Areas of chirp functions are not zero
- Area of linear-phase sinusoid is zero precisely because areas of adjacent positive and negative lobes cancel
  - phase is linear function of coordinate
  - rate of change of phase of linear-phase sinusoid is constant
  - Phase of chirp function (or of any sinusoidal function whose phase is a nonlinear function of x) changes with x at a variable rate, adjacent positive and negative lobes have different "widths" and thus different (and noncancelling) areas.

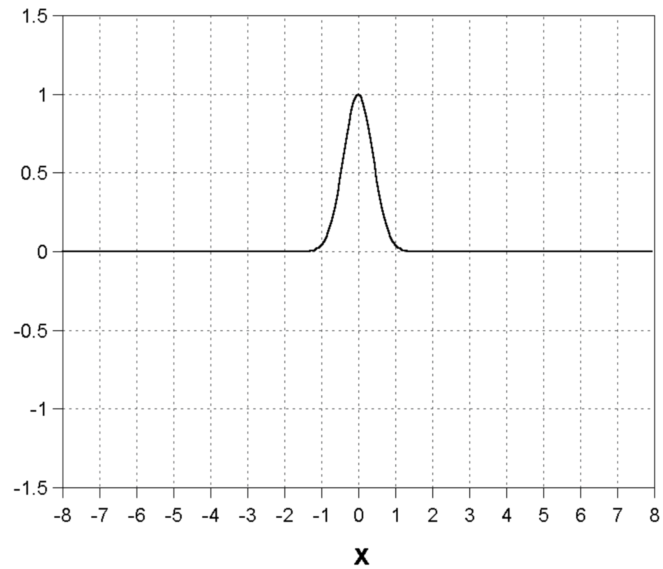
$$\int_{-\infty}^{+\infty} \cos\left[\pi x^2\right]\ dx = \int_{-\infty}^{+\infty} \sin\left[\pi x^2\right]\ dx = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \simeq 0.7071$$

1.14 GAUSSIAN FUNCTION

- familiar "bell curve" of probability theory
- appears in many imaging contexts.

$$GAUS[x] = e^{-\pi x^2}$$

- scale factor of π not used by some authors
  - Peak amplitude is unity (at origin)
  - Decays smoothly as $|x|$ increases, decreases to $e^{-\pi} \simeq 0.043$ at $x = \pm 1$
  - Approaches zero as $|x| \to \infty$.
- Infinite support.



Gaussian function $e^{-\pi x^2}$

- Area is unity

$$\int_{-\infty}^{+\infty} GAUS[x]\ dx = \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$$

- More general form has area $= |b|$

$$GAUS\left[\frac{x - x_0}{b}\right] = e^{-\pi \frac{(x-x_0)^2}{b^2}}$$

- Relate to Gaussian Distribution in Probability
- Random variable n is *normally distributed* with mean $\langle n \rangle$ and standard deviation σ (variance $=$ σ^2) probability density function $p[n]$ is:

$$p[n] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(n-\langle n \rangle)^2}{2\sigma^2}\right)}$$

$$b = \left(\sqrt{2\pi}\right)\sigma \simeq 2.5\sigma \to \sigma \simeq \frac{b}{2.5} = 0.4b$$

$$b^2 = 2\pi\sigma^2 \simeq 6.28\sigma^2 \Longrightarrow \sigma^2 \simeq \frac{b^2}{6.28} \simeq 0.16b^2$$

1.15 "SUPERGAUSSIAN" FUNCTION

- Same form as Gaussian but with integer exponents other than 2:

$$GAUS\,[x;n] \equiv e^{-\pi|x|^n}$$

- parameter n is a positive integer
- Absolute value of coordinate ensures that $GAUS\,[x;n]$ remains finite and even for negative x and odd values of n
- SuperGaussian for $n = 1$ is sum of decaying exponential $e^{-\pi x}\ STEP\,[x]$ and "reversed" replica $e^{-\pi|x|}\ STEP\,[-x]$.
- Amplitude remains near 1 near origin for larger n In the limit $n \to \infty$, the amplitudes of the function in the various regions are:

$$\text{for } |x| < 1: \quad \lim_{n\to 0} \{|x|^n\} = 0 \Longrightarrow \lim_{n\to\infty} \left\{e^{-\pi|x|^n}\right\} = 1$$

$$\text{for } |x| > 1: \quad \lim_{n\to 0} \{|x|^n\} = +\infty \Longrightarrow \lim_{n\to\infty} \left\{e^{-\pi|x|^n}\right\} = 0$$

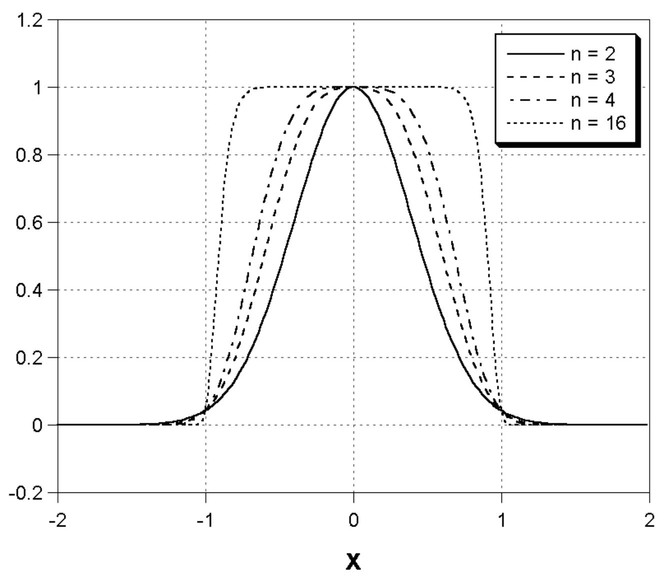
$$\text{for } |x| = 1: \quad \lim_{n\to 0} \{|x|^n\} = 1 \Longrightarrow \lim_{n\to\infty} \left\{e^{-\pi|1|^n}\right\} = e^{-\pi} \simeq 0.04321$$

- Resembles rectangle function with width $b = 2$ for large values of n:

$$\lim_{n\ \to\infty} \{GAUS\,[x,n]\} \simeq RECT\left[\frac{x}{2}\right]$$

- "Endpoint" amplitudes not identical, but isolated values have no effect on any integrals of superGaussian.

$$RECT\,[x] \simeq \lim_{n\ \to\infty} \{GAUS\,[2x,n]\} = \lim_{n\ \to\infty} \left\{e^{-\pi|2x|^n}\right\}$$



Supergaussian functions $f\,[x] = e^{-\pi x^n}$ for $n = 2$, 3, 4, and 16.

1.15.1 Area of superGaussian

- from gamma function

$$\Gamma[x] = \int_0^{+\infty} \alpha^{x-1} \; e^{-\alpha} \; d\alpha$$

- Change variable of integration from α to πu^n:

$$\begin{aligned}\Gamma[x] &= \int_0^{+\infty} (\pi u^n)^{x-1} \;\; e^{-\pi u^n} \;\; d(\pi u^n) \\ &= \int_0^{+\infty} \pi^{x-1} \; u^{nx-n} \; e^{-\pi u^n} \; n\pi \; u^{n-1} \; du \\ &= \int_0^{+\infty} n\pi^x \; u^{nx-n+n-1} \; e^{-\pi u^n} \; du\end{aligned}$$

- Divide both sides of this equation by n:

$$\frac{1}{n} \; \pi^{-x} \; \Gamma[x] = \int_0^{+\infty} u^{nx-1} \; e^{-\pi u^n} \; du$$

- Setting $x = n^{-1}$ to ensure that $u^{nx-1} = u^0 = 1$

$$\int_0^{+\infty} e^{-\pi u^n} \; du = \frac{1}{n} \; \pi^{-n^{-1}} \; \Gamma\left[\frac{1}{n}\right]$$

- Area of symmetric function $e^{-\pi|x|^n}$ is twice:

$$\int_{-\infty}^{+\infty} e^{-\pi|x|^n} dx = \frac{2}{n} \; \left(\pi^{-\frac{1}{n}}\right) \; \Gamma\left[\frac{1}{n}\right] = \frac{2}{\pi^{\frac{1}{n}}} \; \Gamma\left[\frac{n+1}{n}\right]$$

- Area of the superGaussian is proportional to gamma function with a reciprocal-integer argument.

- $n = 1 - 4$:

$$\begin{aligned}n = 1 : \int_{-\infty}^{+\infty} e^{-\pi|x|} \; dx &= 2 \; \pi^{-1} \; \Gamma[1] = \left(\frac{2}{\pi}\right) 0! = \frac{2}{\pi} \simeq 0.6366 \\ n = 2 : \int_{-\infty}^{+\infty} e^{-\pi|x|^2} \; dx &= \frac{2}{2} \; \frac{1}{\sqrt{\pi}} \; \Gamma\left[\frac{1}{2}\right] = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \\ n = 3 : \int_{-\infty}^{+\infty} e^{-\pi|x|^3} \; dx &= \frac{2}{3} \; \left(\frac{1}{\pi^{\frac{1}{3}}}\right) \; \Gamma\left[\frac{1}{3}\right] = \frac{2}{\pi^{\frac{1}{3}}} \; \Gamma\left[\frac{4}{3}\right] \simeq 1.2194 \\ n = 4 : \int_{-\infty}^{+\infty} e^{-\pi|x|^4} \; dx &= \frac{2}{4} \; \frac{1}{\pi^{\frac{1}{4}}} \; \Gamma\left[\frac{1}{4}\right] = \frac{2}{\pi^{\frac{1}{4}}} \; \Gamma\left[\frac{5}{4}\right] \simeq 1.3616\end{aligned}$$

- $n = 2$ confirms that area of "normal" Gaussian function is unity
- Area increases with n
- Limiting value of the area as $n \to +\infty$:

$$\lim_{n\to\infty} \left\{\int_{-\infty}^{+\infty} e^{-\pi|x|^n} dx\right\} = \lim_{n\to\infty} \left\{\frac{2}{\pi^{\frac{1}{n}}} \; \frac{1}{n} \; \Gamma\left[\frac{1}{n}\right]\right\} = \left(\frac{2}{\pi^0}\right) 1 = 2$$

1.16 BESSEL FUNCTIONS $J_n[x]$

- Linear-phase sinusoid are solutions of second-order linear differential equation
- Other useful functions are solutions of other 1-D linear differential equations:

$$x^2 \frac{d^2}{dx^2}(Z_\nu[x]) + x\frac{d}{dx}(Z_\nu[x]) + \left(x^2 - \nu^2\right) Z_\nu[x] = 0,\ x \geq 0,\ \nu \in \Re$$

- Solutions $Z_\nu[x]$ are *Bessel functions*
- Often appears in physical problems involving planar circular symmetry or cylindrical coordinates
  - descriptions of imaging systems constructed from optics with circular cross-sections
- Three independent types of solutions are recognized.
  1. The "Bessel functions of the first kind" with integer and half-integer order ($\nu = n$ or $\frac{n}{2}$ for $n = 0, 1, 2, 3, \cdots$)
     - most relevant in imaging
     - labeled $J_n[x]$
     - finite amplitude for positive x.
  2. "Bessel functions of second kind"
     - also called "Neumann functions"
     - denoted by $N_\nu[x]$
     - indeterminate amplitude at $x = 0$.
  3. "Hankel function"
     - (not to be confused with the "Hankel transform" that will be discussed later)
     - complex-valued linear combination of first two types using scheme analogous to Euler relation:
     - $H_\nu[x] = J_\nu[x] \pm i\ N_\nu[x]$
- Numerical values for the Bessel functions via:
  - "generating function"
  - contour integrals
  - series solution of differential equation for integer indices ($\nu = n$).

