SIMG-716
Linear Imaging Mathematics I
03

1 REPRESENTATIONS of FUNCTIONS

1.1 FUNCTIONS

- Function is a rule that assigns one numerical value (the *dependent variable*) to another numerical value (the *independent variable* or input *coordinate*).
- used to describe all aspects of the imaging process: the input scene f, the imaging system $\mathcal{O}$, and the output image g.
- Number of independent variables required to specify a unique location in that domain is the *dimensionality* of the function
- set of possible values of the independent variable(s) is the *domain* of the function.
- Set of possible values of the dependent variable f is the *range* of the function.

1.1.1 Multidimensional Functions

- More than one independent variable
- If independent variables have same "units" (e.g., $[x, y]$) then may be expressed in various equivalent coordinate systems
  - 2-D Cartesian
  - 2-D polar coordinates
  - 3-D Cartesian
  - 3-D cylindrical
  - 3-D spherical
- Choice of coordinate system depends on the conditions of the specific imaging problem.
- Notational Convention
  - Cartesian coordinates in brackets, e.g., $[x, y]$ and $[x, y, z]$.
  - domains with at least one angle enclosed in parentheses; e.g., (r, θ).
  - may be confused with the common notation for "open-ended" and "closed" intervals of real-valued coordinates
    - combinations of brackets and parentheses specify intervals in the domain.
    - Bracketed coordinates $[-1, +1]$, specify a domain that includes the indicated endpoints
    - parentheses, $(-1, +1)$ indicate that the interval does not include the endpoints
    - infinite domain is $(-\infty, +\infty)$.
    - bracket and a parenthesis used together to specify interval with one included endpoint (the "closed" coordinate) but not the other (the "open" coordinate): $[0, 1)$ indicates an interval of unit length on the real line that includes the origin but not $x = 1$, so that $0 \leq x < 1$.

1.2 Classes of Functions

- based on common characteristics of the domain and/or range.

Examples:

1. domain may be specified either by continuous coordinates or by a discrete set of "samples"
2. coordinates in domain may be real, imaginary, or complex-valued
   - finite (compact) support
   - infinite support
3. range of dependent variable f may be real-, imaginary-, or complex-valued
4. dependent variable f may repeat at regular intervals to create a *periodic* function.
5. set of "zeros" of function: locations where amplitude f is zero.
6. "shape" of function over its domain
   - amplitude is a power of coordinate x
     - *linear* $\Longrightarrow$ amplitude is proportional to coordinate ($f[x] = \alpha x \propto x^1$).
     - *quadratic* function of x has the form $f[x] = \beta x^2 \propto x^2$, a parabolic shape.

All of these classifications for functions (and others yet to be introduced) will be useful in the course of this discussion. We begin by considering the classes based on the continuity of the domain and range.

1.3 FUNCTIONS with CONTINUOUS and DISCRETE DOMAINS

1.
$$f_1[x] \equiv y = 4x$$

   - amplitude of f_1 proportional to input coordinate, and therefore $f_1[x]$ is a linear function
   - domain and range include all real numbers, both domain and range are $(-\infty, +\infty)$.
   - range is "bipolar" because it includes positive and negative values.
   - one zero, located at $x = 0$.

2.
$$f_2[x] = x^2$$

   - quadratic (not linear).
   - domain and range of $f_2[x]$ are continuous
     - domain is $(-\infty, +\infty)$
     - range is $[0, +\infty)$.
   - One zero at $x = 0$.

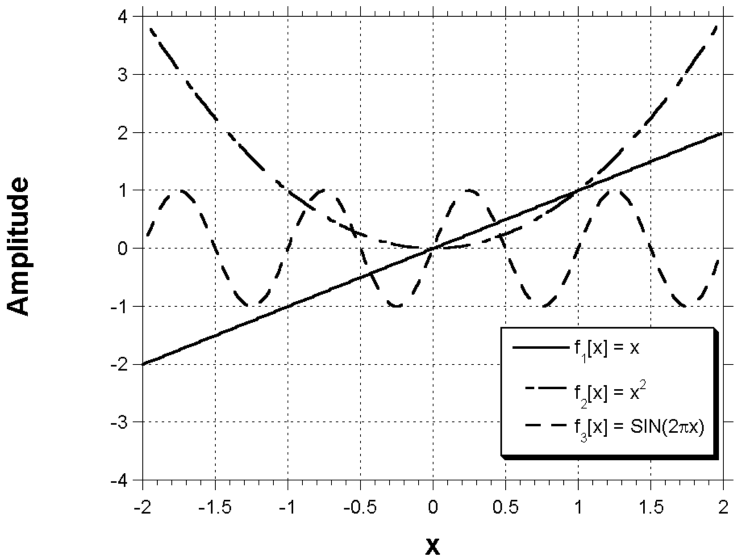
3. 1-D sinusoid with period X_3:

