1051-716-20111 Final Exam Solutions

(Statistics on last page)

1. (Essay) In class, I stated that the recipes for the inverse and matched filters are related, even though the applications seem to be quite different. Write a few paragraphs with appropriate equations that compare and contrast both the applications and the recipes for the two filters. Show how the form of the "approximate" filter for one application may be applied to the other. Feel free to use sketches to illustrate your points.

   Solution: *The transfer function $W[\xi]$ for the ideal inverse filter $w[x]$ is the reciprocal of the transfer function $H[\xi]$ of the original filter $h[x]$:*

$$W[\xi] = \frac{1}{H[\xi]} = H^*[\xi] \cdot \frac{1}{|H[\xi]|^2}$$

   In the second expression, the first term includes the correction for the intrinsic phase of the filter and the second term is a "double amplifier" that corrects for the attenuation of the two terms $H[\xi]$ and $H^*[\xi]$. The associated impulse response is the inverse Fourier transform:

$$w[x] = \mathcal{F}_1^{-1}\{W[\xi]\} = \mathcal{F}_1^{-1}\left\{H^*[\xi] \cdot \frac{1}{|H[\xi]|^2}\right\} = h^*[-x] * \mathcal{F}_1^{-1}\left\{\frac{1}{|H[\xi]|^2}\right\}$$

   The transfer function $M[\xi]$ of the ideal matched filter $m[x]$ is the reciprocal of the spectrum $F[\xi]$ of the "original" test object $f[x]$

$$\begin{aligned} M[\xi] &= \frac{1}{F[\xi]} = F^*[\xi] \cdot \frac{1}{|F[\xi]|^2} \\ m[x] &= f^*[-x] * \mathcal{F}_1^{-1}\left\{\frac{1}{|F[\xi]|^2}\right\} \end{aligned}$$

   To avoid amplifying frequencies where noise is dominant, the "classical" matched filter does not implement the double amplification by deleting the last term; the classical matched filter calculates the autocorrelation of the object centered at its location:

$$\begin{aligned} \hat{M}[\xi] &= \frac{1}{F[\xi]} = F^*[\xi] \cdot 1[\xi] \\ \hat{m}[x] &= f^*[-x] * \delta[x] = f^*[-x] \end{aligned}$$

   We can apply the "classical" approximation to the ideal matched filter by defining:

$$\hat{W}[\xi] = H^*[\xi] \cdot 1[\xi] \implies h^*[-x] * \delta[x] = h^*[-x]$$

   This will correct for the phase without correcting for the attenuated frequencies.

2. A general 1-D complex valued function $f[x] = \text{Re}\{f[x]\} + i \cdot \text{Im}\{f[x]\}$ may be decomposed into four components based on symmetry (even or odd) and complex character (real or imaginary).

   (a) Write down the equations for deriving the four component parts of $f[x]$.

   Solution: *we know the definitions of the even and odd parts:*

$$
\begin{aligned}
f_{even}[x] &= \frac{f[x] + f[-x]}{2} \\
f_{odd}[x] &= \frac{f[x] - f[-x]}{2}
\end{aligned}
$$

and the definitions of the real and imaginary parts:

$f[x] = \text{Re}\{f[x]\} + i \cdot \text{Im}\{f[x]\}$ *where* $\text{Re}\{f[x]\}$ *and* $\text{Im}\{f[x]\}$ *are both real valued*

These lead to the expressions that

$$
\begin{aligned}
\text{Re}\{f[x]\} &= (\text{Re}\{f[x]\})_{even} + (\text{Re}\{f[x]\})_{odd} \\
& \boxed{(\text{Re}\{f[x]\})_{even} = \frac{\text{Re}\{f[x]\} + \text{Re}\{f[-x]\}}{2}} \\
& \boxed{(\text{Re}\{f[x]\})_{odd} = \frac{\text{Re}\{f[x]\} - \text{Re}\{f[-x]\}}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\text{Im}\{f[x]\} &= (\text{Im}\{f[x]\})_{even} + (\text{Im}\{f[x]\})_{odd} \\
& \boxed{(\text{Im}\{f[x]\})_{even} = \frac{\text{Im}\{f[x]\} + \text{Im}\{f[-x]\}}{2}} \\
& \boxed{(\text{Im}\{f[x]\})_{odd} = \frac{\text{Im}\{f[x]\} - \text{Im}\{f[-x]\}}{2}}
\end{aligned}
$$

(b) Determine the character (real, imaginary, or complex and even, odd, or neither) the Fourier transform of each of the four component parts.

We know that the "kernel" $\exp[-2\pi i(\xi x+\eta y)]$ *of the Fourier transform is Hermitian (real part is even – cosine – and imaginary part is odd – sine).*

1. *If the input function* $f[x]$ *is real and even, the product* $f[x]\cdot\exp[-2\pi i(\xi x+\eta y)]$ *has an even real part and odd imaginary part; the area of the odd imaginary part must be zero, so the Fourier transform of an even real function must be even and real.*
2. *If the input function is real and odd, the product* $f[x]\cdot\exp[-2\pi i(\xi x+\eta y)]$ *has an odd real part (odd × even is odd) and even imaginary part (odd × odd is even); the area of the odd real part must be zero, so the Fourier transform of an odd real function must be imaginary and odd.*
3. *If the input function* $f[x]$ *is imaginary and even, the product* $f[x]\cdot\exp[-2\pi i(\xi x+\eta y)]$ *has an even imaginary part and odd real part; the area of the odd reao part must be zero, so the Fourier transform of an even imaginary function must be imaginary and odd.*
4. *If the input function is imaginary and odd, the product* $f[x]\cdot\exp[-2\pi i(\xi x+\eta y)]$ *has an odd imaginary part and even real part (i·odd × i·odd is real and even); the area of the odd imaginary part must be zero, so the Fourier transform of an odd imaginary function must be real and odd.*

$$\boxed{\textit{even and real part of } f[x]\rightarrow \textit{even and real part of } F[\xi]}$$

$$\boxed{\textit{odd and real part of } f[x]\rightarrow \textit{odd and imaginary part of } F[\xi]}$$

$$\boxed{\textit{even and imaginary part of } f[x]\rightarrow \textit{even and imaginary part of } F[\xi]}$$

$$\boxed{\textit{odd and imaginary part of } f[x]\rightarrow \textit{odd and real part of } F[\xi]}$$