1.16.1 Series Solution:

- 1. Assume that $J_n[x]$ has form of power series in x with unknown coefficients:

$$J_n[x] = \sum_{\ell=0}^{+\infty} a_\ell\ x^\ell$$

  2. Insert into differential equation
  3. Evaluate derivatives
  4. Equate terms of same power of x
  5. Juggle terms
  6. Gives two series solutions for $J_n[x]$, one each for $n > 0$ and $n < 0$

7. Most interested in $J_0[x]$ and $J_1[x]$ $\Longrightarrow$series for $n > 0$ considered here

$$J_0[x] = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$
$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{147,456} + \cdots$$

- Compare coefficients to those of $SINC[x]$

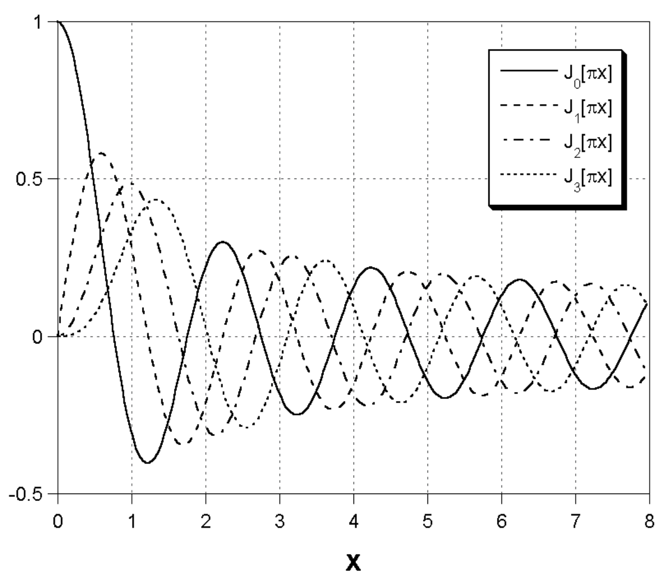
$$\begin{aligned} SINC[x] &= 1 - \frac{(\pi x)^2}{3!} + \frac{(\pi x)^4}{5!} - \frac{(\pi x)^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{\left(\frac{6}{\pi^2}\right)} + \frac{x^4}{\left(\frac{120}{\pi^4}\right)} - \frac{x^6}{\left(\frac{5040}{\pi^6}\right)} + \cdots \\ &= 1 - \frac{x^2}{0.6079} + \frac{x^4}{1.2319} - \frac{x^6}{5.242} + \cdots \end{aligned}$$

  - Coefficients of same order have same algebraic sign
  - Absolute values of coefficients of $J_0[x]$ decrease more quickly with order than $SINC[x]$
  - Extrema of local amplitude of $J_0[x]$ fall off more slowly.

- Amplitude and slope of J_0 at $x = 0$ are unity and zero, respectively
- resembles cosine oscillation modulated by decaying function that happens to be $x^{-\frac{1}{2}}$ instead of x^{-1} in $SINC[x]$.

$$\begin{aligned} \lim_{x \to +\infty} \{J_0[x]\} &= \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{\pi}{4}\right] \\ &= x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \cos\left[2\pi\left(\frac{1}{2\pi}\right)x - \frac{\pi}{4}\right] \end{aligned}$$

- Period of asymptotic form is $X_0 = 2\pi$



The Bessel function of the first kind $J_n[\pi x]$ for $n = 0 - 3$.

- zeros are only approximately uniformly spaced

- three zeros nearest to origin occur at:

Location of zero of $J_0[x]$	Difference Δx
$x_1 \simeq 2.4048 \simeq 0.7655\ \pi$	
	$\}\, x_2 - x_1 \simeq 0.9916\ \pi$
$x_2 \simeq 5.5201 \simeq 1.7571\ \pi$	
	$\}\, x_3 - x_2 \simeq 0.9974\ \pi$
$x_3 \simeq 8.6537 \simeq 2.7546\ \pi$	
$\vdots$	$\vdots$
x_{M-1}	
	$\}\, x_M - x_{M-1} \lesssim 1.0\ \pi$
x_M	

- Intervals between first and second pair of adjacent zeros are approximately 0.9916π and 0.9974π, respectively
- Interval between x_2 and x_3 is larger than that between x_1 and x_2.
- Incremental distance between successive pairs of zeros asymptotically approaches π as $x \to +\infty$.

Series Solution for $J_1[x]$

- $$\begin{aligned} J_1[x] &= \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots \\ &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18,432} + \cdots = \frac{x}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9,216} + \cdots\right) \end{aligned}$$
- Amplitude and slope at origin are equal to coefficients $\alpha_0 = 0$ and $\alpha_1 = \frac{1}{2}$ respectively
- As x increases from zero, amplitude increases from zero and reaches local maximum at $x \simeq 1.8412 \simeq 0.586\pi$.
- Absolute values of the numerical coefficients of $J_1[x]$ decrease more slowly than those of $SINC[x]$
  - absolute values of successive local maxima of $J_1[x]$ decrease more slowly than those of $SINC[x]$ as $x \to \infty$
  - same behavior exhibited by $J_0[x]$.
- First three zeros of $J_1[x]$:

Location of zero of $J_1[x]$	Difference Δx
$x_1 \simeq 3.8317 \simeq 1.219\ \pi$	
	$\}\, x_2 - x_1 \simeq 1.0135\ \pi$
$x_2 \simeq 7.0156 \simeq 2.2331\ \pi$	
	$\}\, x_3 - x_2 \simeq 1.0052\ \pi$
$x_3 \simeq 10.1735 \simeq 3.2383\ \pi$	
$\vdots$	$\vdots$
x_{M-1}	
	$\}\, x_M - x_{M-1} \gtrsim 1.0\ \pi$
x_M	

- Intervals between first pairs of zeros are approximately $1.0135\,\pi$ and $1.0052\,\pi$
- Interval between adjacent zeros of $J_1[x]$ *decreases* for increasing x.
  - Complementary to $J_0[x]$
- Interval between zeros for both $J_0[x]$ and $J_1[x]$ asymptotically approach π as $|x| \to \infty$, though from different directions.
- Asymptotic behavior of $J_1[x]$ is *in quadrature* to $J_0[x]$
  - phase difference of oscillations of two functions is $-\frac{\pi}{2}$ radians:

$$\begin{aligned}\lim_{x\to+\infty}\{J_1[x]\} &= \sqrt{\frac{2}{\pi x}}\;\cos\left[x-\frac{3\pi}{4}\right]\\ &= \sqrt{\frac{2}{\pi x}}\;\cos\left[\left(x-\frac{\pi}{4}\right)-\frac{\pi}{2}\right]=\sqrt{\frac{2}{\pi x}}\;\sin\left[x-\frac{\pi}{4}\right]\end{aligned}$$

1.16.2 General Series Solution for $J_n[x]$:

$$J_n[x]=\sum_{\ell=0}^{+\infty}\frac{(-1)^\ell}{\ell!\,(n+\ell)!}\left(\frac{x}{2}\right)^{n+2\ell}$$

$$\begin{aligned}\lim_{x\to+\infty}\{J_2[x]\} &= \sqrt{\frac{2}{\pi x}}\;\cos\left[x-\frac{5\pi}{4}\right]\\ \lim_{x\to+\infty}\{J_3[x]\} &= \sqrt{\frac{2}{\pi x}}\;\cos\left[x-\frac{7\pi}{4}\right]\end{aligned}$$

1.17 LORENTZIAN FUNCTION

- Named after Dutch physicist Hendrik Lorentz
  - demonstrated significance of curve in study of atomic radiation in early 1900s
- Lorentzian curve is theoretical shape of spectral lines created by atomic absorption or emission
- Our definition:

$$LOR[x] = \frac{2}{1+(2\pi x)^2} = \frac{2}{1+4\pi^2 x^2}$$

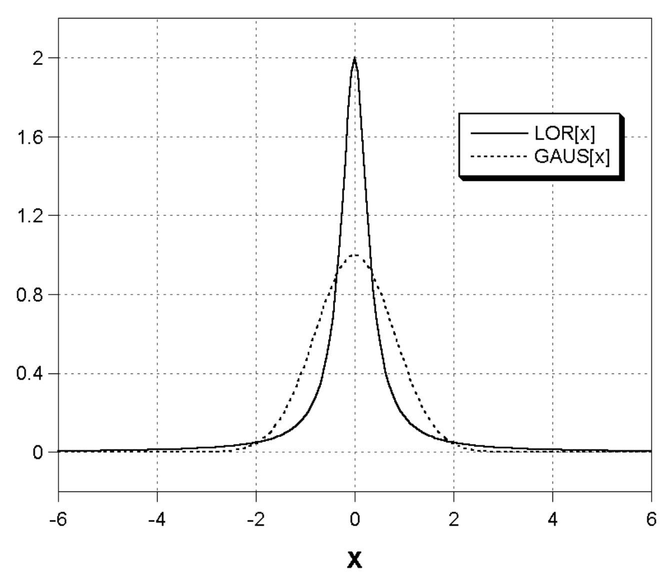
- Amplitude proportional to reciprocal of sum of unit constant and quadratic function of coordinate.
- Quadratic dependence on x ensures that $LOR[x]$ is even.
- Multiplicative factor of 2 ensures that $LOR[x]$ has unit area:

$$Set\ u \equiv 2\pi x,\ dx = \frac{1}{2\pi}du$$

$$\int_{-\infty}^{+\infty} \frac{2}{1+(2\pi x)^2}dx = \int_{-\infty}^{+\infty} \frac{2}{1+u^2}\frac{du}{2\pi} = \frac{1}{\pi}\tan^{-1}[u]\Big|_{u=-\infty}^{u=+\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

- Amplitude decays from $LOR[0] = 2$ at origin through $LOR[\pm 1] \simeq 0.0494$ and on to zero at $x = \pm\infty$.
- Note similarity in amplitude of Lorentzian and Gaussian evaluated at $x = 1$
  - $GAUS[1] = e^{-\pi} \simeq 0.0432$
  - $LOR[\pm 1] \simeq 0.0494$
- $LOR[x]$ decays to zero more rapidly than $GAUS[x]$ for $|x| < 1$, and more slowly for $|x| > 1$

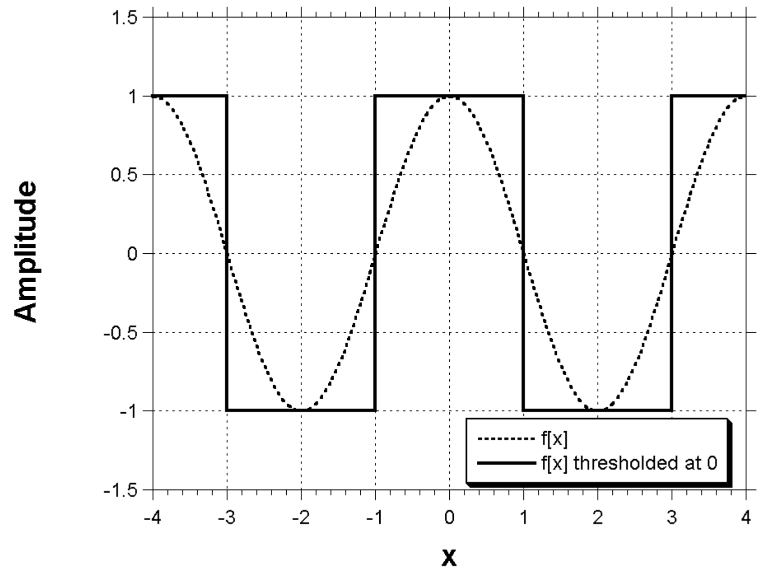


Comparison of the Lorentzian and Gaussian functions. Note that $LOR[x]$ "decays" more quickly than the Gaussian for larger values of $|x|$.

1.18 THRESHOLDED FUNCTIONS

- Square Wave

$$SGN\left[b+\cos\left[2\pi\xi_0 x\right]\right] = \begin{cases} +1 & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] > 0 \\ 0 & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] = 0 \\ -1 & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] < 0 \end{cases}$$



"Thresholding" of a sinusoidal function produces a "square wave".

- Thresholding process interpreted as action of nonlinear imaging system which generates discrete three-state output from continuous input
- Form of lookup table.
- Allowed output amplitudes of 0, 1/2, and 1 by addition and multiplication of the output in fashion analogous to that used to apply the *STEP* function in place of the *SIGNUM* function.