$$f_3[x] = \sin\left[\frac{2\pi x}{X_3}\right]$$

- domain and range are continuous

– domain is $(-\infty, +\infty)$
– range is $[-1, +1]$.

- amplitude f_3 repeats for coordinates x separated by intervals of X_3
  – $f_3[x]$ is *periodic*
  – infinite number of isolated zeros uniformly spaced at intervals of width $\Delta x = \frac{X_3}{2}$.

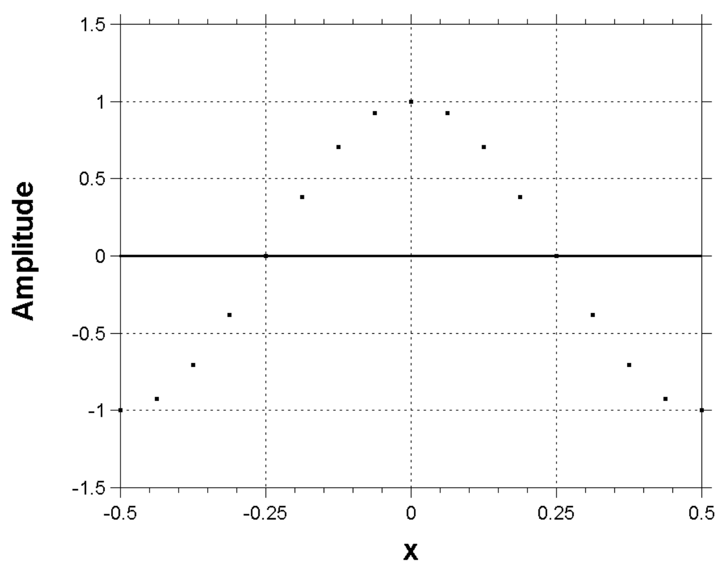


Graphs of the functions in eq.(2-3) with continuous domains and different ranges. Though shown with different line styles, all represent continuous functions.

4. Discrete functions: amplitude is defined only at discrete set of uniformly spaced coordinates

$$f_4[x] = \cos[2\pi n\ \Delta x],\ n = 0, \pm1, \pm2, \cdots$$

- may be constructed from continuous functions by *sampling*, considered later.



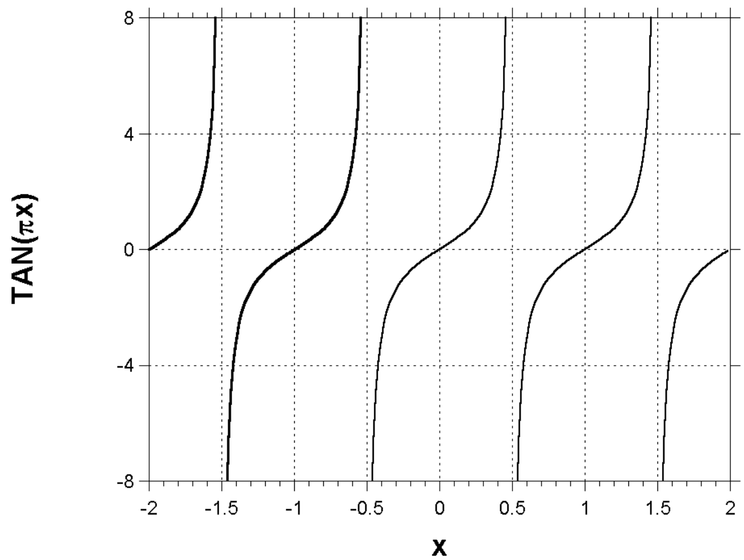
Discrete ("sampled") function, with discrete domain and continuous range.