3. Consider the 2-D convolution:

$$p[x,y] = \left(J_0\left[2\pi\sqrt{\frac{x^2+y^2}{4}}\right]\cdot\exp\left[+i\pi\frac{x}{2}\right]\right) * \left(J_0\left[2\pi\sqrt{\frac{x^2+y^2}{4}}\right]\cdot\exp\left[-i\pi\frac{x}{2}\right]\right)$$

Evaluate the result of this convolution up to a constant; in other words, you need derive only the *form* of the function, not its amplitude. *HINT: Sketch(es) are essential to solve this problem.*

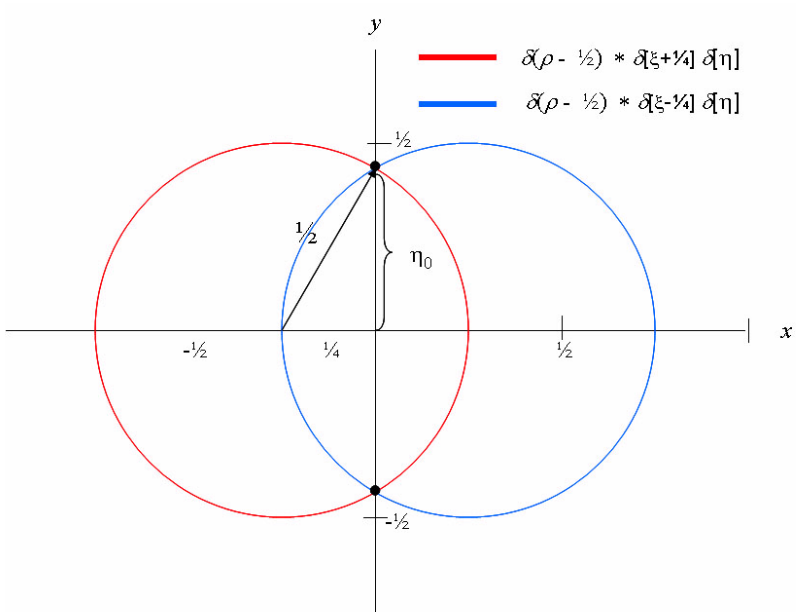
Solution: *as an attempt at clarity, let's define names for the two component functions:*

$$\begin{aligned}
f_1[x,y] &\equiv J_0\left[2\pi\sqrt{\frac{x^2+y^2}{4}}\right]\cdot\exp\left[+i\pi\frac{x}{2}\right] \\
&= J_0\left[2\pi\sqrt{\frac{r^2}{4}}\right]\cdot\exp\left[+2\pi i\frac{x}{4}\right]\cdot 1[y] \\
&= J_0\left[2\pi\cdot r\cdot\frac{1}{2}\right]\cdot\exp\left[+2\pi i\cdot x\cdot\frac{1}{4}\right]\cdot 1[y] \Longrightarrow \rho_0=\frac{1}{2}\ \textit{and}\ \xi_0=\frac{1}{4} \\
f_2[x,y] &\equiv J_0\left[2\pi\sqrt{\frac{x^2+y^2}{4}}\right]\cdot\exp[-2\pi i x] \\
&= J_0\left[2\pi\cdot r\cdot\frac{1}{2}\right]\cdot\exp\left[-2\pi i\cdot x\cdot\frac{1}{4}\right]\cdot 1[y] \\
&= J_0[2\pi\cdot r\cdot\rho_0]\cdot\exp[+2\pi i\cdot x\cdot\xi_0]\cdot 1[y] \\
\rho_0 &= \frac{1}{2},\xi_0=-\frac{1}{2}
\end{aligned}$$

Note that $f_2[x,y]=(f_1[x,y])^*$

$$\begin{aligned}
p[x,y] &= \mathcal{F}_2^{-1}\{\mathcal{F}_2\{f_1[x,y]\}\cdot\mathcal{F}_2\{f_2[x,y]\}\} \\
\mathcal{F}_2\{f_1[x,y]\} &= \mathcal{F}_2\left\{J_0\left[2\pi\cdot r\cdot\frac{1}{2}\right]\right\} * \mathcal{F}_2\left\{\exp\left[+2\pi i\cdot x\cdot\frac{1}{4}\right]\cdot 1[y]\right\} \\
&= \delta\left(\rho-\frac{1}{2}\right) * \left(\delta\left[\xi+\frac{1}{4}\right]\cdot\delta[\eta]\right) \\
&= \delta\left(\sqrt{\left(\xi+\frac{1}{4}\right)^2+\eta^2}-\frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_2\{f_2[x,y]\} &= \delta\left(\rho-\frac{1}{2}\right) * \left(\delta\left[\xi-\frac{1}{4}\right]\cdot\delta[\eta]\right) \\
&= \delta\left(\sqrt{\left(\xi-\frac{1}{4}\right)^2+\eta^2}-\frac{1}{2}\right)
\end{aligned}$$



Spectra of the two component functions $f_1[x,y]$ and $f_2[x,y]$, which are seen to be ring delta functions that intersect at two locations on the η-axis. This shows that the space-domain representation of the convolution will be a cosine along the y-axis.