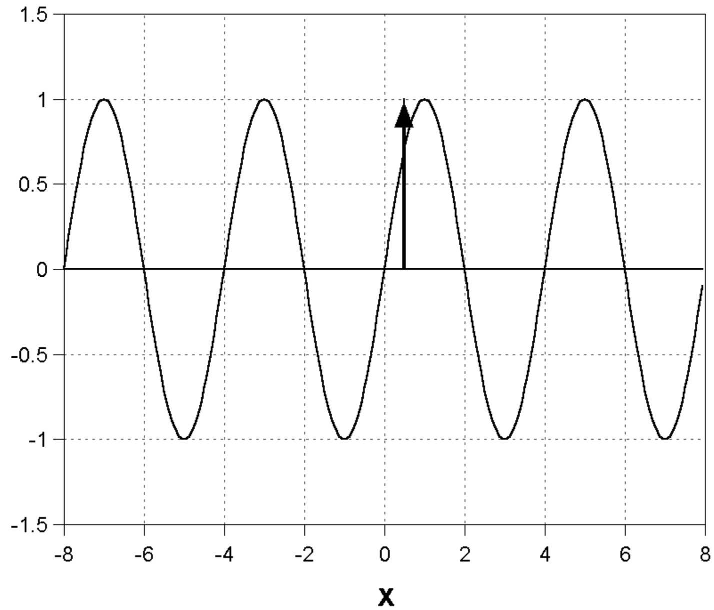
$$\frac{1}{2}\left(1+SGN\left[b+\cos\left[2\pi\xi_0 x\right]\right]\right) = \begin{cases} 1 & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] > 0 \\ \frac{1}{2} & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] = 0 \\ 0 & \text{for} \quad b+\cos\left[2\pi\xi_0 x\right] < 0 \end{cases}$$

- Thresholding process applied to quadratic phase yields 1-D "zone plate"

$$f\left[x\right] = \frac{1}{2}\left(1+SGN\left[\cos\left[\frac{\pi x^2}{\alpha^2}+\phi_0\right]\right]\right) = STEP\left[\cos\left[\frac{\pi x^2}{\alpha^2}+\phi_0\right]\right]$$

1.19 1-D DIRAC DELTA FUNCTION

- Commonly called *impulse* function
- Name honors physicist P.A.M. Dirac, who introduced notation in quantum mechanics
- Dirac delta function does not have "proper" definition that assigns a specific finite amplitude to each coordinate x.
- Strictly $\delta[x]$ is not a function at all
  - notation of $\delta[x]$ is meaningful only within integrand of integral
  - More properly called *delta distribution*.
- Properties of Dirac delta function make an essential tool for solving problems involving many types of physical systems
  - classical mechanics
  - quantum mechanics
  - electrodynamics
- Conditions are:
  1. infinitesimal support
  $$\delta[x - x_0] = 0 \text{ for } x \neq x_0$$
  2. unit area.
  $$\int_{-\infty}^{+\infty} \delta[x - x_0]\, dx = 1$$
- Cannot be depicted in conventional graphical way
- "Image" of characteristics conveyed by arrow (or "spike") located at x_0 with height of the arrowhead's tip above the x-axis equal to area
- Base of the arrow *always* rests on x-axis, even in those cases where $\delta[x - x_0]$ has been added to "proper" finite-amplitude function $f[x]$



Graphical depiction of $f[x] = \delta[x] + COS\left[2\pi\frac{x}{4} - \frac{\pi}{2}\right]$. Because the "height" of the tip of the arrow represents the area of the Dirac delta function, its "base" always lies on the x-axis.

- Fairly common error misconstrues Dirac delta function and a finite discontinuous function $f_0[x]$:

$$\text{(a)} \quad f_0[x] = 0 \text{ for } x \neq 0$$
$$\text{(b)} \quad f_0[x] = 1 \text{ for } x = 0$$

  - (probably results from early exposure to "discrete Dirac delta function" – sampled approximation.

$$\int_{x_0-\frac{1}{2}}^{x_0+\frac{1}{2}} \left(f[x-x_0] + 1[x]\right)\, dx = 1$$
$$\int_{x_0-\frac{1}{2}}^{x_0+\frac{1}{2}} \left(\delta[x-x_0] + 1[x]\right)\, dx = 2$$

- Details of the functional form of continuous $\delta[x]$ do not matter in most situations
- $\delta[x]$ often defined as limit of any of several sequences of functions, differ in some details
- Requirement of unit area is satisfied by several of special functions already defined.
- Scaled rectangle:

$$\int_{-\infty}^{+\infty} \frac{1}{|b|} RECT\left[\frac{x}{b}\right]\, dx = 1$$

- In limit $b \to 0$

$$\lim_{b\to 0}\left\{\frac{1}{|b|} RECT\left[\frac{x}{b}\right]\right\} = 0 \text{ for } x \neq 0$$

- Symmetry of $RECT\left[\frac{x}{b}\right]$ implies that Dirac delta function is symmetric:

$$\delta[x] = \lim_{b\to 0}\left\{\frac{1}{|b|} RECT\left[\frac{x}{b}\right]\right\} = \lim_{b\to 0}\left\{\frac{1}{|-b|} RECT\left[\frac{x}{-b}\right]\right\} = \delta[-x]$$

- Leads to observation that order of arguments of shifted Dirac delta function is immaterial:

$$\delta\left[x-x_0\right]=\delta\left[-\left(x-x_0\right)\right]=\delta\left[x_0-x\right]$$

- Other function sequences that converge to Dirac delta function (based on unit-area special functions)

$$\begin{aligned}
\delta[x] &= \lim_{b\to 0}\left\{\frac{1}{|b|}TRI\left[\frac{x}{b}\right]\right\} \\
&= \lim_{b\to 0}\left\{\frac{1}{|b|}SINC\left[\frac{x}{b}\right]\right\} \\
&= \lim_{b\to 0}\left\{\frac{1}{|b|}SINC^2\left[\frac{x}{b}\right]\right\} \\
&= \lim_{b\to 0}\left\{\frac{1}{|b|}GAUS\left[\frac{x}{b}\right]\right\} \\
&= \lim_{b\to 0}\left\{\frac{1}{|b|}LOR\left[\frac{x}{b}\right]\right\}
\end{aligned}$$

$$f[x]=\lim_{b\to 0}\left\{\frac{1}{|b|}SINC\left[\frac{x}{b}\right]\right\}=\lim_{b\to 0}\left\{\frac{1}{|b|}\left(\frac{\sin\left[\frac{\pi x}{b}\right]}{\left[\frac{\pi x}{b}\right]}\right)\right\}=\lim_{b\to 0}\left\{\frac{\sin\left[\frac{\pi x}{b}\right]}{\pi x}\right\}$$

  - Scale factor b may be expressed in terms of reciprocal $N=\frac{1}{b}$,

$$f[x]=\lim_{N\to\infty}\left\{\frac{\sin[N\pi x]}{\pi x}\right\}=\lim_{N\to\infty}\left\{\frac{\sin\left[2\pi\left(\frac{N}{2}\right)x\right]}{\pi x}\right\}$$

  - Apply L'Hôspital's Rule to show that amplitude at origin is undefined:

$$f[x]=\frac{\lim\limits_{N\to\infty}\{N\pi\}}{\lim\limits_{N\to\infty}\{\pi\}}=\lim_{N\to\infty}\{N\}=\infty$$

- Limiting behavior of Lorentzian:

$$\begin{aligned}
\delta[x] &= \lim_{b\to 0}\left\{\frac{1}{|b|}LOR\left[\frac{x}{b}\right]\right\}=\lim_{b\to 0}\left\{\frac{1}{|b|}\frac{2}{1+\left(2\pi\left(\frac{x}{b}\right)\right)^2}\right\} \\
&= \lim_{b\to 0}\left\{\frac{\left(\frac{b}{2\pi}\right)\frac{1}{\pi}}{\left(\frac{b}{2\pi}\right)^2+x^2}\right\}=\lim_{\epsilon\to 0}\left\{\frac{\frac{\epsilon}{\pi}}{\epsilon^2+x^2}\right\}\text{ where }\epsilon\equiv\frac{b}{2\pi}.
\end{aligned}$$

1.19.1 Area of $\delta[x]$ Over Indefinite Limit

$$\int_{-\infty}^{x}\delta[\alpha]\;d\alpha=\left\{\begin{array}{ccc}0 & \text{for} & x<0 \\ 1 & \text{for} & x>0\end{array}\right\}$$

- Apply Cauchy Principal Value at origin:

$$\int_{-\infty}^{x}\delta[\alpha]\;d\alpha=\left\{\begin{array}{ccc}1 & \text{for} & x>0 \\ \frac{1}{2} & \text{for} & x=0 \\ 0 & \text{for} & x<0\end{array}\right\}=STEP[x]$$

- Differentiate both sides and apply fundamental theorem of calculus:

$$\frac{d}{dx}STEP[x] = \frac{d}{dx}\int_{-\infty}^{x} \delta[\alpha]\,d\alpha = \delta[x] - \delta[-\infty]$$
$$= \delta[x] - 0 = \delta[x]$$

- Means that derivative of step function is Dirac delta function :.

$$\frac{d}{dx}STEP[x - x_0] = \delta[x - x_0]$$

- Very (most?) important representation is obtained by summing complex sinusoids with unit amplitudes and zero phase over all spatial frequencies:

$$\int_{-\infty}^{+\infty} e^{+2\pi i\xi x} d\xi = \int_{-\infty}^{+\infty} \cos[2\pi\xi x]\; d\xi + i\int_{-\infty}^{+\infty} \sin[2\pi\xi x]\; d\xi$$

  - Demonstrate by evaluating integral over arbitrary finite and symmetric limits:

$$\int_{-B}^{+B} e^{+2\pi i\xi x} d\xi = \frac{1}{2\pi i x}\left(e^{+2\pi\xi x}\right)\Big|_{\xi=-B}^{\xi=B} = \frac{1}{\pi x}\frac{\left(e^{+2\pi i x B} - e^{-2\pi i x B}\right)}{2i}$$
$$= \frac{1}{\pi x}\sin[2\pi Bx] = \frac{2B}{2\pi Bx}\sin[2\pi Bx] = 2B\; SINC[2Bx]$$

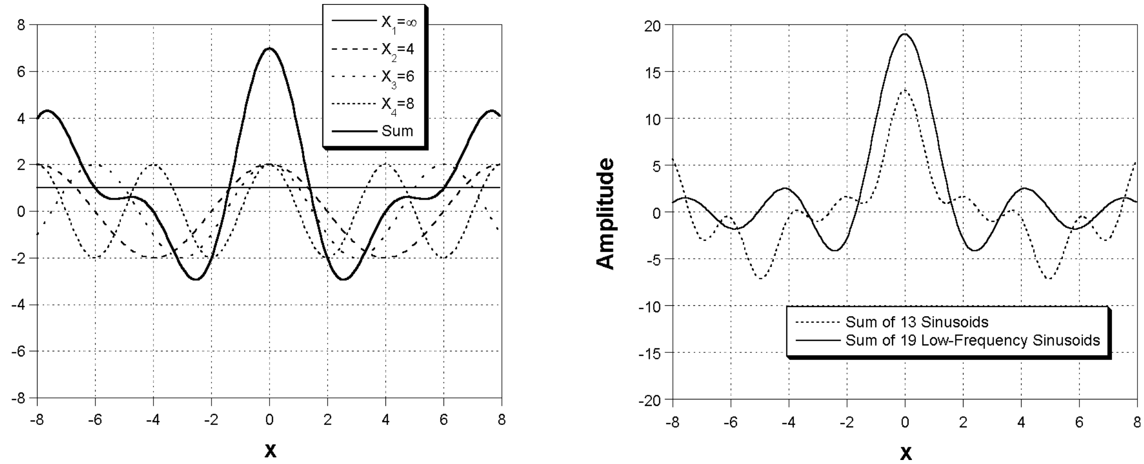
  - In limit $B \to +\infty$, this is a valid representation of Dirac delta function
  - Integral of original complex Hermitian function over symmetric limits yields real-valued result due to cancellation of areas in the antisymmetric imaginary part for positive and negative x.

$$\delta[-x] = \int_{-\infty}^{+\infty} e^{-2\pi i\xi[-x]} d\xi = \int_{-\infty}^{+\infty} e^{+2\pi i\xi x} d\xi = \delta[x]$$

- Integral form of $\delta[x]$ used to derive equivalent expression for Dirac delta function scaled by "width parameter" b.

$$\delta\left[\frac{x}{b}\right] = \int_{-\infty}^{+\infty} e^{-2\pi i\xi\left(\frac{x}{b}\right)} d\xi = \int_{-\infty}^{+\infty} e^{-2\pi i\left(\frac{\xi}{b}\right)x} d\xi$$
$$= \int_{-\infty}^{+\infty} e^{-2\pi i\alpha x}\; |b|\; d\alpha; (\text{for } \xi \equiv \alpha b \Longrightarrow d\xi = |b|\; d\alpha)$$
$$= |b|\int_{-\infty}^{+\infty} e^{-2\pi i\alpha x} d\alpha = |b|\;\delta[x]$$

- "scaling property" of 1-D Dirac delta function
- Scaling "width" and scaling "amplitude" of $\delta[x]$ by factor b are equivalent operations.