5. Discontinuous Range

- "transition" coordinates where derivative is not defined

$$f_5\left[x\right] = \tan\left[\frac{2\pi x}{X}\right]$$

- amplitude "jumps" from $f_5 \to +\infty$ at $x = +\frac{X}{4} - \epsilon$ to $f_5 \to -\infty$ at $x = +\frac{X}{4} + \epsilon$, where ϵ is a small positive real number $\epsilon \simeq 0$
- amplitude not defined at $x = \frac{X}{4}$ and its derivative is not finite there.



"Discontinuous" function $\tan\left[\pi x\right]$ with continuous domain and continuous range.

1.4 CONTINUOUS and DISCRETE RANGES

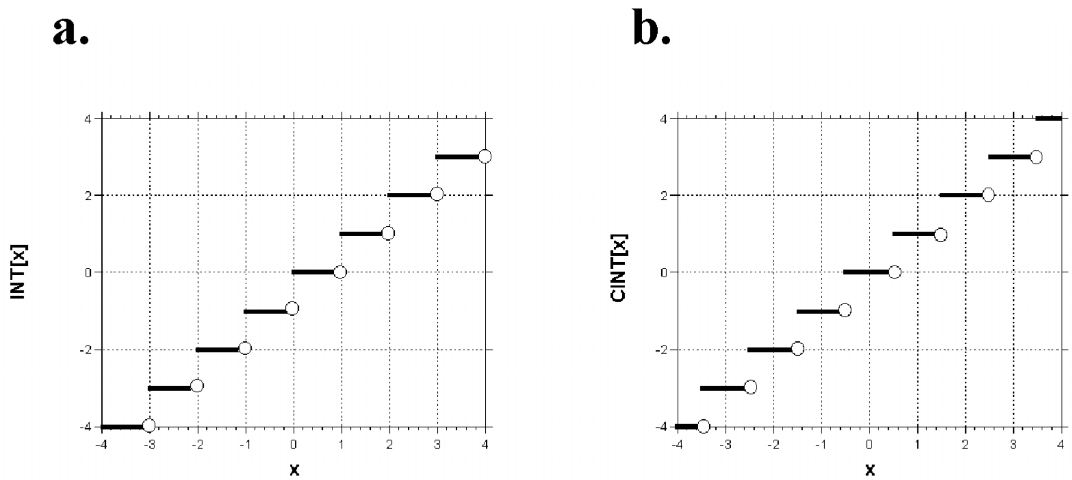
- set of all allowed values of dependent variable f
- different "flavors" of range exist for different functions
  - may have an infinite or finite extent
  - may have all real values between extrema or only some discrete set.
  - may be "continuous" or "discrete"

1.

$$f_6\left[x\right] = INT\left[x\right]$$

- domain is entire real line $(-\infty, +\infty)$
- range includes discrete (though infinite) set of integers $(0, \pm 1, \pm 2, \cdots)$
- common variant is $CINT[\ ]$, which "rounds" the value to the nearest integer.

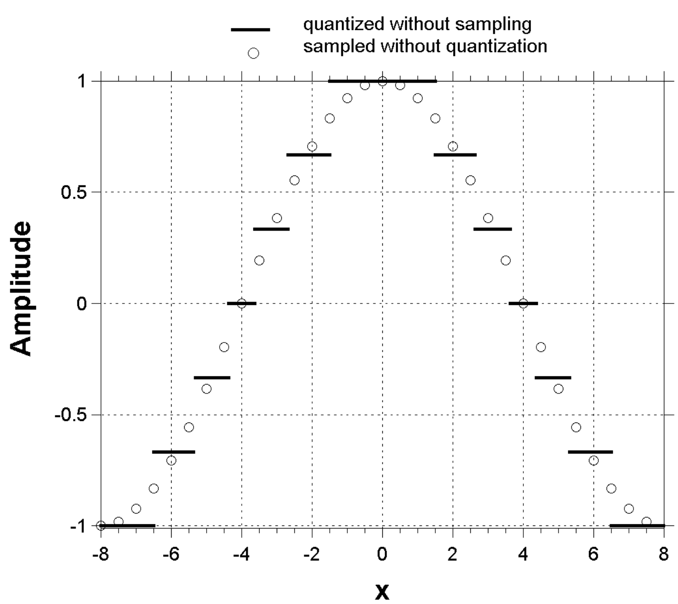
$$f_7\left[x\right] = CINT\left[x\right] \equiv INT\left[x + \frac{1}{2}\right]$$



(a) “Greatest integer” function (truncation of amplitude) compared to (b) “closest integer” function $CINT[x]$ that “rounds” the amplitude to the nearest integer.

