SIMG-716
Linear Imaging Mathematics I
01 - Introduction

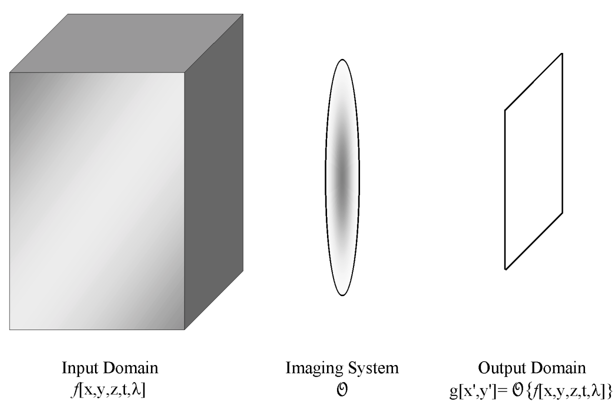
1 Mathematical descriptions of imaging systems

$$\text{Input to Imaging System: } f[x,y,z,\lambda,t] \text{ or } f[x,y,z,\nu,t]$$

$$\text{Output of Imaging System: } g[x',y',z',\lambda,t] \to g[x',y']$$

$$\begin{aligned} \text{Action of Imaging System is specified by the operator } \mathcal{O} &: \\ \mathcal{O}\{f[x,y,z,t,\lambda\} &= g[x',y'] \end{aligned}$$

1. Representations are mathematical functions
   - Rules that assign a single number, the amplitude, to each location in the multidimensional domain
   - amplitude may be complex, e.g., $f[x,y] = a[x,y] + i\ b[x,y]$, where $i \equiv \sqrt{-1}$
2. Mathematical description of system action is a relationship between the two functional representations
   - Action may relate input and output at same location (point operation): e.g., $g[x,y] = -f[x,y]$
   - Action at one output location may combine input amplitudes over a local region of the input (neighborhood operation), or over the entire input function (global operation)



Schematic of imaging system that acts on a time-varying input with three spatial dimensions and color, $f[x,y,z,t,\lambda]$ to produce 2-D monochrome (gray scale) image $g[x',y']$.

2 Three Imaging "Tasks"

1. *Forward* or *direct* problem:

   find the mathematical expression for the image $g[x', \cdots]$ given complete knowledge of the input object $f[x, \cdots]$ and the system $\mathcal{O}$;

2. *Inverse problem*:

   the input $f[x, \cdots]$ is evaluated from the measured image $g[x', \cdots]$ and the known action of the system $\mathcal{O}$

3. *System analysis* problem:

   the action of the operator $\mathcal{O}$ must be determined from the input $f[x, \cdots]$ and the image $g[x', \cdots]$.

   - (similar in form to the inverse problem in many cases)

Variants:

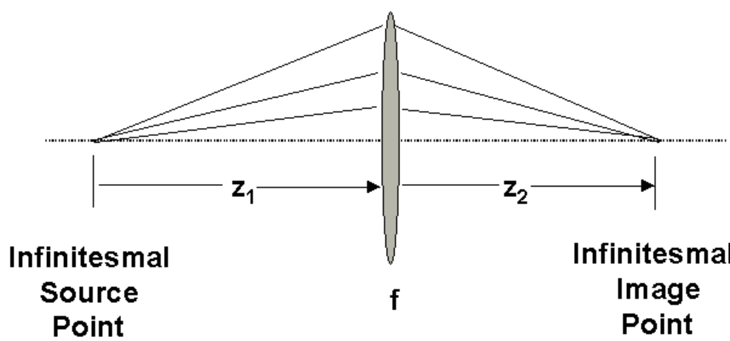
1. knowledge of some or all of f, g, and $\mathcal{O}$ may be incomplete
2. Output image has been contaminated by random noise in inverse or system analysis task.

3 Examples of Imaging Tasks:

1. Optics
   (a) Ray (geometrical) Optics
   (b) Wave (physical) Optics
   (c) System Analysis of HST
   (d) Aberrations due to Atmosphere
2. Medical Imaging
   (a) Gamma-ray imaging (nuclear medicine)
      i. Pinhole Camera
      ii. Multiple Apertures
      iii. Coded Apertures
   (b) Radiography (X-ray imaging)
      i. Information loss in one image
      ii. Multiple-angle radiography (computed tomography)

3.1 Optical Models of Imaging

3.1.1 Imaging Tasks in Ray Optics Model:



Perfect optical imaging system in "ray optics" model with no aberrations. All rays from a single source point converge to "image" point.