Approximations of $\delta[x]$ obtained by summing sinusoidal functions with small spatial frequencies.

1.20 SIFTING PROPERTY of the DIRAC DELTA FUNCTION

- Most significant property of Dirac delta function is ability to evaluate amplitude of another function at any coordinate
- Responsible for most important applications.
- Some authors use mathematical statement of sifting property as definition of Dirac delta function.
- Infinitesimal support of $\delta[x]$ allows area of product of $\delta[x]$ and $f[x]$ to be evaluated by method similar to approximation of integrals as summations of rectangular areas.

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x]\ dx = \lim_{b\to 0}\left\{\frac{1}{|b|}\int_{-\infty}^{+\infty} RECT\left[\frac{x}{b}\right]\ f[x]\ dx\right\}$$

- Area of limiting rectangle is width b multiplied by amplitude evaluated at origin:

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x]\ dx = \lim_{b\to 0}\left\{\frac{1}{|b|}(|b|\ f[0])\right\} = f[0]$$

- Integral of product of $f[x]$ and $\delta[x]$ has "sifted" out specific amplitude $f[0]$ from $f[x]$
- Reason for name *sifting* property
- Straightforward to translate Dirac delta function by adding a term of $-x_0$ to argument of $RECT$ function;
- Analogous integral "sifts" out amplitude of $f[x]$ at x_0:

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x-x_0]\ dx = \int_{-\infty}^{+\infty} f[x]\ \delta[x_0-x]\ dx = f[x_0]$$

- Sifting property may be used to derive yet another useful result for well-behaved $f[x]$:
  - so-called "property of Dirac delta function in products"

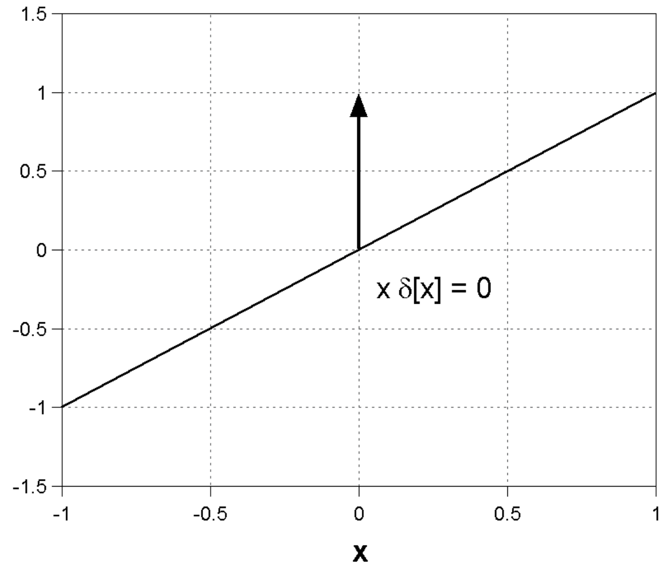
$$\int_{-\infty}^{+\infty} f[x]\ \delta[x-x_0]\ dx = f[x_0] = f[x_0]\times 1$$

$$= f[x_0]\int_{-\infty}^{+\infty}\delta[x-x_0]\ dx = \int_{-\infty}^{+\infty} f[x_0]\ \delta[x-x_0]\ dx$$

$$\boxed{f[x]\ \delta[x-x_0] = f[x_0]\ \delta[x-x_0]}$$

- Substitution of $f[x]=x$ and $x_0=0$ produces the special case:

$$x\ \delta[x] = 0\ \delta[x] = 0$$



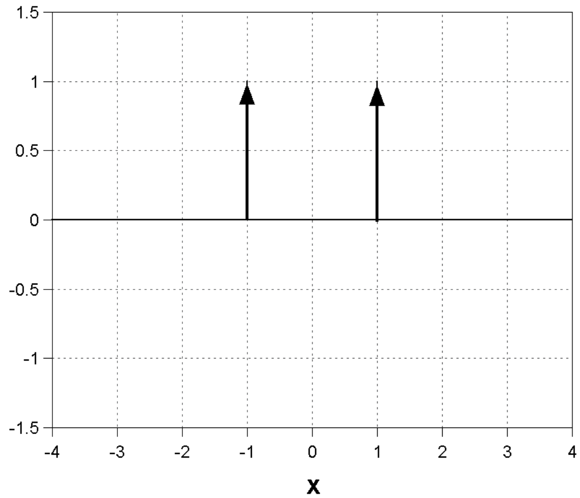
Demonstration of sifting property of $\delta[x]$ applied to $f[x] = x$.

1.21 RELATIVES of DIRAC DELTA FUNCTION

Three additional special functions based on Dirac delta function

1. even pair of Dirac delta functions

   - Unit-amplitude Dirac delta functions located at $x = \pm 1$
   - Notation: $\delta\delta[x] \equiv \delta[x+1] + \delta[x-1]$

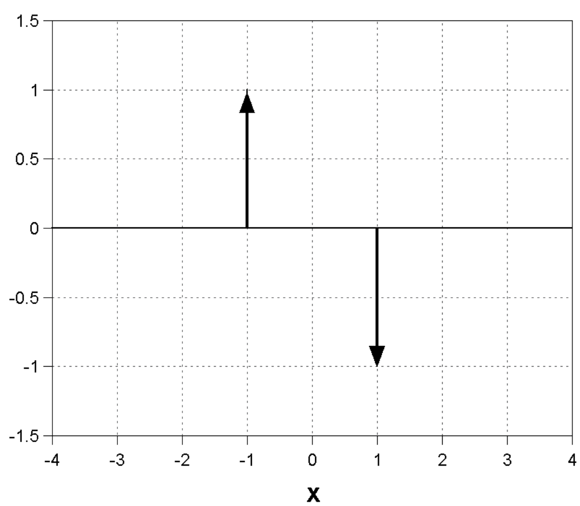


Even pair of Dirac delta functions: $f[x] = \delta[x+1] + \delta[x-1]$.

   - $\int_{-\infty}^{+\infty} \delta\delta[x] \; dx = 2$
   - $\delta\delta\left[\frac{x}{b}\right] = \delta\left[\frac{x}{b}+1\right] + \delta\left[\frac{x}{b}-1\right] = \delta\left[\frac{x+b}{b}\right] + \delta\left[\frac{x-b}{b}\right] = |b| \;\; (\delta[x+b] + \delta[x-b])$

2. odd pair of Dirac delta functions

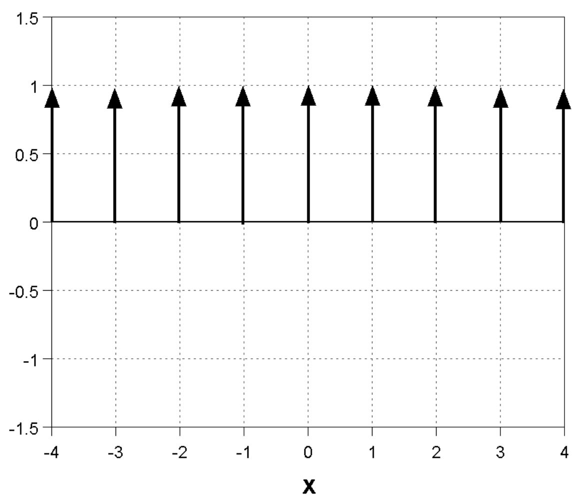
   - Unit-amplitude Dirac delta functions located at $x = \pm x$ with areas ∓ 1
   - Notation:$^{\delta}{}_{\delta}[x] = \delta[x+1] - \delta[x-1]$
   - $^{\delta}{}_{\delta}\left[\frac{x}{b}\right] = |b| \;\; (\delta[x+b] - \delta[x-b])$
   - $\frac{1}{|b|}{}^{\delta}{}_{\delta}\left[\frac{x}{b}\right] = \delta[x+b] - \delta[x-b]$



Odd pair of Dirac delta functions: $f[x] = \delta[x+1] - \delta[x-1]$.

3. $COMB$ function

- infinite set of uniformly spaced Dirac delta functions
- $COMB[x] \equiv \sum_{n=-\infty}^{+\infty} \delta[x-n]$
- $\int_{-\infty}^{+\infty} COMB[x]\, dx = \infty$
- $\sum_{n=-\infty}^{+\infty} \delta[x - x_0 - n] = COMB[x - x_0]$



$$f[x] = COMB[x] \equiv \sum_{n=-\infty}^{+\infty} \delta[x-n]$$

$$\begin{aligned} COMB\left[\frac{x}{b}\right] &= \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x}{b} - n\right] = \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x-nb}{b}\right] \\ &= \sum_{n=-\infty}^{+\infty} |b|\ \delta[x-nb] = |b| \sum_{n=-\infty}^{+\infty} \delta[x-nb] \end{aligned}$$

$$\frac{1}{|b|}\ COMB\left[\frac{x}{b}\right] = \sum_{n=-\infty}^{+\infty} \delta[x-nb]$$

- Use $COMB$ to "sample" a continuous function

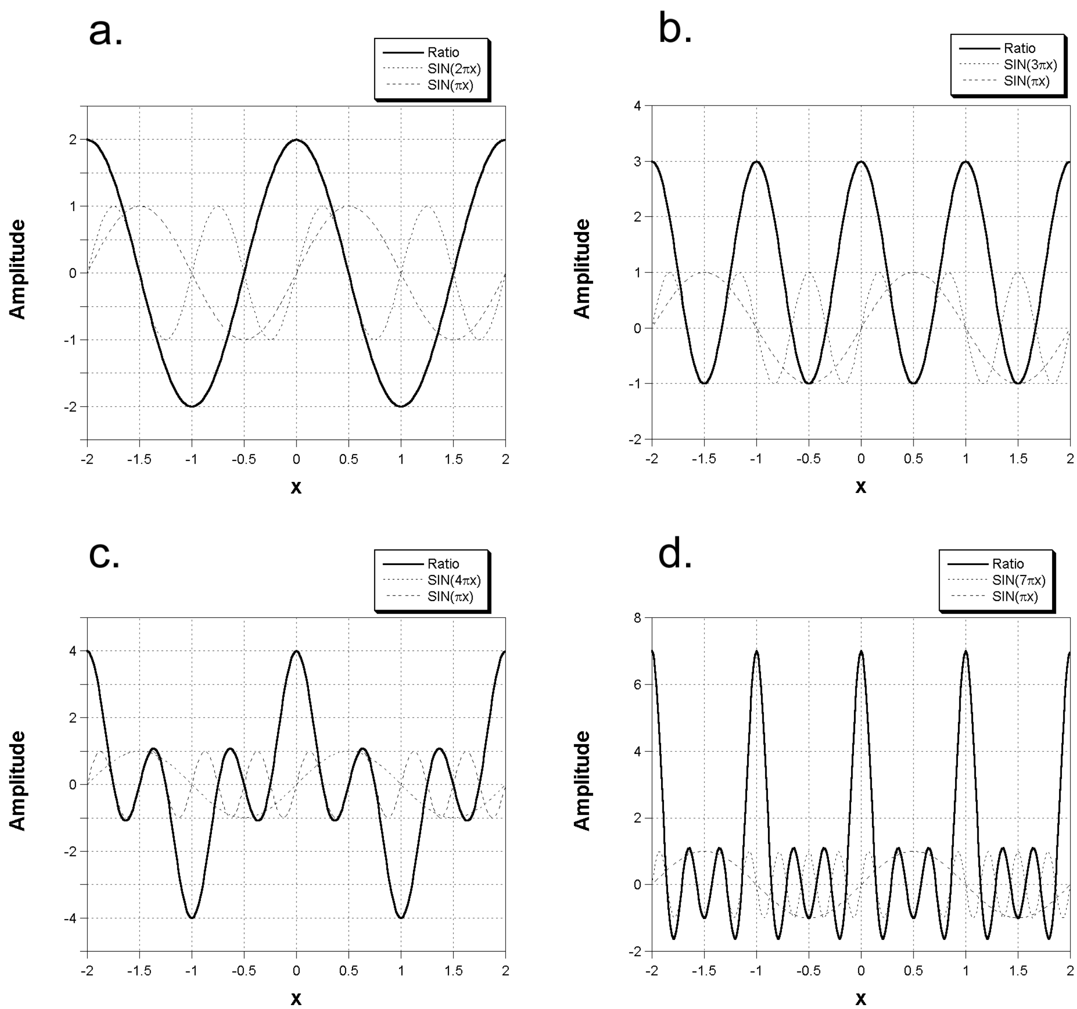
$$\begin{aligned} \cos[2\pi\xi_0 x + \phi_0] \ \left(\frac{1}{\Delta x}\ COMB\left[\frac{x}{\Delta x}\right]\right) &= \cos[2\pi\xi_0 x + \phi_0] \left(\sum_{n=-\infty}^{+\infty} \delta[x - n\ \Delta x]\right) \\ &= \sum_{n\ =\ -\infty}^{+\infty} \cos[2\pi\xi_0 x + \phi_0]\ \ \delta[x - n\ \Delta x] \\ &= \sum_{n\ =\ -\infty}^{+\infty} \cos[2\pi\xi_0 (n\ \Delta x) + \phi_0]\ \ \delta[x - n\ \Delta x] \end{aligned}$$