- truncation or rounding may be applied to amplitude of any function $f[x]$ to convert from a continuous to discrete range

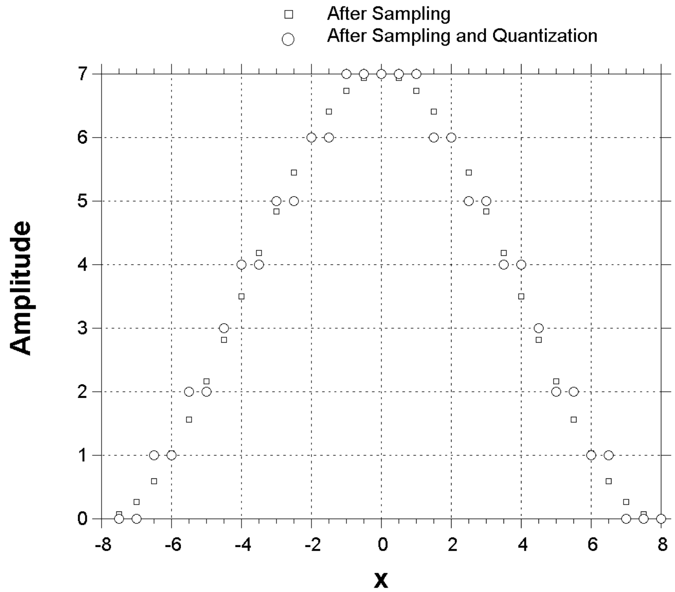
$$g[x] = CINT\{f[x]\} \Longrightarrow g = CINT[f]$$



$COS\left[\frac{2\pi x}{16}\right]$ after independent quantization to 7 levels and independent sampling at integer coordinates.

1.5 DISCRETE DOMAIN and RANGE – DIGITAL FUNCTIONS

Both domain and range are discrete



Results from the cascade of sampling and subsequent quantization of nonnegative sinusoid.

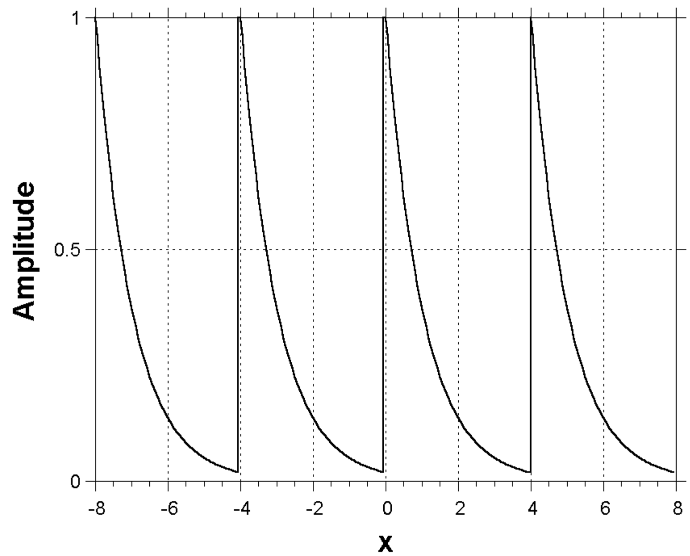
1.6 PERIODIC, APERIODIC, and HARMONIC FUNCTIONS

1. criterion for a one-dimensional (1-D) function $f[x]$ to be periodic:

$$f[x_0] = f[x_0 + nX_0]$$

   where n is any integer and X_0 (the period of $f[x]$) is the smallest possible increment of the independent variable x such that the requirement is satisfied.