1. Point source of energy emits geometrical "rays" of light that propagate in straight lines to infinity in all directions.
2. Imaging "system" is an optical "element" that interacts with any ray it intercepts.
3. Interaction mechanism of system is a physical process (usually refraction or reflection, sometimes diffraction) that "diverts" the ray from its original direction.
4. Rays are collected by a sensor

Mathematical Model of Ray-Optics Imaging:

1. *System* is specified by the focal length $\mathbf{f}$
2. *Input* by the "object distance" z_1
3. *Output* by the "image distance" z_2
4. *Imaging Equation* relates the three quantities

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{\mathbf{f}}$$

Solutions to Imaging Tasks:

1. Direct Task: Given z_1 and $\mathbf{f}$, find z_2:

$$\left(\frac{1}{\mathbf{f}} - \frac{1}{z_1}\right)^{-1} = z_2$$

2. Inverse Task: Given z_2 and $\mathbf{f}$, find z_2:

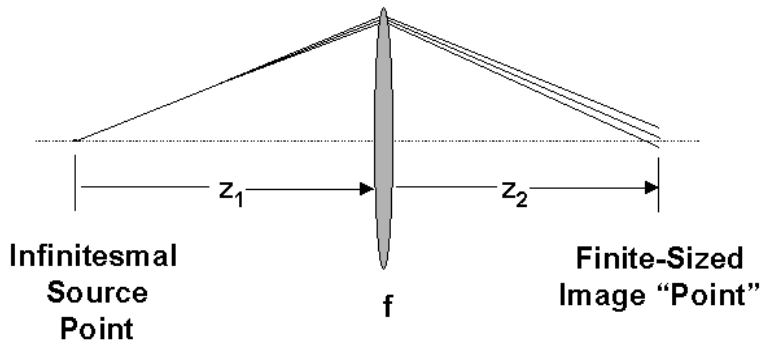
$$\left(\frac{1}{\mathbf{f}} - \frac{1}{z_2}\right)^{-1} = z_1$$

3. System Analysis Task: Given z_1 and z_2, find $\mathbf{f}$:

$$\left(\frac{1}{z_1} + \frac{1}{z_2}\right)^{-1} = \mathbf{f}$$

- Imaging equation "pairs up" object and image planes located at distances z_1 and z_2 for a "fixed system" with focal length **f.**
- Imaging system "maps" source planes at distances $\{z_1\}$ to image planes.$\{z_2\}$
- We rarely seem to think about action of lens in other cases, e.g.,
  - on 3-D objects, so that several object distances exist that are mapped to different image distances
  - on "out-of-focus" planar objects located at a distance z_1 that does not satisfy the imaging equation for a fixed focal length $\mathbf{f}$ and image distance z_2, even though there are many obvious situations where we might want to calculate the appearance of such an image.

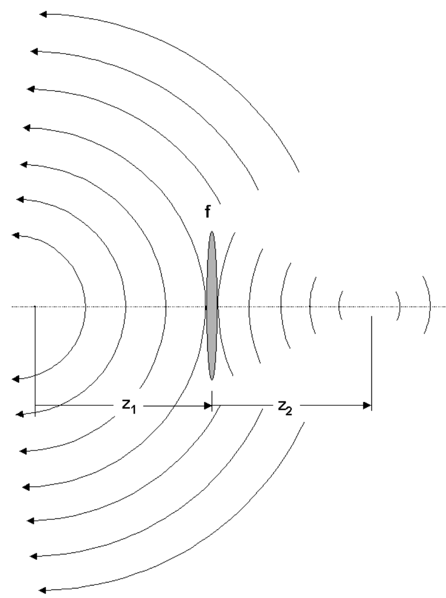
3.1.2 Wave Optics Model:



"Ray" model of optical imaging including diffraction; rays "spread" while propagating to lens. Image is not a single image "point", but rather "blurry" spot.