- $COMB$ function expressed as ratio of functions:

$$\delta[x] = \lim_{N\to\infty}\left\{\frac{\sin[N\pi x]}{\pi x}\right\} = \lim_{N\to\infty}\left\{\frac{\sin\left[2\pi\left(\frac{N}{2}\right)x\right]}{\pi x}\right\}$$

- Ratio of two terms that have null amplitude and unit slope at all integer values of x.

$$COMB[x] = \lim_{odd\ N\to\infty}\left\{\frac{\sin[N\pi x]}{\sin[\pi x]}\right\}$$



Approximations for $COMB[x]$ as ratios of sine waves: (a) $\frac{\sin[2\pi x]}{\sin[\pi x]} = 2\ \cos[\pi x]$; (b) $\frac{\sin[3\pi x]}{\sin[\pi x]} = 2\ \cos[2\pi x] + 1$; (c) $\frac{\sin[4\pi x]}{\sin[\pi x]}$; (d) $\frac{\sin[7\pi x]}{\sin[\pi x]}$. Note that the numerator must be an odd multiple of πx to obtain an approximation of $COMB(x)$.

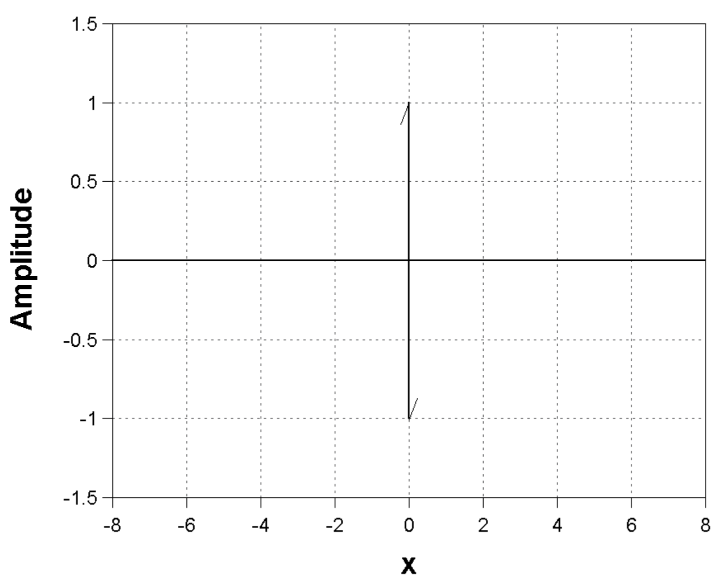
1.22 DERIVATIVES of the DIRAC DELTA FUNCTION

- $$\begin{aligned}\frac{d\delta[x]}{dx} &\equiv \delta^{(1)}[x] = \lim_{b\to 0}\left\{\frac{d}{dx}\left(\frac{1}{|b|}RECT\left[\frac{x}{b}\right]\right)\right\}\\ &= \lim_{b\to 0}\left\{\frac{1}{|b|}\frac{d}{dx}\left(STEP\left[x+\frac{b}{2}\right]-STEP\left[x-\frac{b}{2}\right]\right)\right\}\\ &= \lim_{b\to 0}\left\{\frac{1}{|b|}\left(\delta\left[x+\frac{b}{2}\right]-\delta\left[x-\frac{b}{2}\right]\right)\right\}\\ &= \lim_{b\to 0}\left\{\frac{1}{|b|}\left(\delta\left[\frac{\left(\frac{2x}{b}+1\right)}{\left(\frac{2}{b}\right)}\right]-\delta\left[\frac{\left(\frac{2x}{b}-1\right)}{\left(\frac{2}{b}\right)}\right]\right)\right\}\\ &= \lim_{b\to 0}\left\{\frac{1}{|b|}\frac{2}{|b|}\delta_\delta\left[\frac{2x}{b}\right]\right\}\\ &= \lim_{b\to 0}\left\{\frac{2}{|b|^2}\delta_\delta\left[\frac{x}{\left(\frac{b}{2}\right)}\right]\right\}\end{aligned}$$
- $\delta^{(1)}[x]$ is odd because $\delta[x]$ is even
- Integral of $\delta^{(1)}[x]$ over symmetric limits must be zero.
- Differentiate $f[x]$ within sifting property:

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}\left(\int_{-\infty}^{+\infty} f[\alpha]\ \delta[x-\alpha]\ d\alpha\right) = \int_{-\infty}^{+\infty} f[\alpha]\ \frac{d}{dx}\left(\delta[x-\alpha]\right)\ d\alpha\\ &= \int_{-\infty}^{+\infty} f[\alpha]\ \delta^{(1)}[x-\alpha]\ d\alpha\end{aligned}$$

- Generalize to n^{th} order

$$\int_{-\infty}^{+\infty} f[\alpha]\ \frac{d^n}{dx^n}\left(\delta[x-\alpha]\right)\ dx = \int_{-\infty}^{+\infty} f[x]\ \left(\delta^{(n)}[x-\alpha]\right)\ dx = \frac{d^n f}{dx^n} = f^{(n)}[x]$$



$\delta'[x]$ represented as "doublet" of Dirac delta functions.

1.23 DIRAC DELTA FUNCTION with FUNCTIONAL ARGUMENT $\delta[g[x]]$

- $\delta\left[\frac{x}{b}\right]$, where $g[x] = \frac{x}{b}$
- Functional form may be evaluated when $g[x]$ satisfies certain conditions.
- Expression appears frequently in some imaging applications
  - (e.g., computed tomography to derive inverse Radon transform)
- First criterion for Dirac delta function $\Longrightarrow \delta[g[x]] = 0$ wherever $g[x] \neq 0$
  - Example: $g[x] = 2 + \cos(2\pi x) \Longrightarrow \delta[g[x]] = 0[x]$ because $g[x] \neq 0$.
- Second criterion of unit area considered in case where $g[x]$ has a single zero located at x_0 $\Longrightarrow g[x_0] = 0$
  - If derivatives exist and are finite in vicinity of x_0, then $g[x]$ may be expanded in Taylor series:

$$\begin{aligned} g[x] &= \sum_{n=0}^{+\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n g[x]}{dx^n} \right)\Bigg|_{x=x_0} (x - x_0)^n \\ &= g[x_0] + (x - x_0) \left(\frac{dg}{dx} \right)\Bigg|_{x=x_0} + \frac{(x - x_0)^2}{2} \left(\frac{d^2 g}{dx^2} \right)\Bigg|_{x=x_0} + \cdots \\ &= 0 + (x - x_0) \left(\frac{dg}{dx} \right)\Bigg|_{x=x_0} + \frac{(x - x_0)^2}{2} \left(\frac{d^2 g}{dx^2} \right)\Bigg|_{x=x_0} + \cdots, \textit{ because } g[x_0] = 0 \end{aligned}$$

  - Amplitude of $\delta[g[x]]$ is zero except at $x = x_0 \Longrightarrow$ area of $\delta[g[x]]$ is concentrated in vicinity of x_0
  - Evaluate area by restricting domain of integral to $x_0 \pm \varepsilon$, where ε is arbitrarily small positive real number:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta[g[x]]\ dx &\simeq \int_{x_0-\epsilon}^{x_0+\epsilon} \delta[g[x]]\ dx \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \delta\left[(x - x_0) \left(\frac{dg}{dx} \right)\Bigg|_{x=x_0} + \frac{(x - x_0)^2}{2} \left(\frac{d^2 g}{dx^2} \right)\Bigg|_{x=x_0} + \cdots \right] dx \end{aligned}$$

  - Restriction of domain of x to neighborhood of x_0 ensures that only smallest-order term of Taylor series with nonzero amplitude need be retained
    - If first derivative of $g[x]$ is nonzero and finite at x_0, then $(x - x_0) >> (x - x_0)^n$ for $n > 1$
  - Simplify:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta[g[x]]\ dx &\simeq \int_{x_0-\epsilon}^{x_0+\epsilon} \delta\left[(x - x_0) \left(\frac{dg}{dx} \right)\Bigg|_{x=x_0} \right] dx \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{\delta[x - x_0]}{\left(\left| \frac{dg}{dx} \right| \right)\Big|_{x=x_0}} dx \\ &= \left(\left| \frac{dg}{dx} \right|^{-1} \right)\Bigg|_{x=x_0} \int_{x_0-\epsilon}^{x_0+\epsilon} \delta[x - x_0]\ dx \\ &= \left(\left| \frac{dg}{dx} \right|^{-1} \right)\Bigg|_{x=x_0} \end{aligned}$$

- Requirements for Dirac delta function are satisfied if :

$$\delta[g[x]] = \frac{\delta[x - x_0]}{\left(\left|\frac{dg}{dx}\right|\right)\Big|_{x=x_0}}, \text{ where } g[x_0] = 0 \text{ and } \left.\frac{dg}{dx}\right|_{x=x_0} \neq 0.$$

- If both $g[x_0] = 0$ and $\left.\frac{dg}{dx}\right|_{x=x_0} = 0$, but second derivative is nonzero and finite, then all terms in Taylor series of order three and higher are neglected:

$$\begin{aligned} \delta[g[x]] &= \delta\left[\frac{(x - x_0)^2}{2}\left(\frac{d^2 g}{dx^2}\right)\Big|_{x=x_0}\right] \\ &= \frac{2\delta\left[(x - x_0)^2\right]}{\left|\left(\frac{d^2 g}{dx^2}\right)\Big|_{x=x_0}\right|} \end{aligned}$$

- Now stuck; further simplification requires evaluation of $\delta\left[x^2\right]$, which has same qualities used to derive expression that $g[x_0] = g[0] = 0$ and $\left(\frac{dg}{dx}\right)\Big|_{x_0=0} = 0$.

- In more general case where $g[x]$ has N zeros and derivative is nonzero and finite at each, evaluate series at each zero and sum:

$$\delta[g[x]] = \sum_{n=1}^{\infty} \frac{\delta[x - x_n]}{\left|\left(\left.\frac{dg}{dx}\right|_{x=x_n}\right)\right|} \text{ where } g[x_n] = 0 \text{ and } \left.\frac{dg}{dx}\right|_{x=x_n} \neq 0$$

1. $g_1[x] = x^2$
   - one zero at $x = 0$
   - derivative is zero at $x = 0$
   - produces worthless result $\delta\left[x^2\right] = \delta\left[x^2\right]$.