From sketch, we see that the intersection of the two ring delta functions occur on the η axis (meaning that $\xi = 0$). We can find the intersections using trivial trigonometry:

$$\eta_0 = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{3}}{4} \cong 0.433$$

So the intersection of the two rings creates a pair of Dirac delta functions:

$$\delta\left(\sqrt{\left(\xi + \frac{1}{4}\right)^2 + \eta^2} - \frac{1}{2}\right) \cdot \delta\left(\sqrt{\left(\xi - \frac{1}{4}\right)^2 + \eta^2} - \frac{1}{2}\right)$$
$$= A_0\left(\delta[\xi] \cdot \delta[\eta + -\eta_0] + \delta[\xi] \cdot \delta[\eta - \eta_0]\right)$$

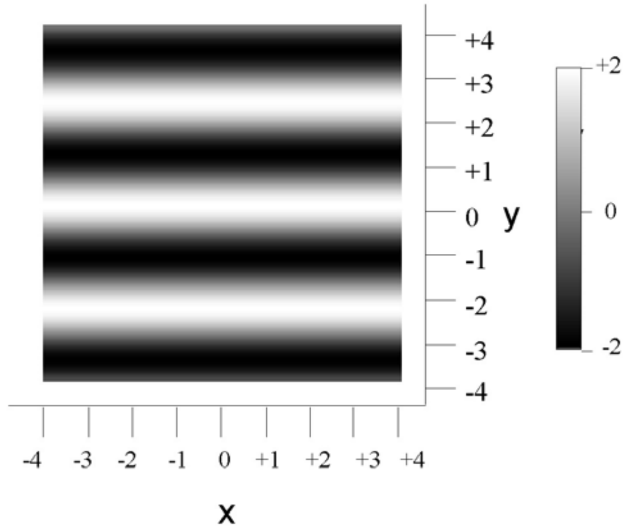
where $\eta_0 = \frac{\sqrt{3}}{4}$

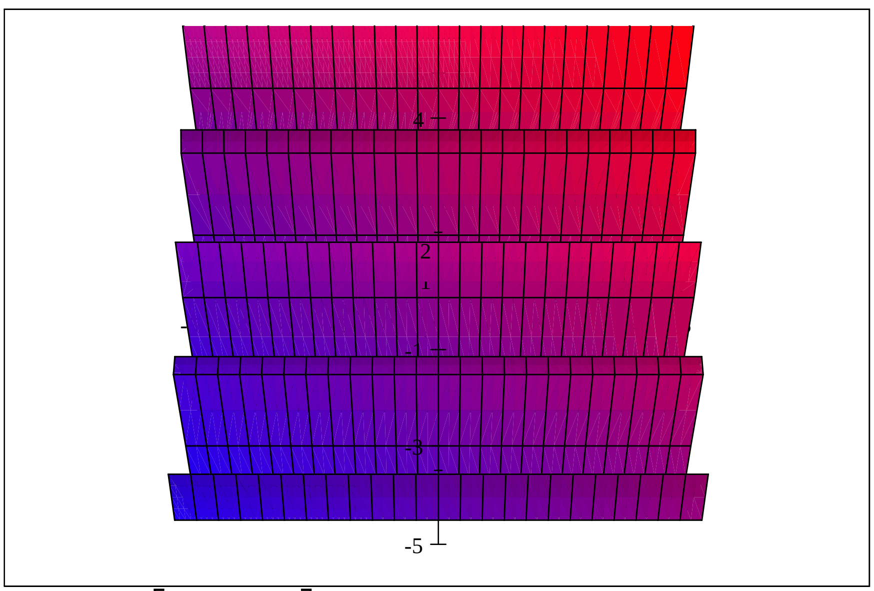
$$p[x,y] = \mathcal{F}_2^{-1}\left\{A_0\left(\delta[\xi] \cdot \delta[\eta + -\eta_0] + \delta[\xi] \cdot \delta[\eta - \eta_0]\right)\right\} = 2 \cdot 1[x] \cdot A_0 \cdot \cos[2\pi\eta_0 y]$$

$$p[x,y] \propto 2 \cdot 1[x] \cdot \cos\left[2\pi\frac{\sqrt{3}}{4}y\right] = 2 \cdot 1[x] \cdot \cos\left[2\pi\frac{y}{\left(\frac{4}{\sqrt{3}}\right)}\right]$$

We did not calculate the exact weighting of the Dirac delta functions in the intersection.

$$p[x,y] = 2 \cdot 1[x] \cdot \cos\left[2\pi \frac{y}{\left(\frac{4}{\sqrt{3}}\right)}\right]$$





$p\left[x,y\right] = 2 \cdot 1\left[x\right] \cdot \cos\left[2\pi \frac{y}{\left(\frac{4}{\sqrt{3}}\right)}\right]$ *where the x axis runs left-right and the y axis runs bottom-top*

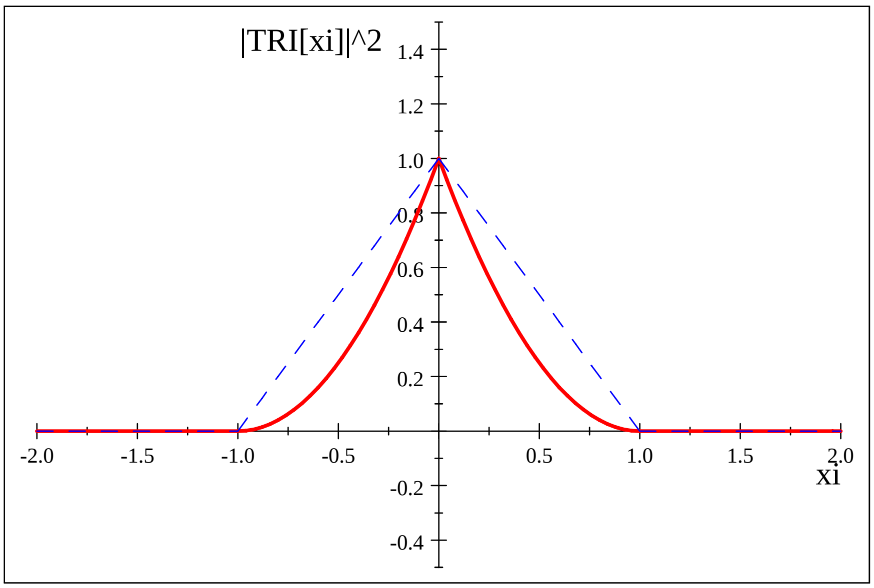
4. Determine the area of $f[x] = (SINC[x])^4$