- Rays may be modeled, but diffraction creates "spread" of ray, or "multiple rays"
- Difficult to quantify action of imaging system with diffraction on rays
- More convenient to describe the light as a "wave" instead of a "ray" in models of optical systems w/diffraction.
  - Each source point emits spherical waves of e-m radiation
    - propagates outward from the source at velocity of light.
    - radiation at all points on one wave surface emitted at same instant of time in a direction perpendicular to wave surface, thus corresponding to the ray in the simpler model.

Wave Model:



Wave model of light propagation. Several spherical waves are shown that were emitted by point source at times separated by identical intervals. Lens intercepts portion of each wave and alters curvatures. Waves propagate until converging (more or less) to a finite-sized image point.

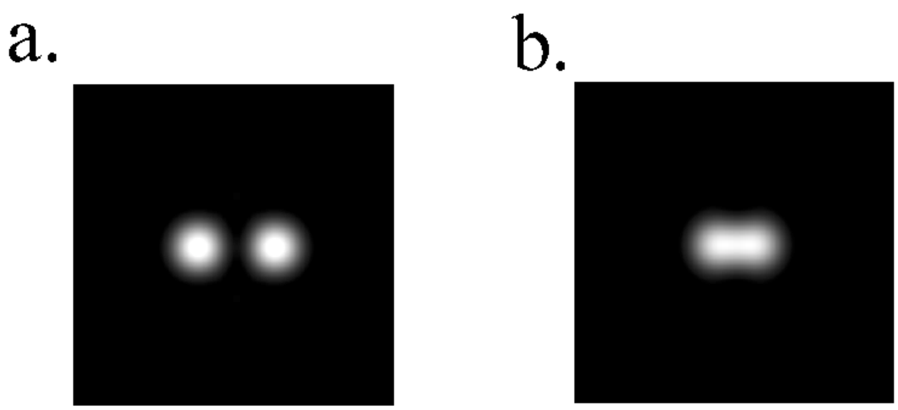
- Function that describes spherical wave is valid everywhere, not than just along individual ray.
- Optical element (typically a lens or mirror) intercepts portion of each spherical wave and changes its radius of curvature
- Picture suggests another interpretation of the action of optical system
  - System attempts to produce "replica" of source at new "image" location
  - Success of reproduction is demonstrated by size of image of point source
    - size *decreases* as fidelity improves.
  - Quality of image produced by "flawless" optical system (without aberrations) *improves* if system intercepts larger portion of outgoing wave.
    - size of diffracted "point image" decreases as size of optic increases (assuming no aberrations).

Simulations of images from optical system in wave model (with diffraction)

Images of point source through aberration-free optics with diameters d_1 and $2\,d_1$, respectively.

Concept of "Resolution"

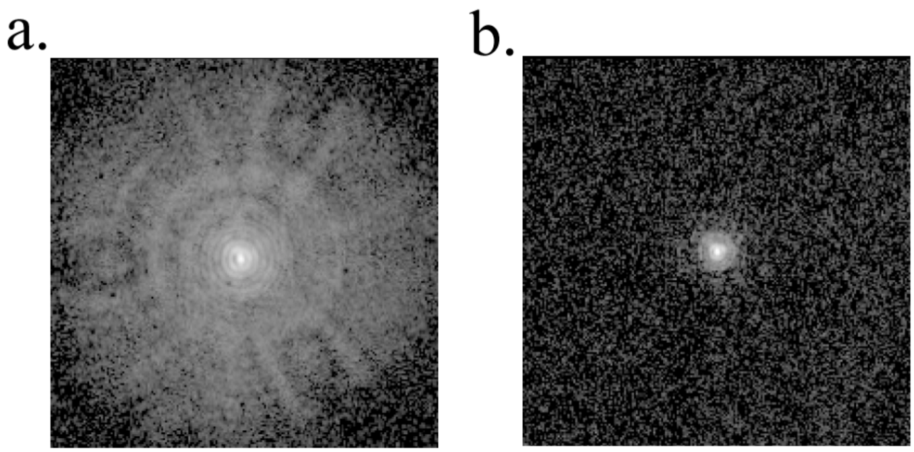
1. • Overlapping "blurred" images may be difficult or impossible to distinguish.
   • Details of Image are "averaged" together.