- a nonnull function $f[x]$ is periodic if amplitudes are identical at all coordinates separated by integer multiples of some distance X_0.
- may have very irregular form; the criterion for periodicity merely requires that the amplitude repeat at regular intervals.
- *aperiodic function* is any function that does not satisfy the criterion for periodicity
- Functions of dimension two (or larger) may be periodic over all coordinates or over just a subset.



Function that obeys requirement for "periodicity" because $f[x] = f[x + 4n]$ for any integer value of n.

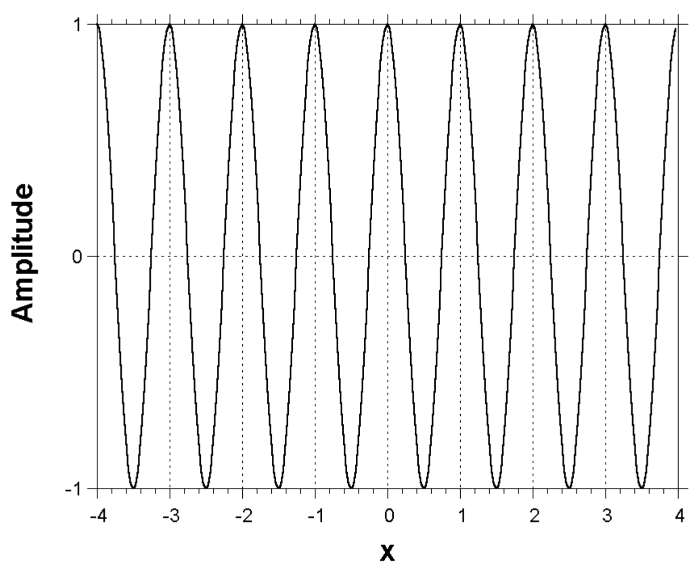
2. *Harmonic* functions produced by oscillatory motions

- common in electromagnetism (and therefore in optics), acoustics, classical and quantum mechanics
- harmonic function is composed of a single sinusoid

$$f[x] = A_0 \cos[\Phi[x]] = A_0 \cos\left[\frac{2\pi x}{X_0} + \phi_0\right]$$

$\Phi[x]$ is an *angle* (measured in radians) called the *phase angle*.

- $\Phi[x]$ must be a linear or a constant function of x for $f[x]$ to be harmonic.
- constant part ϕ_0 is angular argument at $x = 0$ and rightly should be called the *initial phase*
- concept of "phase" is generalized to nonsinusoidal, and even nonperiodic, functions



"Harmonic" function $f[x] = \cos[2\pi x]$, which is composed of a single sinusoidal component.

1.6.1 Harmonic Function of Space AND Time

$$f[x,t] = A_0 \cos[\Phi[x,t]] = A_0 \cos\left[\frac{2\pi x}{X_0} - \frac{2\pi t}{T_0} + \phi_0\right]$$

- Amplitude is A_0 wherever phase $\Phi[x,t]$ is integer multiple of 2π radians.
- Any location in 2-D space-time domain for which $\Phi[x,t] = 2\pi \times n$ is a maximum of the sinusoid.
- As t increases, the spatial position x of this particular maximum also must change to maintain the same phase.
  - x of the point with zero phase must increase as t increases
  - "point of constant phase" of the wave moves toward $x = +\infty$
  - Sinusoids of this form are called "traveling waves" in physics.
- rate of energy transfer of $f[x,t]$ is proportional to $(f[x,t])^2$, which is the "power" .
- period X_0 of traveling wave is the interval of x over which the phase changes by 2π radians
- X_0 is a "length" (e.g., mm).

Spatial Frequency:

- Often convenient to recast phase in terms of reciprocal of period $\xi_0 \equiv X_0^{-1}$
- specifies number of periods ("cycles") of sinusoid in one unit of independent variable x
- Because it is a "rate" of oscillation, ξ_0 is the *spatial frequency.*
- Equivalent expression for sinusoid is:

$$f[x] = A_0 \ \cos[\Phi[x]] = A_0 \ \cos[2\pi\xi_0 x + \phi_0]$$

- Common to measure ξ_0 in units of "cycles per mm".