SOLUTION:

$$\begin{aligned} Parseval\text{'s Theorem} &: \int_{-\infty}^{+\infty} |f[x]|^2 \; dx = \int_{-\infty}^{+\infty} |F[\xi]|^2 \; d\xi \\ (SINC[x])^4 &= |f[x]|^2 \implies f[x] = SINC^2[x] \\ &\implies F[\xi] = TRI[\xi] \end{aligned}$$

$$\int_{-\infty}^{+\infty} \left|SINC^2[x]\right|^2 \; dx = \int_{-\infty}^{+\infty} |TRI[\xi]|^2 \; d\xi$$

$$|TRI[\xi]|^2 = \begin{cases} 0 & \text{if} \quad \xi < -1 \\ (1+\xi)^2 & \text{if} \quad -1 \leq \xi \leq 0 \\ (1-\xi)^2 & \text{if} \quad 0 \leq \xi \leq 1 \\ 0 & \text{if} \quad \xi > 1 \end{cases}$$



$TRI[\xi]$ (blue dash) and $|TRI[\xi]|^2$ (red solid)

$$
\begin{aligned}
\int_{-\infty}^{+\infty} |TRI\,[\xi]|^2 \; d\xi &= \int_{-1}^{0} (1+\xi)^2 \; d\xi + \int_{0}^{+1} (1-\xi)^2 \; d\xi \\
&= \int_{-1}^{0} \left(1+2\xi+\xi^2\right) \; d\xi + \int_{0}^{+1} \left(1-2\xi+\xi^2\right) \; d\xi \\
&= \int_{-1}^{+1} 1 \; d\xi + 2 \cdot \left(\int_{-1}^{0} \xi \; d\xi - \int_{0}^{1} \xi \; d\xi \right) + \int_{-1}^{+1} \xi^2 \; d\xi \\
&= (+1-(-1)) + 2 \cdot \left(\left. \frac{\xi^2}{2} \right|_{\xi=-1}^{\xi=0} - \left. \frac{\xi^2}{2} \right|_{\xi=0}^{\xi=1} \right) + \left. \frac{\xi^3}{3} \right|_{\xi=-1}^{\xi=+1} \\
&= 2 + 2 \cdot \left(-\frac{1}{2} - \frac{1}{2} \right) + \left(\frac{1}{3} - \left(-\frac{1}{3} \right) \right) \\
&= 2 - 2 \cdot (-1) + \frac{2}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\boxed{\int_{-\infty}^{+\infty} (SINC\,[x])^4 \; dx = \frac{2}{3}}
$$

MANY MANY of you tried to solve using the central ordinate theorem (NOT the "central limit theorem" as many stated). The problem here is that you have to find the central ordinate of $TRI\,[\xi] * TRI\,[\xi]$*, which results in the same calculation, but MANY of you assumed (I imagine, they did not "derive") that:*

$$
TRI\,[\xi] \cdot TRI\,[\xi] = TRI\,[\xi]
$$

which clearly isn't true.

5. Evaluate and graph the real part, imaginary part, magnitude, and phase of $F[\xi] = \mathcal{F}_1\{f[x]\}$, where $f[x] = -i \cdot \dfrac{1}{1+x^2}$

SOLUTION: *The linearity of the Fourier transform ensures that*

$$F[\xi] = \mathcal{F}_1\left\{-i \cdot \frac{1}{1+x^2}\right\} = -i \cdot \mathcal{F}_1\left\{\frac{1}{1+x^2}\right\}$$

Now recall the known transform

$$r[x] \equiv \exp[-x] \cdot STEP[x] \implies R[\xi] = \frac{1}{1+2\pi i\xi}$$

$$\implies \mathcal{F}_1\{r[-x]\} = \mathcal{F}_1\{\exp[+x] \cdot STEP[-x]\} = R[-\xi] = \frac{1}{1+2\pi i(-\xi)} = \frac{1}{1-2\pi i\xi}$$

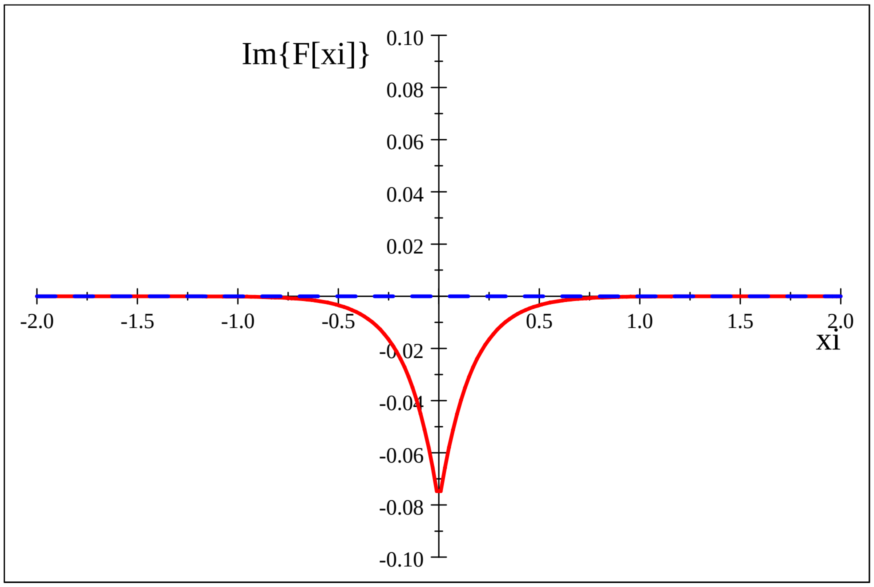
$$\begin{aligned}
\mathcal{F}_1\{r[x]+r[-x]\} &= \mathcal{F}_1\{\exp[-x] \cdot STEP[x] + \exp[+x] \cdot STEP[-x]\} = \mathcal{F}_1\{\exp[-|x|]\} \\
R[\xi]+R[-\xi] &= \frac{1}{1+2\pi i\xi} + \frac{1}{1-2\pi i\xi} = \frac{(1-2\pi i\xi)+(1+2\pi i\xi)}{(1-2\pi i\xi)\cdot(1+2\pi i\xi)} = \frac{2}{1+(2\pi\xi)^2}
\end{aligned}$$