Effect of diffraction on ability to resolve objects and fine structure in image. Original object is pair of point objects (e.g., double stars) at two different separations viewed through aberration-free system. In (a), object clearly consists of two disjoint stars; less convincing in (b).

3.1.3 Analysis of HST

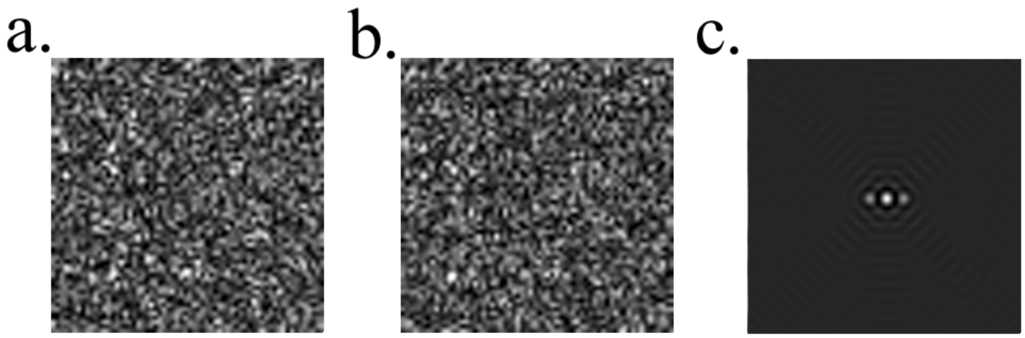
- System Analysis Task: Given knowledge of input $f[x, y]$ and output $g[x, y]$, find the mathematical equation for the action of the system $\mathcal{O}$.
- Find formula for corrector optics ($COSTAR$).



Images of star from HST before and after COSTAR. (a) exhibits spherical aberration of primary mirror.

3.1.4 Imaging by Ground-Based Telescopes:

1. Aberrations in optical propagation induced by the atmosphere
   (a) Localized "tilts" and "defocus" to incoming plane wave
      i. local variations in index of refraction
   (b) Varies with time
   (c) Resolution limited by atmosphere, not aperture
2. Modify the system to deduce new information about the object from the image



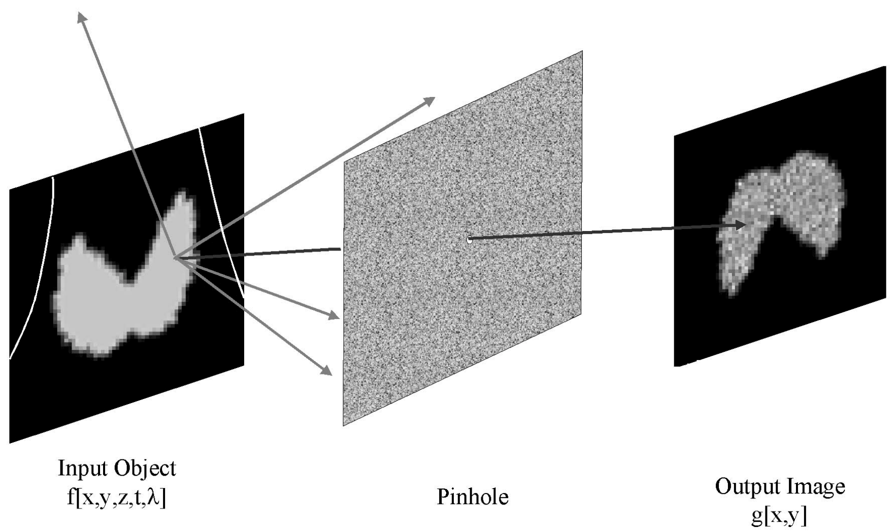
Simulation of stellar speckle interferometry: (a) image obtained from single star through a large telescope and turbulent atmosphere. Note "speckles" due to localized refraction ; (b) image of double star through same atmosphere; (c) image of double star after image processing to compute "autocorrelation" of original object.