2. $g_2[x] = x^2 - 1$
   - two zeros located at $x = \pm 1$
   - respective slopes are ± 2:

$$\begin{aligned} \delta\left[x^2 - 1\right] &= \frac{\delta[x + 1]}{|-2|} + \frac{\delta[x - 1]}{|+2|} \\ &= \frac{1}{2}\left[\delta[x + 1] + \delta[x - 1]\right] \\ &= \frac{1}{2}\delta\delta[x] \end{aligned}$$

3. $g_3[x] = \sin[2\pi\xi_0 x]$
   - zeros at integer multiples of $\frac{1}{2\xi_0}$
   - slopes evaluated at zeros are $\pm 2\pi\xi_0$

- Equivalent expression is:

$$\begin{aligned}
\delta\left[\sin\left[2\pi\xi_0 x\right]\right] &= \frac{\cdots+\delta\left[x+\frac{2}{2\xi_0}\right]+\delta\left[x+\frac{1}{2\xi_0}\right]+\delta\left[x\right]+\delta\left[x-\frac{1}{2\xi_0}\right]+\delta\left[x-\frac{2}{2\xi_0}\right]+\cdots}{\left|\pm 2\pi\xi_0\right|}\\
&= \frac{1}{2\pi\left|\xi_0\right|}\sum_{n=-\infty}^{+\infty}\delta\left[x-\frac{n}{2\xi_0}\right]\\
&= \frac{1}{2\pi\xi_0}COMB\left[\frac{2\xi_0 x-n}{2\xi_0}\right]\\
&= \frac{1}{\pi}\,COMB\left[2\xi_0 x\right]
\end{aligned}$$

- Useful equivalent expression for $COMB\left[x\right]$ in terms of $\sin\left(\pi x\right)$ by setting $\xi_0=\frac{1}{2}$:

$$COMB\left[x\right]=\pi\ \delta\left[\sin\left[\pi x\right]\right]=\pi\ \delta\left[\sin\left[2\pi\frac{x}{2}\right]\right]$$

2 1-D COMPLEX-VALUED SPECIAL FUNCTIONS

2.1 Sum of Weighted 1-D Functions

- Complex-valued 1-D function may be created by assigning examples of individual real-valued special functions as real and imaginary parts.

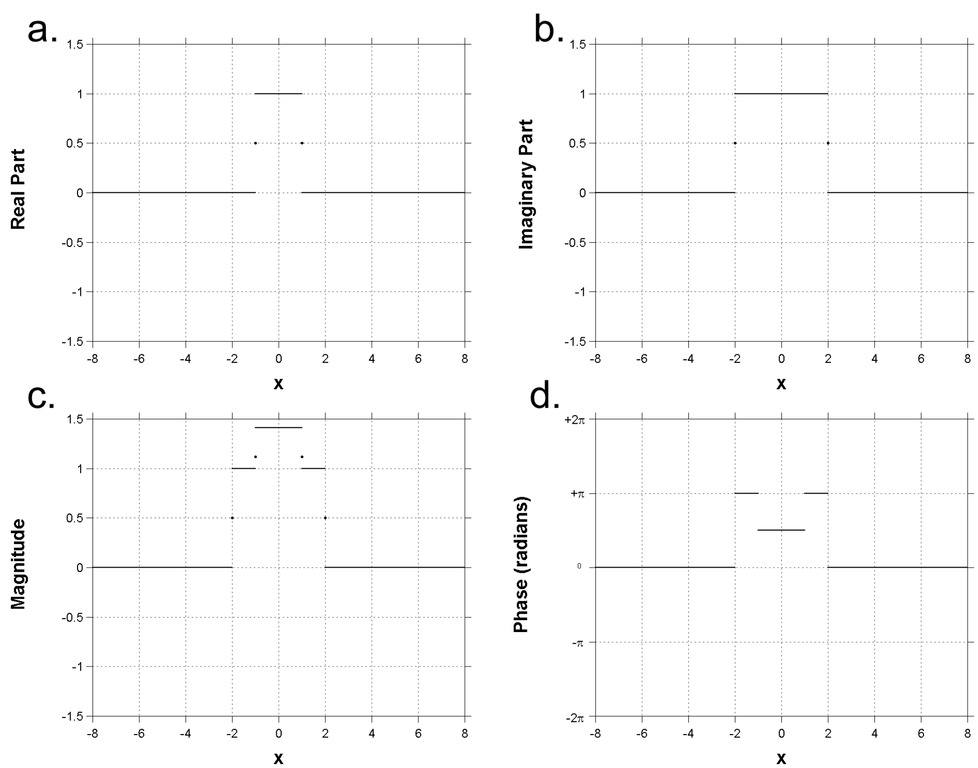
$$f[x] = f_R[x] + i\ f_I[x] = RECT[x] + i\ RECT\left[\frac{x}{2}\right]$$

  – Magnitude and phase via:

$$|f[x]| \equiv \sqrt{(f_R[x])^2 + (f_I[x])^2} = RECT\left[\frac{x}{2}\right] + \left(\frac{\sqrt{2}-1}{2}\right) RECT[x]$$

$$\Phi\{f[x]\} \equiv \tan^{-1}\left[\frac{f_I[x]}{f_R[x]}\right]$$

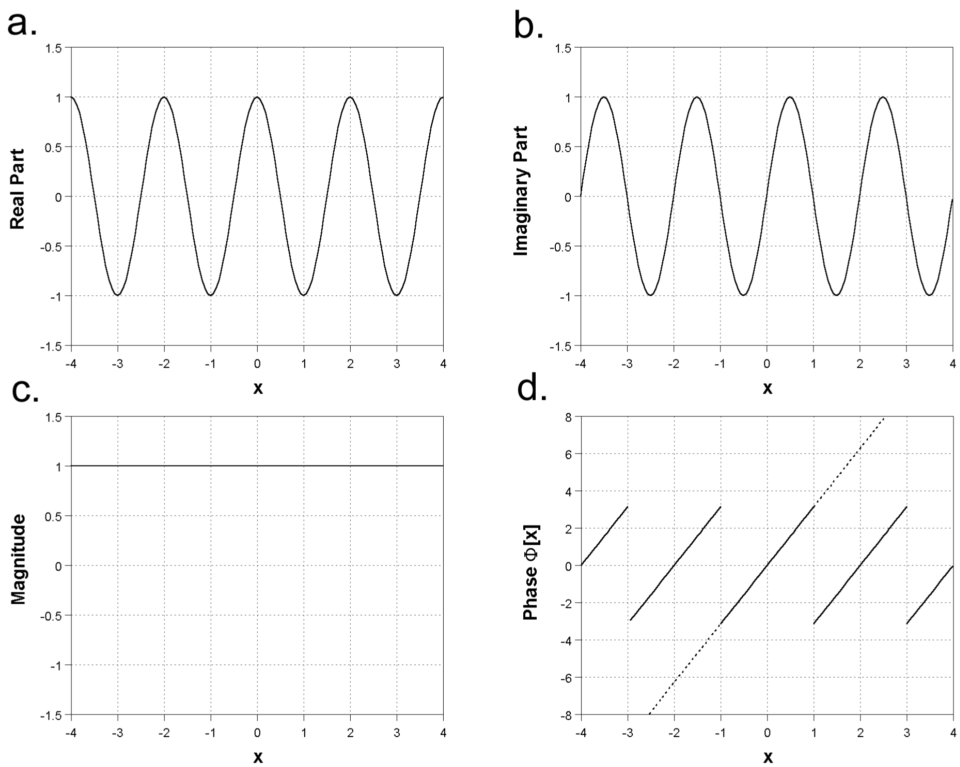
  – Particularly note algebraic sign of phase angles



Complex function constructed from real functions: $f[x] = RECT[x] + i\ RECT\left[\frac{x}{2}\right]$, (a) Real part; (b) imaginary part; (c) magnitude; and (d) phase.

2.2 COMPLEX LINEAR-PHASE SINUSOID

$$
\begin{aligned}
e^{\pm 2\pi i \xi_0 x} &= \cos[2\pi\xi_0 x] \pm i\ \sin[2\pi\xi_0 x] \\
&= \cos[2\pi\xi_0 x] \pm i\ \cos\left[2\pi\xi_0 x - \frac{\pi}{2}\right]
\end{aligned}
$$



Complex sinusoid $f[x] = e^{+2\pi \frac{x}{2}}$: (a) $\Re\left\{e^{+2\pi\frac{x}{2}}\right\} = \cos[\pi x]$, (b)$\Im\left\{e^{+2\pi\frac{x}{2}}\right\} = \sin[\pi x]$, (c) $\left|e^{+2\pi\frac{x}{2}}\right| = 1[x]$, and (d) phase with both range $[-\pi, +\pi)$ and $[-\infty, +\infty)$.

2.3 COMPLEX QUADRATIC PHASE EXPONENTIAL – COMPLEX "CHIRP"

$$e^{\pm i\pi\left(\frac{x}{\alpha}\right)^2} = \cos\left[\frac{\pi x^2}{\alpha^2}\right] \pm i\ \sin\left[\frac{\pi x^2}{\alpha^2}\right]$$

$$\xi\left[x\right] = \pm\frac{1}{2\pi}\frac{\partial\Phi}{\partial x} = \pm\frac{x}{\alpha^2}$$

- area is complex valued.
- Gaussian and complex chirp functions are defined by single functional form throughout domain
- "widths" may be scaled by complex-valued analogues of b.

$$e^{\pm i\pi\left(\frac{x}{\sqrt{\mp i}}\right)^2} = e^{\pm i\pi\left(\frac{x^2}{\mp i}\right)} = e^{-\pi x^2} = GAUS\left[x\right]$$

- Also possible to express chirp function in form of Gaussian:

$$\begin{aligned} e^{\pm i\pi x^2} &= e^{-(\mp i)\pi x^2} = e^{-\pi\left(x\sqrt{\mp i}\right)^2} \\ &= GAUS\left[\left(\sqrt{\mp i}\right)x\right] \\ &= GAUS\left[\frac{x}{\sqrt{\pm i}}\right] \end{aligned}$$

2.4 "SUPERCHIRP" FUNCTION

$$e^{\pm i\pi x^n} = \cos\left[\pi x^n\right] \pm i\ \sin\left[\pi x^n\right]$$

- Even for even n
- Hermitian for odd n
- Symmetry assured if use argument $|x|$

$$f[x] = e^{\pm i\pi |x|^n} = \cos\left[\pi\,|x|^n\right] \pm i\ \sin\left[\pi\,|x|^n\right] \Longrightarrow f[-x] = f[x]$$

- Construct a Hermitian superchirp for even value of n by forcing imaginary part to be odd via multiplication by $SGN[x]$:

$$f[x] = \cos\left[\pi x^n\right] \pm\ \ SGN[x]\ \ \sin\left[\pi\,|x|^n\right] \Longrightarrow f^*[-x] = f[x]$$

- Symmetric superchirp may be expressed in terms of corresponding n^{th}-order superGaussian:

$$e^{\pm i\pi |x|^n} = e^{-\pi\left((\mp i)^{\frac{1}{n}}|x|\right)^n} = GAUS\left[\left((\mp i)^{\frac{1}{n}}\,|x|\right);n\right]$$

- Area

$$\begin{aligned}\int_0^{+\infty} e^{\pm i\pi x^n}dx &= \frac{1}{n}\ \Gamma\left[\frac{1}{n}\right]\ \pi^{-\frac{1}{n}}\ e^{\pm\frac{i\pi}{2n}} \\ &= \left(\frac{1}{\pi}\right)^{\frac{1}{n}}\ e^{\pm\frac{i\pi}{2n}}\ \Gamma\left[1+\frac{1}{n}\right]\end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^n}dx = 2\Gamma\left[1+\frac{1}{n}\right]\ \pi^{-\frac{1}{n}}\ e^{\pm\frac{i\pi}{2n}}$$