Now use the "transform of a transform" Theorem:

$$\begin{aligned}
\mathcal{F}_1\left\{\frac{2}{1+(2\pi x)^2}\right\} &= \exp[-|\xi|] \\
&\implies \mathcal{F}_1\left\{\frac{1}{1+(2\pi x)^2}\right\} = \frac{1}{2} \cdot \exp[-|\xi|]
\end{aligned}$$

Scaling Theorem:

$$\begin{aligned}
\mathcal{F}_1\left\{\frac{1}{1+x^2}\right\} &= \mathcal{F}_1\left\{\frac{1}{1+\left(2\pi\left(\frac{x}{2\pi}\right)\right)^2}\right\} = \frac{1}{2\pi} \cdot \frac{1}{2} \cdot \exp[-|2\pi\xi|] \\
F[\xi] &= \mathcal{F}_1\left\{-i \cdot \frac{1}{1+x^2}\right\} = -i \cdot \mathcal{F}_1\left\{\frac{1}{1+x^2}\right\} = -i \cdot \frac{1}{2\pi} \cdot \frac{1}{2} \cdot \exp[-|2\pi\xi|]
\end{aligned}$$



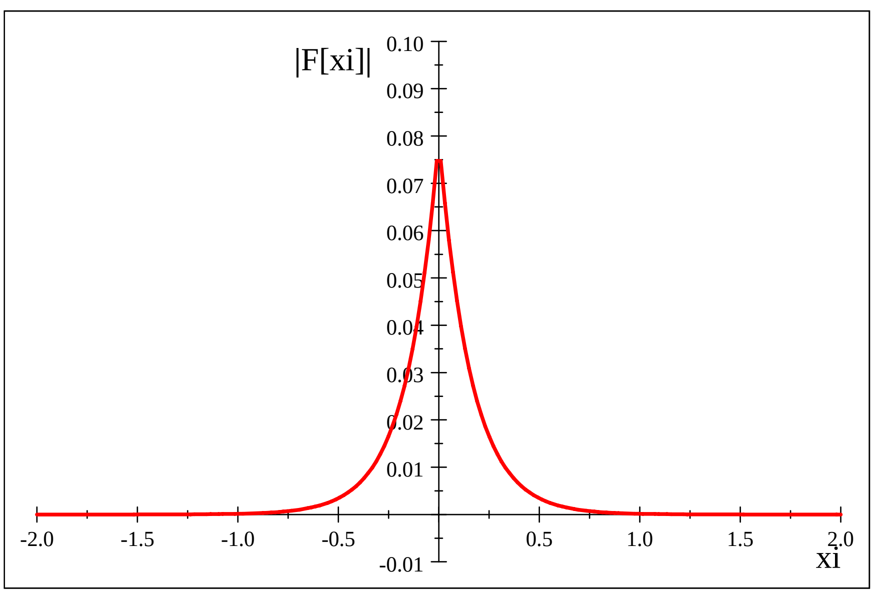
$\mathrm{Re}\{F[\xi]\}$ *in blue dashed line and* $\mathrm{Im}\{F[\xi]\}$ *in red solid line*

$$\boxed{\mathrm{Re}\left\{\mathcal{F}_1\left\{i\cdot\frac{1}{1+x^2}\right\}\right\}=0\,[\xi]}$$

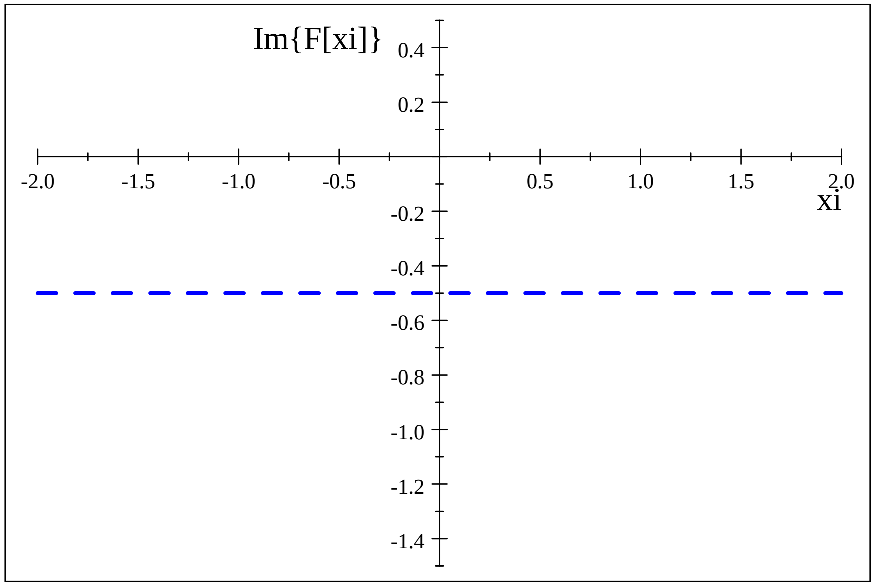
$$\boxed{\mathrm{Im}\left\{\mathcal{F}_1\left\{i\cdot\frac{1}{1+x^2}\right\}\right\}=\frac{1}{2\pi}\cdot\frac{1}{2}\cdot\exp\left[-\left|2\pi\xi\right|\right]\leq 0}$$

$$\boxed{\left|\mathcal{F}_1\left\{-i\cdot\frac{1}{1+x^2}\right\}\right|=\sqrt{\left(\mathrm{Re}\left\{F\left[\xi\right]\right\}\right)^2+\left(\mathrm{Im}\left\{F\left[\xi\right]\right\}\right)^2}=\left|\mathrm{Im}\left\{F\left[\xi\right]\right\}\right|}$$

$$\boxed{\Phi\left\{F\left[\xi\right]\right\}=-\frac{\pi}{2}\cdot 1\,[\xi]}$$



$\mathrm{Re}\{F[\xi]\}$ *in blue dashed line and* $\mathrm{Im}\{F[\xi]\}$ *in red solid line*