3.2 Medical Imaging Applications

3.2.1 Gamma-Ray Imaging

Kinetic energy of gamma rays emitted by technetium is ≃140 keV

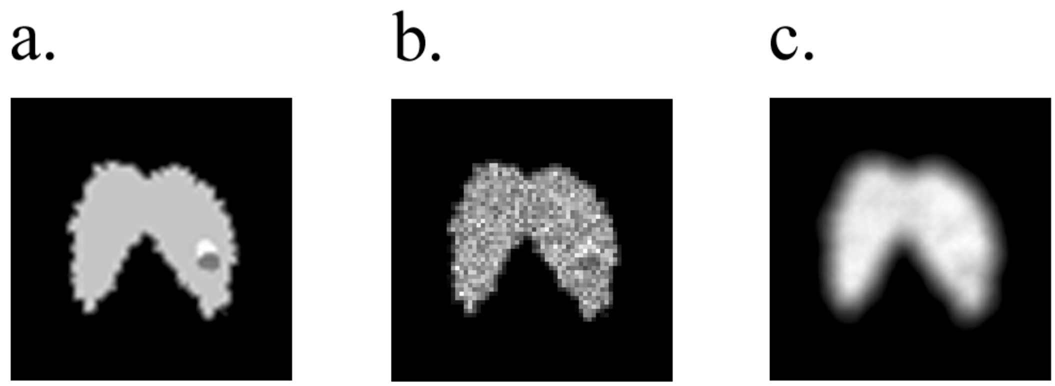
- $E \simeq 140keV \simeq 2.24 \times 10^{-7}$ *ergs* per photon
- Gamma rays have very short wavelength: $\lambda_\gamma = \frac{hc}{E} \simeq 9 \times 10^{-12}m << \lambda_{vis} \simeq 5 \times 10^{-7}m$
- Gamma-ray photons pass through optical refractors virtually without deviation, and reflect only at very shallow (i.e., "grazing") angles of incidence (e,g, *Chandra* Orbiting Observatory)



Schematic of gamma-ray pinhole imaging. Planar object $f[x,y]$ emits energetic gamma-ray photons in all directions. A small percentage travel along paths that pass through pinhole in lead plate to expose sensor and form planar image $g[x,y]$.

Resolution of Gamma-Ray Imaging System

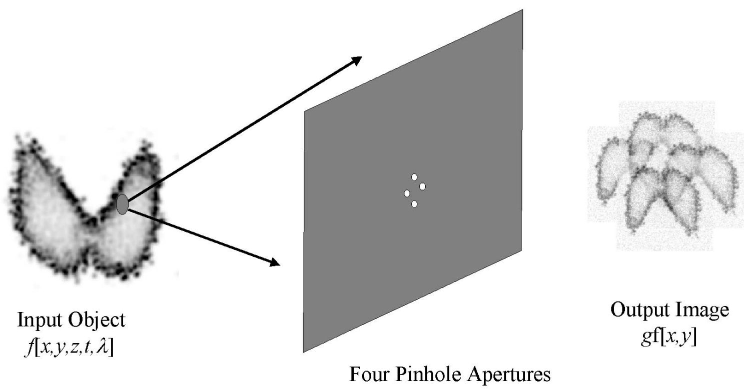
1.
    - Spatial resolution depends on d, diameter of pinhole
        - smaller hole $\Longrightarrow$ better spatial resolution
    - "Brightness" resolution depends on number of photons counted
        - Poisson noise in number of counted photons
        - larger hole $\Longrightarrow$ better "brightness" resolution
    - A QUANDARY!
2. Modify the system to address the problem!



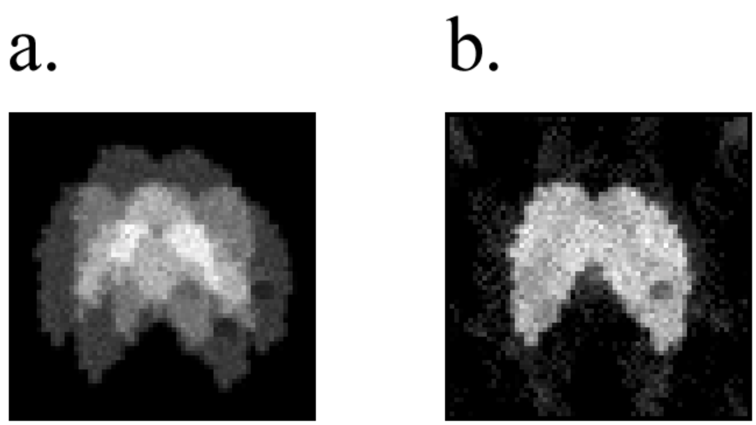
Simulation of effect of pinhole diameter d on spatial resolution. (a) $f[x, y]$ is simulated thyroid with "hot" & "cold" spots (b) Simulated image obtained using small pinhole w/ added noise due to small number of counted photons. (c) Simulated image with large d; noise is reduced by averaging but image is "blurred" due to exposure of different points on sensor by photons from same object point.