- Area of Hermitian superchirp (odd n) must be real valued because area of odd imaginary part is zero:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\pm i\pi x^n}dx &= \Re\left\{2\ \pi^{-\frac{1}{n}}\ \Gamma\left[1+\frac{1}{n}\right]\ e^{\pm\frac{i\pi}{2n}}\right\} \\ &= 2\ \Gamma\left[1+\frac{1}{n}\right]\ \pi^{-\frac{1}{n}}\ \cos\left[\frac{\pi}{2n}\right] \quad \text{(odd n)}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\pm i\pi x^2}dx &= e^{\pm\frac{i\pi}{4}}\ \pi^{-\frac{1}{2}}\ \frac{2}{|2|}\ \Gamma\left[\frac{1}{2}\right] \\ &= e^{\pm\frac{i\pi}{4}}\left(\frac{1}{\sqrt{\pi}}\right)\times 1\times\sqrt{\pi} = e^{\pm\frac{i\pi}{4}} \\ &= \left(\frac{1}{\sqrt{2}}\right)(1\pm i) \\ &\simeq 0.707\ \ (1\pm i)\end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^3} dx = \Re\left\{2\ e^{\pm\frac{i\pi}{6}}\ \pi^{-\frac{1}{3}}\ \Gamma\left[\frac{4}{3}\right]\right\}$$
$$\simeq \cos\left[\frac{\pi}{6}\right]\ \frac{1}{\pi^{\frac{1}{3}}} \cdot 2 \cdot 0.893$$
$$\simeq 1.056$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^3} dx = 2\ e^{\pm\frac{i\pi}{6}}\ \pi^{-\frac{1}{3}}\ \Gamma\left[\frac{4}{3}\right]$$
$$\simeq 1.219\ e^{\frac{\mp i\pi}{6}}$$
$$\simeq 1.056 \pm i\ 0.609$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^4} dx = 2\ e^{+\frac{i\pi}{8}}\ \pi^{-\frac{1}{4}}\ \Gamma\left[\frac{5}{4}\right]$$
$$= \frac{1}{2} \cdot 3.6256$$
$$\simeq 1.2580 \pm i\ 0.5211$$

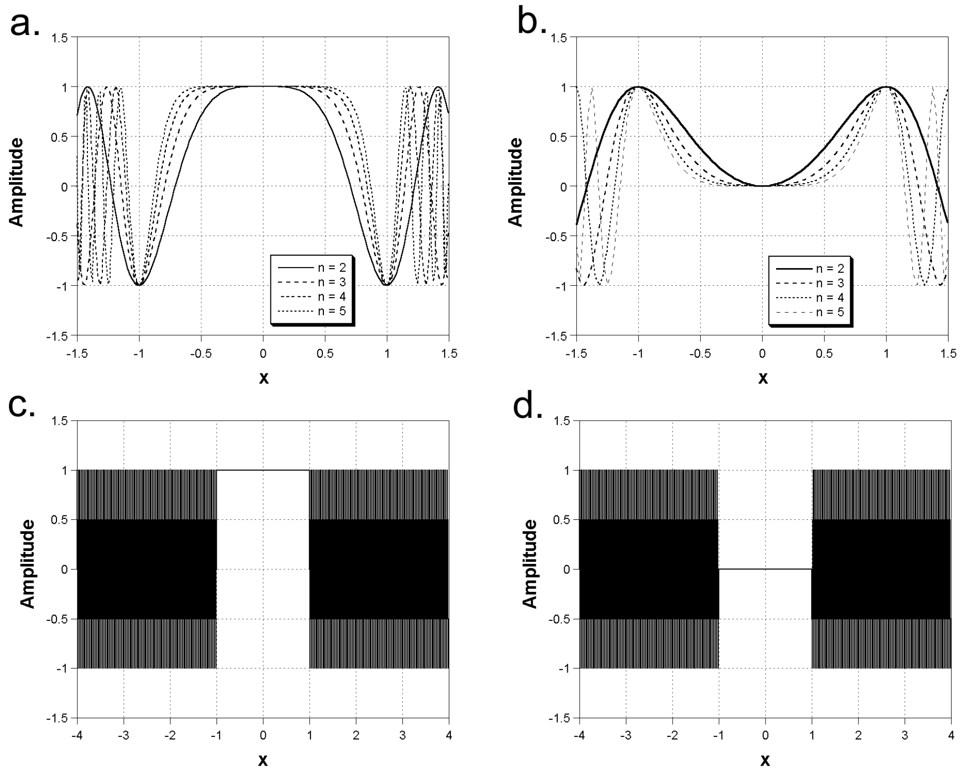
$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^5} dx = \Re\left\{2 \cdot e^{\pm\frac{i\pi}{10}}\ \pi^{-\frac{1}{5}}\ \Gamma\left[\frac{6}{5}\right]\right\}$$
$$= \cos\left[\frac{\pi}{10}\right] \cdot \pi^{-\frac{1}{5}} \cdot 2 \cdot 0.9182$$
$$\simeq 1.3891$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^5} dx = 2\ e^{\pm\frac{i\pi}{10}}\ \pi^{-\frac{1}{5}}\ \Gamma\left[\frac{6}{5}\right]$$
$$\simeq 1.2891 \pm i\ 0.4513$$

- Trends:
  - real part of area increases with n,
  - magnitude of area of imaginary parts of symmetric chirps decrease with increasing n.

$$\lim_{n\to\infty}\left\{\int_{-\infty}^{+\infty} e^{-i\pi x^n} dx\right\} = \lim_{n\to\infty}\left\{e^{+\frac{i\pi}{n}}\ \pi^{-\frac{1}{n}}\ \frac{2}{n}\ \Gamma\left[\frac{1}{n}\right]\right\}$$
$$\Longrightarrow e^0 \cdot \pi^0 \cdot 2 \cdot \lim_{n\to\infty}\left\{\frac{1}{n}\ \Gamma\left[\frac{1}{n}\right]\right\} = 2 + i\ 0$$

n	$\int_{-\infty}^{+\infty} e^{\pm i\pi x^n} dx$	$\int_{-\infty}^{+\infty} e^{\pm i\pi \lvert x\rvert^n} dx$
2	$\left(\frac{1}{\sqrt{2}}\right)(1 \pm i) \simeq 0.707\ (1 \pm i)$	$\left(\frac{1}{\sqrt{2}}\right)(1 \pm i) \simeq 0.707\ (1 \pm i)$
3	$\simeq 1.056$	$\simeq 1.056 \pm i\ 0.609$
4	$\simeq 1.2580 \pm i\ 0.5211$	$\simeq 1.2580 \pm i\ 0.5211$
5	$\simeq 1.2891$	$\simeq 1.2891 \pm i\ 0.4513$



"Superchirp" functions $e^{+i\pi x^n}$ for various values of n: (a) $\Re\left\{e^{+i\pi x^n}\right\} = \cos\left[\pi x^n\right]$, (b) $\Im\left\{e^{+i\pi x^n}\right\} = \sin\left[\pi x^n\right]$, (c) $\lim_{n\to\infty}\left(\Re\left\{e^{+i\pi x^n}\right\}\right) \simeq RECT\left[\frac{x}{2}\right]$, (d) and $\lim_{n\to\infty}\left(\Im\left\{e^{+i\pi x^n}\right\}\right) \simeq 0\left[x\right]$.

2.5 COMPLEX-VALUED LORENTZIAN FUNCTION

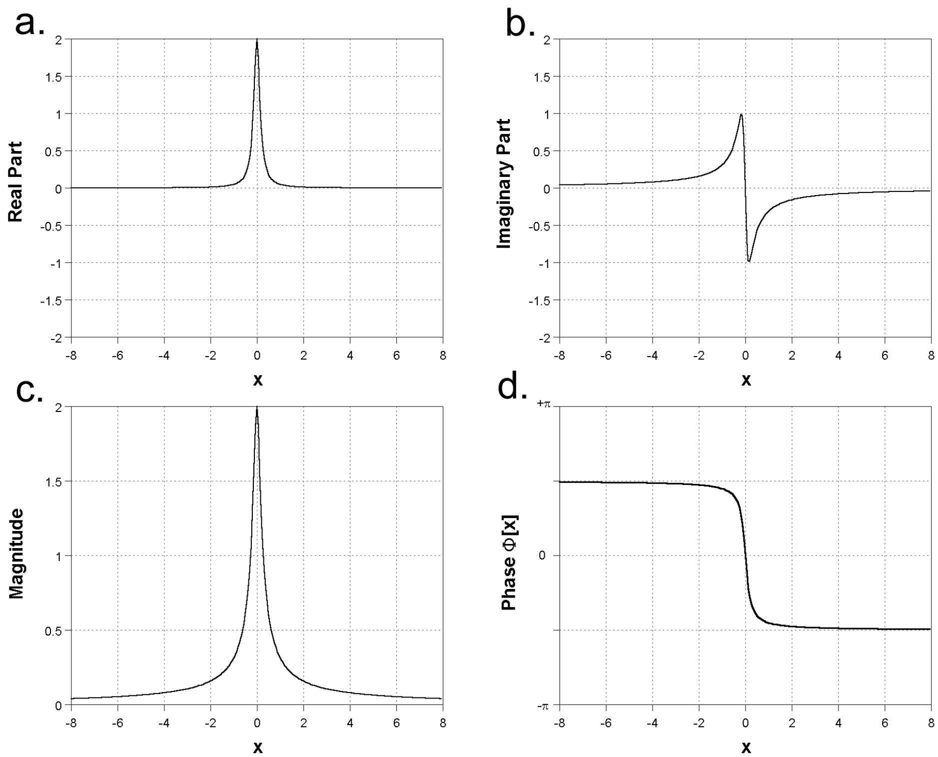
- $$\begin{aligned} f[x] &= LOR[x] \quad (1-2\pi ix) \\ &= \frac{2 \ (1-2\pi ix)}{1+(2\pi x)^2} \\ &= \frac{2}{1+(2\pi x)^2} - i\frac{4\pi x}{1+(2\pi x)^2} \\ &\equiv CLOR[x] \end{aligned}$$

$$\begin{aligned} |CLOR[x]| &= \left|\frac{2 \ (1-2\pi ix)}{1+(2\pi x)^2}\right| \\ &= \sqrt{\frac{4 \ \left(1+(2\pi x)^2\right)}{\left(1+(2\pi x)^2\right)^2}} \\ &= \sqrt{\frac{4}{\left(1+(2\pi x)^2\right)}} \\ &= \sqrt{2 \ LOR[x]} \end{aligned}$$

- Magnitude of complex Lorentzian "decays" more slowly than real-valued Lorentzian.
- Phase is arctangent of ratio of imaginary and real parts.
- Identical denominators cancel to leave inverse tangent of $-2\pi x$:.

$$\begin{aligned} \Phi\{CLOR[x]\} &= \Phi\left\{\frac{2 \ (1-2\pi ix)}{1+(2\pi x)^2}\right\} \\ &= \tan^{-1}\left[-\frac{\left(\frac{4\pi x}{1+(2\pi x)^2}\right)}{\left(\frac{2}{1+(2\pi x)^2}\right)}\right] \\ &= \tan^{-1}[-2\pi x] \\ &= -\tan^{-1}[2\pi x] \end{aligned}$$

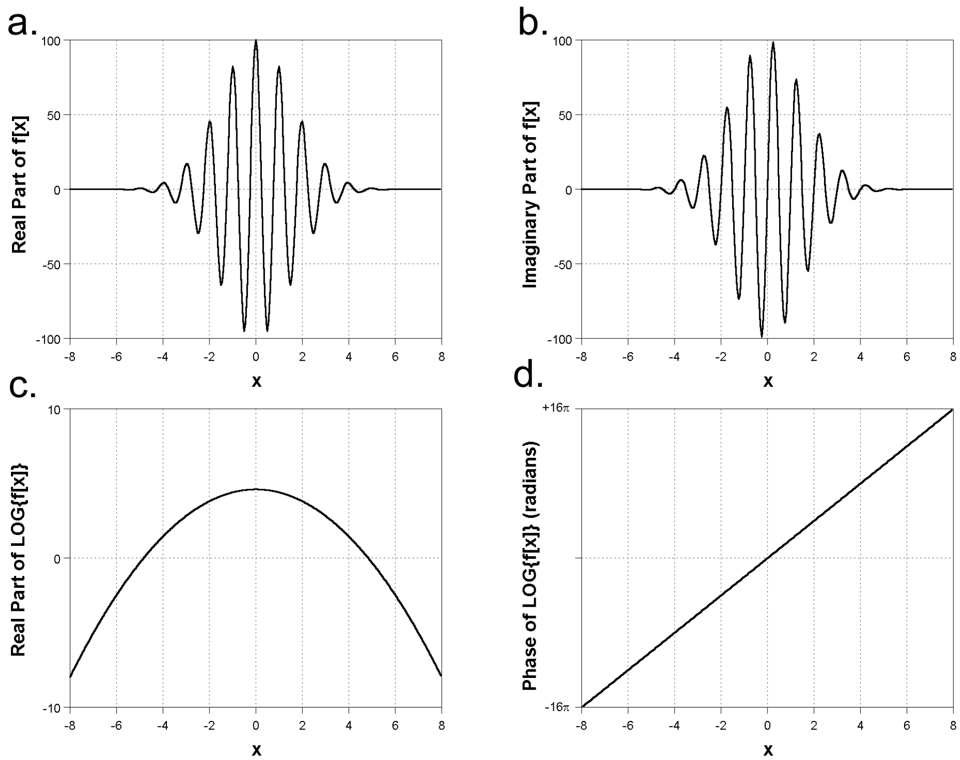
- Phase of complex Lorentzian function follows curve of inverse tangent function.