$\Phi\{F[\xi]\}$ *(blue dashed line) in units of* π *radians*

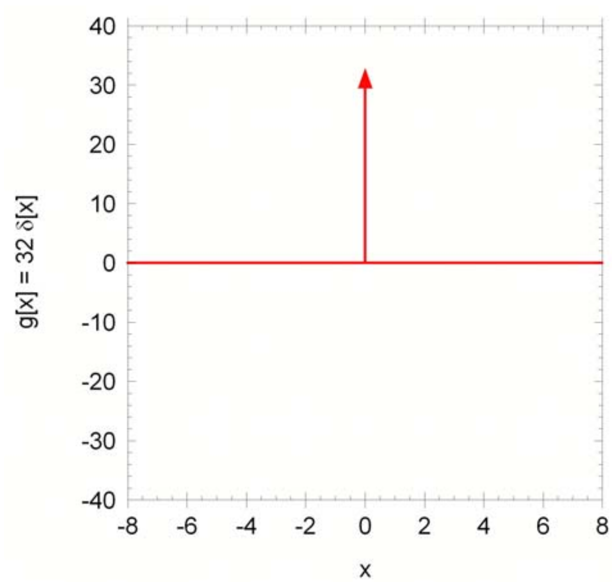
6. Evaluate and graph the autocorrelation of $f[x] = (2+2i) \cdot \exp\left[+i\pi\left(\frac{x}{2}\right)^2\right]$

Solution:

$$\begin{aligned}
f[x] \bigstar f[x] &= f[x] * f^*[-x] = \mathcal{F}_1^{-1}\left\{|F[\xi]|^2\right\} \\
f[x] &= (2+2i) \cdot \exp\left[+i\pi\left(\frac{x}{2}\right)^2\right] \\
&\Longrightarrow F[\xi] = (2+2i) \cdot \mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{2}\right)^2\right]\right\} \\
&= (2+2i) \cdot |2| \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi(2\xi)^2\right]
\end{aligned}$$

$$\begin{aligned}
|F[\xi]|^2 &= \left|(2+2i) \cdot |2| \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi(2\xi)^2\right]\right|^2 \\
&= |2 \cdot (1+i) \cdot 2|^2 \cdot \left|\exp\left[+i\frac{\pi}{4}\right]\right|^2 \cdot \left|\exp\left[-i\pi(2\xi)^2\right]\right|^2 \\
&= 16 \cdot |1+i|^2 \cdot 1 \cdot 1[\xi] \\
&= 16 \cdot 2 \cdot 1[\xi] = 32 \cdot 1[\xi] \\
g[x] &= f[x] \bigstar f[x] = 32 \cdot \mathcal{F}_1^{-1}\{1[\xi]\} = 32 \cdot \delta[x]
\end{aligned}$$

This is a Dirac delta function at the origin with area of 32 units.



7. Consider the input signal $f[x] = COMB\left[\frac{x}{2}\right]$ which is filtered by $h[x] = RECT[x]$

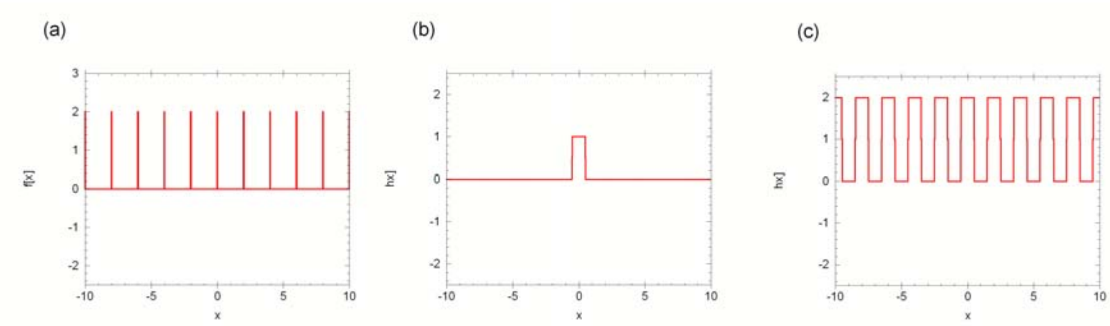
(a) Evaluate and sketch the output $g[x] = f[x] * h[x]$
SOLUTION:

$$\begin{aligned}
g[x] &= COMB\left[\frac{x}{2}\right] * RECT[x] \\
&= \left(\sum_{n=-\infty}^{+\infty} \delta\left[\frac{x}{2} - n\right]\right) * RECT[x] \\
&= \left(\sum_{n=-\infty}^{+\infty} \delta\left[\frac{x-2n}{2}\right]\right) * RECT[x] \\
&= 2 \cdot \left(\sum_{n=-\infty}^{+\infty} \delta[x-2n]\right) * RECT[x] \\
&= 2 \cdot \sum_{n=-\infty}^{+\infty} RECT[x-2n]
\end{aligned}$$

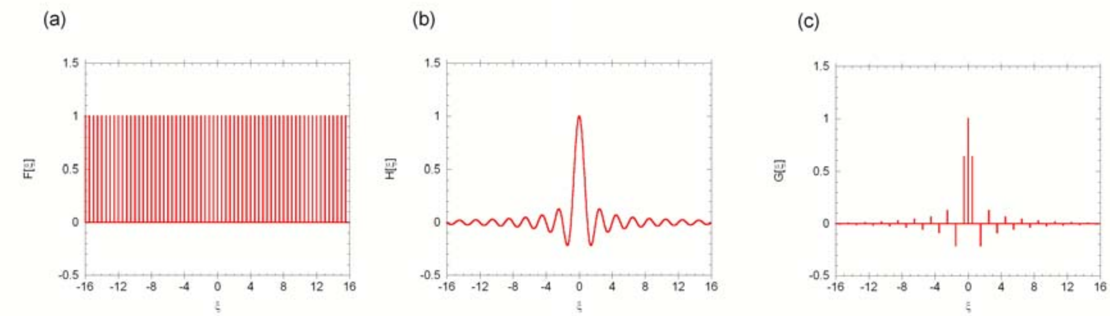
This is a sequence of unit-width rectangle functions at the even integer values of x. The spectrum $G[\xi]$ is:

$$\begin{aligned}
G[\xi] &= \mathcal{F}_1\left\{COMB\left[\frac{x}{2}\right]\right\} \cdot \mathcal{F}_1\{RECT[x]\} \\
&= 2 \cdot COMB[2\xi] \cdot SINC[\xi] \\
&= 2 \cdot \left(\sum_{k=-\infty}^{+\infty} \delta[2\xi - k]\right) \cdot SINC[\xi] \\
&= 2 \cdot \left(\sum_{k=-\infty}^{+\infty} \delta\left[2\left(\xi - \frac{k}{2}\right)\right]\right) \cdot SINC[\xi] \\
&= \frac{2}{2} \cdot \left(\sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{2}\right]\right) \cdot SINC[\xi] \\
& \left(\sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{2}\right]\right) \cdot SINC\left[\frac{k}{2}\right]
\end{aligned}$$

which is the SINC function evaluated at the half-integer values of ξ, i.e., at $\xi = 0. \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}$, etc.



(a) $f[x] = COMB\left[\frac{x}{2}\right]$, *(b)* $h[x] = RECT[x]$, *(c)* $g[x] = f[x] * h[x]$



(a) $F[\xi] = 2 \cdot COMB[2\xi]$, *(b)* $H[\xi] = SINC[\xi]$, *(c)* $G[\xi] = F[\xi] \cdot H[\xi]$

(b) Specify and sketch the transfer function $W[\xi]$ of the pseudoinverse filter for $h[x]$

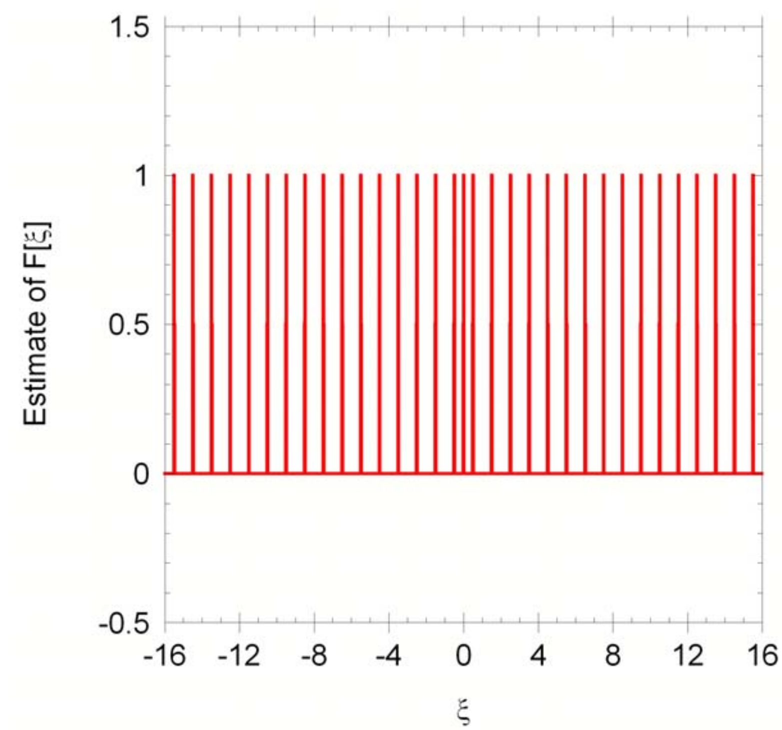
SOLUTION: *The transfer function of the inverse filter* $W[\xi]$ *is:*

$$\begin{aligned} H[\xi] &= \mathcal{F}_1\{h[x]\} = SINC[\xi] \\ W[\xi] &= (SINC[\xi])^{-1} \end{aligned}$$

and of the pseudoinverse filter:

$$\hat{W}[\xi] \equiv \begin{cases} (SINC[\xi])^{-1} & \text{if } \xi \neq \pm 1, \pm 2 \pm 3, \cdots \\ 0 & \text{if } \xi = \pm 1, \pm 2, \pm 3, \cdots \end{cases}$$

so the pseudoinverse filter "blocks" the integer frequencies, but corrects the amplitudes at the half-integer frequencies.



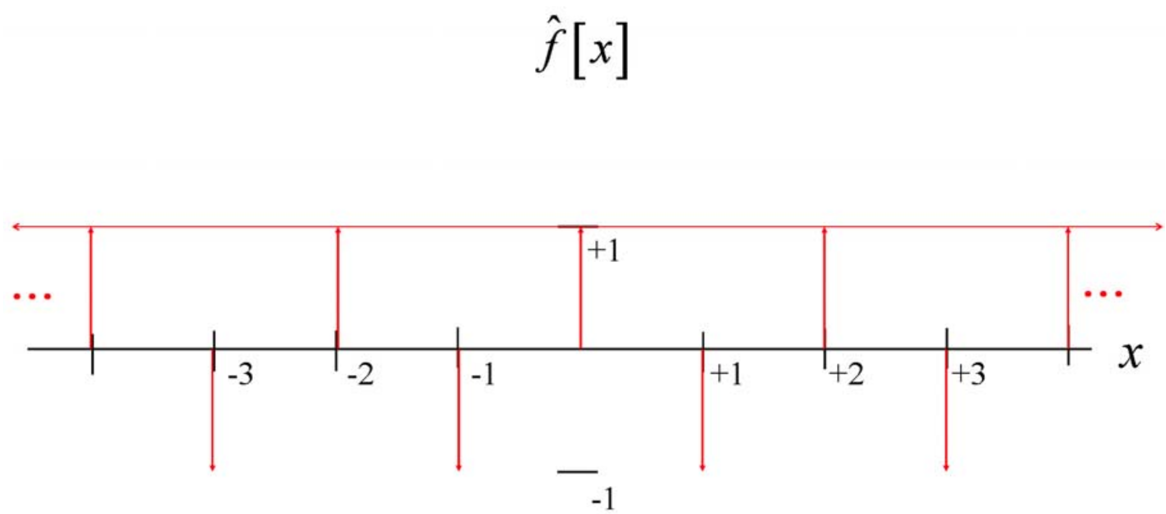
$\hat{F}[\xi]$

(c) Evaluate and sketch the estimate of $f[x]$ that would be recovered by applying the pseudoinverse filter to $g[x]$