Angular Spatial Frequency

- based upon number of radians of phase within one unit of length
- denoted by k_0 or σ_0
- often called *wavenumber* of spatial wave

$$f[x] = A_0 \ \cos[\Phi[x]] = A_0 \ \cos[k_0 x + \phi_0], \text{ where } k_0 = 2\pi\xi_0.$$

- angular spatial frequency most conveniently defined as spatial derivative of phase:

$$\text{Angular Spatial Frequency } k_0 = \frac{\partial\Phi[x]}{\partial x} \quad \text{[radians/unit length]}$$

  - angular spatial frequency is constant (function is harmonic) if $\Phi[x]$ is (at most) a linear function of x
- *spatial frequency* includes scale factor of 2π radians per cycle:

$$\text{Spatial Frequency } \xi_0 = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} \quad \text{[cycles/unit length]}$$

Temporal Frequencies

- parameter corresponding to ξ_0 is the *temporal frequency*, often indicated by ν_0
- units of "cycles per time interval" (e.g., "cycles per second" or *Hertz*).
- units of angular temporal frequency ω_0 are "radians per unit time", so that the conversion is $\omega_0 = 2\pi\nu_0$

$$\text{Angular Temporal Frequency } \omega_0 = \frac{\partial\Phi[t]}{\partial t} \quad \text{radians/unit time}$$
$$\text{Temporal Frequency } \nu_0 = \frac{1}{2\pi}\frac{\partial\Phi[t]}{\partial t} \quad \text{cycles/unit time}$$

Negative Frequencies

- ξ_0 defined as derivative $\Longrightarrow$ negative values are "allowed"
- phase *decreases* as corresponding coordinate (x or t) increases

$$f[x,t] = A_0 \;\cos[\Phi[x,t]] = A_0 \;\cos[2\pi\xi_0 x_0 - 2\pi\nu_0 t_0 + \phi_0]$$

- angular temporal frequency ω_0 is negative if position variable x *decreases* as t increases; the points of constant phase move toward $-\infty$.

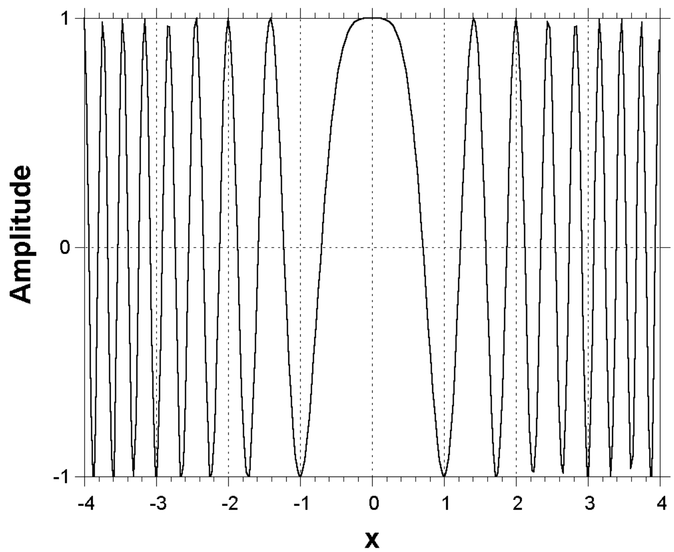
1.6.2 Specifying Sinusoids

- any sinusoidal function $f[x]$ may be specified by three parameters
  1. amplitude A_0
  2. period X_0 (or spatial frequency ξ_0 or angular spatial frequency k_0)
  3. initial phase ϕ_0.
- This equivalence is basis for alternate representation of a function obtained via Fourier transform.
- Independent variable x and its corresponding frequency ξ_0 (or the time t and temporal frequency ν_0) are *conjugate* variables.

1.6.3 Anharmonic Sinusoids

- Harmonic function is sinusoid whose frequency (spatial or temporal) does not vary with location (in space or time, or both)
- Anharmonic sinusoid: phase includes terms of order larger than 1

$$\begin{aligned} f[x] &= A_0 \;\cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2 + \phi_0\right] \\ \xi &= \xi[x] = \frac{1}{2\pi}\frac{\partial\Phi}{\partial x} = \frac{1}{2\pi}\left(\frac{2\pi x}{\alpha_0^2}\right) = \frac{x}{\alpha_0^2} \propto x \end{aligned}$$



Sinusoidal function with quadratic phase, $f[x] = \cos[\pi x^2]$.