Multiple Pinhole Imaging of Gamma Rays

1. Collect images through several small pinholes at once
2. Images may overlap
3. Postprocess to reconstruct image of object with improved signal-to-noise ratio (SNR)



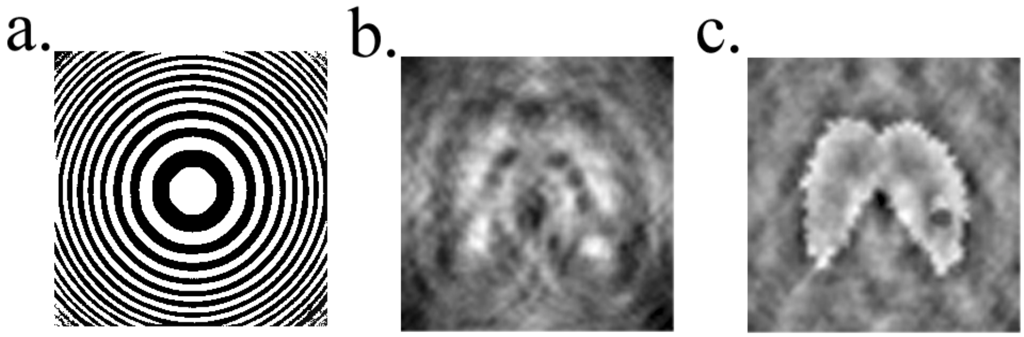
Gamma-ray camera with four pinholes, producing four overlapping images.



Simulation of "raw" output and subsequently processed gamma-ray images from camera with four pinholes (a) Raw image is the sum of four overlapping images. (b) The result after processing the image to "merge" the four overlapping images.

Coded Aperture Imaging of Gamma Rays

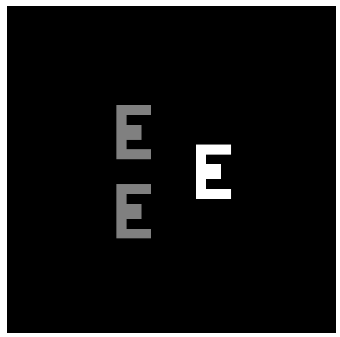
1. Drill out lead plate to open more "pinholes"
2. Pinholes merge together to form regions of "open space"
   (a) 50% of the lead removed in this example
   (b) Many more photons transmitted to detector
   (c) Images from "pinholes" overap.
3. Pattern of detected photons is processed by a mathematical algorithm based on the pattern of pinholes
4. Postprocess to "reconstruct" (approximation of) original object with improved SNR.

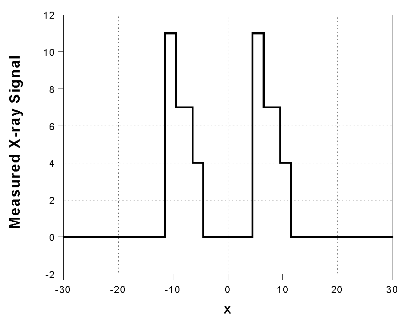


Imaging through "multiple" pinholes in configuration shown in (a) ("coded aperture"); (b) "Image" obtained through this aperture; (c) "Reconstructed" image obtained by postprocessing of (b) using knowledge of (a).

3.2.2 Radiography (X-Ray Imaging)

1. X rays propagate in straight lines (as rays)
2. Absorption by interactions with object reduces the flux of x rays
3. Transmitted x rays are measured by detector
   (a) 3-D object $f[x,y,z]$ is measured as a 2-D image, e.g., $g[x,y]$
   (b) Example shows a 2-D object $f[x,y]$ that is imaged as the 1-D function $g[x]$
4. All "depth" information about the absorption is lost; absorption could have occurred anywhere along the path.