Complex Lorentzian function $CLOR[x] = \frac{2\ (1-2\pi ix)}{1+(2\pi x)^2}$: (a) $\Re\{CLOR[x]\} = \frac{2}{1+(2\pi x)^2}$; (b) $\Im\{CLOR[x]\} = \frac{-2\pi x}{1+(2\pi x)^2}$; (c) $|CLOR[x]| = \sqrt{\frac{4}{1+(2\pi x)^2}}$; and (d) phase $\Phi\{CLOR[x]\} = -TAN^{-1}[2\pi x]$.

2.6 LOGARITHM of the COMPLEX AMPLITUDE

$$\begin{aligned} LOG\,[f\,[x]] &= LOG\left[|f\,[x]|\;\; e^{i\Phi\{f[x]\}}\right] \\ &= LOG\,[|f\,[x]|] + LOG\left[e^{i\Phi\{f[x]\}}\right] \\ &= LOG\,[|f\,[x]|] + i\;\Phi\,\{f\,[x]\} \end{aligned}$$



$\log\left(f\,[x]\right) = \log\left(100\; e^{(+2\pi i x)}\; e^{-\pi\left(\frac{x}{4}\right)^2}\right)$: (a) $\Re\left\{(f\,[x])\right\} = 100\;\cos\left[2\pi x\right] e^{-\pi\left(\frac{x}{4}\right)^2}$, (b) $\Im\left\{(f\,[x])\right\} = 100\;\sin\left[2\pi x\right]\; e^{-\pi\left(\frac{x}{4}\right)^2}$, (c) $\Re\left\{\log_e\left(f\,[x]\right)\right\} = \log_e|f\,[x]| = \log_e\left|100\pi\left(\frac{x}{4}\right)^2\right|$, (d) $\Im\left\{\log_e\left(f\,[x]\right)\right\} = \Phi\left\{f\,[x]\right\} = 2\pi x$.

3 A. APPENDIX: Series Solution for Bessel Functions

3.1 A.1. Series Solution for $J_0[x]$

1. Substitute $\nu = 0$ into differential equation:

$$x^2 \frac{d^2}{dx^2}(J_0[x]) + x\frac{d}{dx}(J_0[x]) + x^2 J_0[x] = 0 \quad \text{(A1)}$$

   The power-series solution for $J_0[x]$ has the form:

$$J_0[x] = \sum_{\ell=0}^{+\infty} a_\ell \, x^\ell \quad \text{(A2)}$$

2. After substitution of eq.(A2) in eq.(A1) and evaluating the derivatives of each term:relationship satisfied by coefficient a_ℓ for each power of x:

$$\begin{aligned} x^2 \frac{d^2}{dx^2}\left(\sum_{\ell=0}^{+\infty} a_\ell \, x^\ell\right) + x\frac{d}{dx}\left(\sum_{\ell=0}^{+\infty} a_\ell \, x^\ell\right) + x^2 \sum_{\ell=0}^{+\infty} a_\ell \, x^\ell \\ = x^2 \sum_{\ell=0}^{+\infty} a_\ell \, (\ell(\ell-1)) \, x^{\ell-2} + \left(\sum_{\ell=0}^{+\infty} a_\ell \, \ell \, x^\ell\right) + \left(\sum_{\ell=0}^{+\infty} a_\ell \, x^{\ell+2}\right) \\ = \sum_{\ell=0}^{+\infty} a_\ell \, (\ell(\ell-1)) \, x^\ell + a_\ell \, \ell \, x^\ell + a_\ell \, x^{\ell+2} = 0 \end{aligned} \quad \text{(A3)}$$

3. Collect coefficients of identical powers of x:

$$\sum_{\ell=0}^{+\infty} a_\ell (\ell(\ell-1)) x^\ell + a_\ell \, \ell \, x^\ell + a_\ell \, x^{\ell+2} = \sum_{\ell=0}^{+\infty} (a_\ell [\ell(\ell-1)] + a_\ell \, \ell + a_{\ell-2}) x^\ell = 0 \quad \text{(A4)}$$

   Coefficient of each power of x must be zero.

4. Result is recursion relation for a_ℓ:

$$a_\ell [\ell(\ell-1)] + a_\ell \, \ell + a_{\ell-2} = 0 \Longrightarrow a_\ell = -\frac{a_{\ell-2}}{[\ell(\ell-1)] + \ell} = -\frac{a_{\ell-2}}{\ell^2} \quad \text{(A5)}$$

- Relates only coefficients that differ in power by 2
- Sign of each coefficient is the opposite of the next one in the series.
- Coefficients determined by boundary conditions.
- Zeroth-order coefficient is amplitude of function at origin, $a_0 = J_0[0]$, assumed to be unity
- Subsequent coefficients for even powers are:

$$
\begin{aligned}
a_2 &= -\frac{1}{2^2} = -\frac{1}{4} && \text{(A6a)}\\
a_4 &= -\frac{a_2}{4^2} = -\frac{\left(-\frac{1}{4}\right)}{16} = +\frac{1}{64} && \text{(A6b)}\\
a_6 &= -\frac{a_4}{6^2} = -\frac{\left(+\frac{1}{64}\right)}{36} = -\frac{1}{2304} && \text{(A6c)}\\
a_8 &= -\frac{a_6}{8^2} = -\frac{\left(-\frac{1}{2304}\right)}{64} = -\frac{1}{147,456} && \text{(A6d)}\\
&\vdots\\
a_{2\ell} &= (-1)^\ell \frac{1}{\left((2\ell)^2 \;\; [2(\ell-1)]^2 \;\; (2[\ell-2])^2 \cdots 2^2\right)}\\
&= (-1)^\ell \frac{1}{(2^2)^\ell (\ell!)^2} = \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2} && \text{(A6e)}
\end{aligned}
$$

- Coefficients of odd powers assumed to be zero
- $J_0[x]$ is even function.
- The power series for zero-order Bessel function of the first kind is:

$$
\begin{aligned}
J_0[x] &= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1}{(\ell!)^2} \left(\frac{x}{2}\right)^{2\ell}\\
&= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456} - \cdots && \text{(A7)}
\end{aligned}
$$

- Magnitudes decrease very rapidly with order
  - Extrema of $J_0[x]$ decrease much more slowly with increasing x than $SINC[x]$.

3.2 A.2. Series Solution for $J_1[x]$

- Substitute $v = 1$:

$$x^2 \frac{d^2}{dx^2}(J_1[x]) + x \frac{d}{dx}(J_1[x]) + \left(x^2 - 1\right) J_1[x] = 0 \tag{A8}$$

- Power-series solution for $J_1[x]$ has form:

$$J_1[x] = \sum_{\ell=0}^{+\infty} b_\ell \, x^\ell \tag{A9}$$

$$\begin{aligned} x^2 \frac{d^2}{dx^2}\left(\sum_{\ell=0}^{+\infty} b_\ell \, x^\ell\right) + x \frac{d}{dx}\left(\sum_{\ell=0}^{+\infty} b_\ell \, x^\ell\right) + \left(x^2-1\right)\sum_{\ell=0}^{+\infty} b_\ell \, x^\ell \\ = x^2 \sum_{\ell=0}^{+\infty} b_\ell \left(\ell\left(\ell-1\right)\right) x^{\ell-2} + \left(\sum_{\ell=0}^{+\infty} b_\ell \, \ell \, x^\ell\right) + \left(\sum_{\ell=0}^{+\infty} b_\ell \, x^{\ell+2} - \sum_{\ell=0}^{+\infty} b_\ell \, x^\ell\right) \\ = \sum_{\ell=0}^{+\infty} \left(b_\ell \left(\ell\left(\ell-1\right)\right) + b_\ell \, \ell + b_{\ell-2} - b_\ell\right) x^\ell = 0 \\ = \sum_{\ell=0}^{+\infty} \left(b_\ell \left(\left(\ell\left(\ell-1\right)\right) + \ell - 1\right) + b_{\ell-2}\right) x^\ell = 0 \end{aligned} \tag{A10}$$

- Recursion relation:

$$b_\ell = -\frac{b_{\ell-2}}{\ell^2 - \ell + \ell - 1} = -\frac{b_{\ell-2}}{\ell^2 - 1} \tag{A11}$$

- Boundary conditions determine first two coefficients
  - $J_1[0] = 0 \Longrightarrow$ all even-order coefficients vanish
  - First-order coefficient is slope at origin, set to $\frac{1}{2}$

Remaining odd-order coefficients determined by recursion relation in eq.(A10):

$$b_1 = \frac{1}{2} \tag{A12a}$$

$$b_3 = -\frac{b_1}{(3^2-1)} = -\frac{\left(\frac{1}{2}\right)}{2 \cdot 4} = -\frac{1}{16} \tag{A12b}$$

$$b_5 = -\frac{b_3}{(5^2-1)} = \frac{(-1)^2}{(5^2-1)\cdot(3^2-1)\cdot 2} = \frac{(-1)^2}{(4\cdot 6)\cdot(4\cdot 2)\cdot 2} = \frac{(-1)^2}{(2^2\cdot 4^2\cdot 6)} = +\frac{1}{384} \tag{A12c}$$

$$\begin{aligned} b_7 &= -\frac{b_5}{(7^2-1)} = \frac{(-1)^3}{(7^2-1)\cdot(5^2-1)\cdot(3^2-1)\cdot 2} = \frac{(-1)^3}{(8\cdot 6)\cdot(4\cdot 6)\cdot(4\cdot 2)\cdot 2} \\ &= -\frac{\left(\frac{1}{384}\right)}{48} = -\frac{1}{18,432} \end{aligned} \tag{A12d}$$

$$\vdots$$

$$\begin{aligned} b_{2\ell+1} &= (-1)^\ell \frac{1}{\left((2\ell+1)^2-1\right)\cdot\left((2\ell-1)^2-1\right)\cdot\left((2\ell-3)^2-1\right)\cdot\cdots\cdot 2} \\ &= \frac{(-1)^\ell}{\left(2^2\cdot 4^2\cdot\cdots\cdot(2\ell)^2\cdot(2\ell+2)\right)} \end{aligned} \tag{A12e}$$

- Power series for the first-order Bessel function of the first kind:

$$J_1[x] = \frac{x}{2} - \frac{x^3}{(2^2 \cdot 4)} + \frac{x^5}{(2^2 \cdot 4^2 \cdot 6)} - \frac{x^7}{(2^2 \cdot 4^2 \cdot 6^2 \cdot 8)} + \cdots$$
$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18,432} + \cdots \quad \text{(A13)}$$

- Magnitudes decrease rapidly as power of x increases.
- Comparing series for $J_1[x]$ and $J_0[x]$ in eq.(A7):

$$\frac{d}{dx}(J_0[x]) = 0 - \frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{384} + \frac{x^7}{18,432} - \cdots = J_1[x] \quad \text{(A14)}$$

- General expression valid for all positive integer values of n:

$$J_n[x] = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!\,(n+\ell)!} \left(\frac{x}{2}\right)^{n+2\ell} \quad \text{(A15)}$$