1.7 SYMMETRY PROPERTIES of FUNCTIONS

1.7.1 *even* and *odd* parts

- $$f_e[x] = f_e[-x]$$
  $$f_o[x] = -f_o[-x]$$
  $$f[x] = f_e[x] + f_o[x]$$

- even part of $f[x]$ is *symmetric* with respect to the origin
- odd part is *antisymmetric*
- Easy to show that even and odd parts of $f[x]$ are evaluated from $f[x]$ via:

$$f_e[x] \equiv \frac{1}{2}(f[x] + f[-x])$$
$$f_o[x] \equiv \frac{1}{2}(f[x] - f[-x]).$$

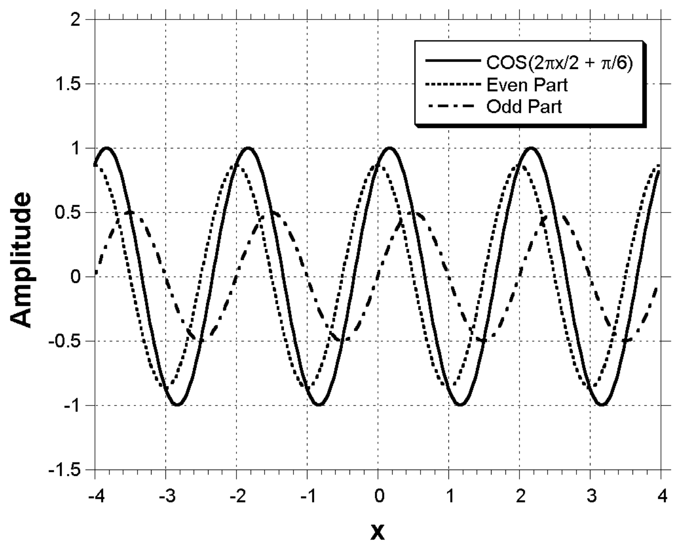
- $\cos[2\pi\xi_0 x]$ is an even harmonic function
- $\sin[2\pi\xi_0 x]$ is odd
- General harmonic function with spatial frequency ξ_0 and initial phase ϕ_0 is decomposed into constituent even and odd parts via:

$$\cos[\alpha \pm \beta] = \cos[\alpha] \ \cos[\beta] \mp \sin[\alpha] \ \sin[\beta]$$

$$\begin{aligned} f[x] = \cos[2\pi\xi_0 x + \phi_0] &= \cos[2\pi\xi_0 x] \ \cos[\phi_0] - \sin[2\pi\xi_0 x] \ \sin[\phi_0] \\ &= \cos[\phi_0] \ \cos[2\pi\xi_0 x] + (-\sin[\phi_0]) \ \sin[2\pi\xi_0 x] \end{aligned}$$

$$f_e[x] = \cos[\phi_0] \ \cos[2\pi\xi_0 x]$$
$$f_o[x] = (-\sin[\phi_0]) \ \sin[2\pi\xi_0 x]$$

- sum of two sinusoids with same frequency and arbitrary amplitudes and phases yields a sinusoid with that same frequency.



Sinusoidal function $f[x] = \cos\left[2\pi\frac{x}{2} + \frac{\pi}{6}\right]$ decomposed into its even and odd parts: $f_e[x] = \frac{\sqrt{3}}{2}\cos[\pi x]$ and $f_o[x] = \frac{1}{2}\sin[\pi x]$.

1.7.2 Symmetry of 2-D Functions:

$$
\begin{aligned}
f[x,y] &= f_e[x,y] + f_o[x,y] \\
f_e[x,y] &\equiv \frac{1}{2}\left(f[x,y] + f[-x,-y]\right) \\
f_o[x,y] &\equiv \frac{1}{2}\left(f[x,y] - f[-x,-y]\right)
\end{aligned}
$$

$$
\begin{aligned}
f_e(r,\theta) &= f_e(r,\theta \pm n\pi) \\
f_o(r,\theta) &= -f_o(r,\theta \pm n\pi)
\end{aligned}
$$