Simulation of radiographic imaging. The object is a 2-D function $f[x,z]$ that describes the x-ray attenuation "map", where "black" and "white" represent complete and no attenuation, respectively. The 1-D image is $g[x]$, shown at bottom. Note that different structures in object (two "E"s on left, single denser "E" on right) produce same feature in image.

3.2.3 Computed Tomographic Radiography (CT)

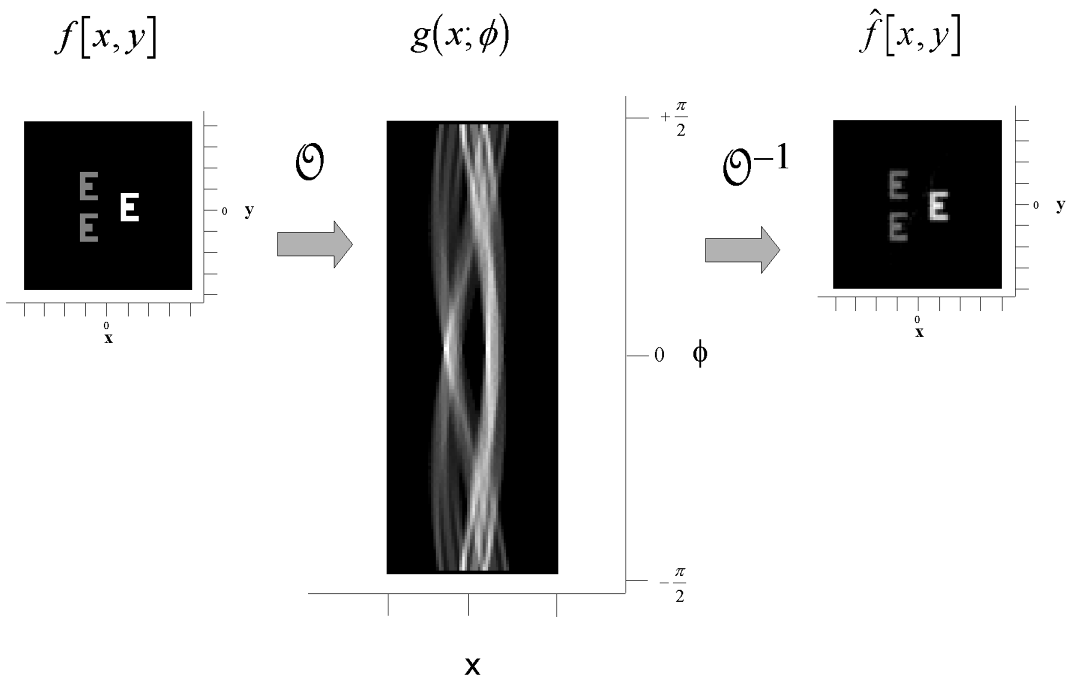
1. Modify the X-ray imaging system

2. Measure x-ray absorption of object over complete set of azimuth angles.

   (a) Example shown for 2-D object $f[x, y]$

   (b) 1-D image $g[x]$ measured for each azimuth angle ϕ, thus producing a new 2-D "hybrid" image $g(x, \phi)$

3. Recorded image $g(x, \phi)$ is postprocessed to compute an estimate of the original object $\hat{f}[x, y]$.



Simulation of x-ray computed tomography system. X-ray transmission is measured at each of the set of angles ϕ to compute the "image" $g(x; \phi)$. An estimate $\widehat{f}[x, y]$ is computed from knowledge of $g(x; \phi)$ and the imaging system.

4 Goals of Course:

1. Develop an intuitive grasp of mathematical methods for describing the action of a general linear system on signals of one or more spatial dimensions.

   - i.e., we want to "develop images" of the mathematics

2. Develop a consistent mathematical formalism for characterizing linear imaging systems; requires derivation of equations used to describe:

   - the action of the imaging system
   - its effect on the *quality* of output image

3. Derive representations of images that are defined over:

   - a continuous domain, convenient for describing:
     - realistic objects

- – realistic imaging systems
     - resulting images
   - a discrete domain, essential for computer modeling of
     - objects
     - systems
     - images
   - continuous range
   - discrete range
   - continuous and discrete ranges
     - Representations in discrete coordinates (that is, using *sampled* functions) are essential for modeling general objects, images, and systems in a computer. Discrete images and systems usually are represented most conveniently as vectors and matrices