SOLUTION: *The estimate of the output spectrum includes the DC term and the*

$$\begin{aligned}
\hat{F}[\xi] &= (F[\xi]) - (\textit{the amplitudes at the nonzero integer frequencies}) \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{2}\right]\right) - \left(\sum_{k=-\infty}^{+\infty} \delta[\xi - k]\right) + \delta[\xi] \\
&= \delta[\xi] + COMB\left[\xi - \frac{1}{2}\right] \\
\hat{f}[x] &= 1[x] + COMB[x] \cdot \exp\left[+2\pi i x \cdot \frac{1}{2}\right] \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n]\right) \cdot \exp\left[+2\pi i x \cdot \frac{1}{2}\right] \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n] \cdot \exp\left[+2\pi i x \cdot \frac{1}{2}\right]\right) \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n] \cdot \exp\left[+2\pi i \cdot n \cdot \frac{1}{2}\right]\right) \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n] \cdot \exp[+i\pi n]\right) \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n] \cdot (\exp[+i\pi])^n\right) \\
&= 1[x] + \left(\sum_{n=-\infty}^{+\infty} \delta[x-n] \cdot (-1)^n\right)
\end{aligned}$$

which does not much resemble the original function $f[x] = COMB[x]$



(d) Explain any artifacts present in the estimate of $f[x]$

SOLUTION: *Because the spectrum of the original function* $f[x] = COMB\left[\frac{x}{2}\right]$ *consists of only a discrete set of frequencies at half integers and since the transfer function* $H[\xi]$ *"blocks" the nonzero integer frequencies, only the DC term and the odd half-integers appear in the spectrum, with their original values.*

LESSON FROM THIS PROBLEM – SKETCH THE RESULTS OF EACH STEP IN THE PROCESS IF YOU ARE UNSURE!

8. Derive $f[x,y]$ if $F[\xi.\eta] = \left(\sqrt{\left[(\xi-1)^2 + (\eta+1)^2\right]}\right)^{-1}$; show the steps in the derivation, do not just write down a result.

Solution: *Note that:*

$$\begin{aligned}\left[(\xi-1)^2+(\eta+1)^2\right]^{-\frac{1}{2}} &= \left[\xi^2+\eta^2\right]^{-\frac{1}{2}} * \delta[\xi-1,\eta+1] \\ &= \frac{1}{\rho} * \delta[\xi-1,\eta+1]\end{aligned}$$

$$\begin{aligned}f[x,y] &= \mathcal{F}_2^{-1}\{F[\xi.\eta]\} = \mathcal{F}_2^{-1}\left\{\frac{1}{\rho}\right\} \cdot \mathcal{F}_2^{-1}\{\delta[\xi-1,\eta+1]\} \\ &= \mathcal{H}_0^{-1}\left\{\frac{1}{\rho}\right\} \cdot \mathcal{F}_2^{-1}\{\delta[\xi-1,\eta+1]\}\end{aligned}$$

$$\begin{aligned}\mathcal{F}_2^{-1}\{\delta[\xi-1,\eta+1]\} &= \iint_{-\infty}^{+\infty} \delta[\xi-1,\eta+1] \cdot \exp[+2\pi i(\xi x+\eta y)]\ d\xi\ d\eta \\ &= \iint_{-\infty}^{+\infty} \delta[\xi-1] \cdot \delta[\eta+1] \cdot \exp[+2\pi i\xi x]\exp[+2\pi i\eta y]\ d\xi\ d\eta \\ &= \int_{-\infty}^{+\infty} \delta[\xi-1] \cdot \exp[+2\pi i\xi x]\ d\xi \cdot \int_{-\infty}^{+\infty} \delta[\eta+1] \cdot \exp[+2\pi i\eta y]\ d\eta \\ &= \exp[+2\pi i \cdot 1 \cdot x] \cdot \exp[+2\pi i(-1)y] \\ &= \exp[+2\pi i(x-y)]\end{aligned}$$

$$\begin{aligned}\mathcal{H}_0^{-1}\left\{\frac{1}{\rho}\right\} &= \int_0^{+\infty} \frac{1}{\rho} \cdot 2\pi\rho \cdot J_0(2\pi\rho r)\ d\rho \\ &= 2\pi \cdot \int_0^{+\infty} J_0(2\pi\rho r)\ d\rho \\ &= 2\pi \cdot \frac{1}{2} \cdot \int_{-\infty}^{+\infty} J_0(2\pi\rho r)\ d\rho \\ &= \pi \cdot \int_{u=-\infty}^{u=+\infty} J_0(u)\ \frac{du}{2\pi r} \\ &= \frac{1}{2r} \int_{u=-\infty}^{u=+\infty} J_0(u)\ du = \frac{1}{2r} \cdot 2 \\ &= \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}\end{aligned}$$

$$
\begin{aligned}
f[x,y] &= \mathcal{H}_0^{-1}\left\{\frac{1}{\rho}\right\}\cdot\mathcal{F}_2^{-1}\left\{\delta[\xi-1,\eta+1]\right\} \\
&= \frac{1}{\sqrt{x^2+y^2}}\cdot\exp\left[+2\pi i\,(x-y)\right] \\
&= \frac{1}{\sqrt{x^2+y^2}}\cdot\left(\cos\left[+2\pi\,(x-y)\right]+i\cdot\sin\left[+2\pi\,(x-y)\right]\right)
\end{aligned}
$$

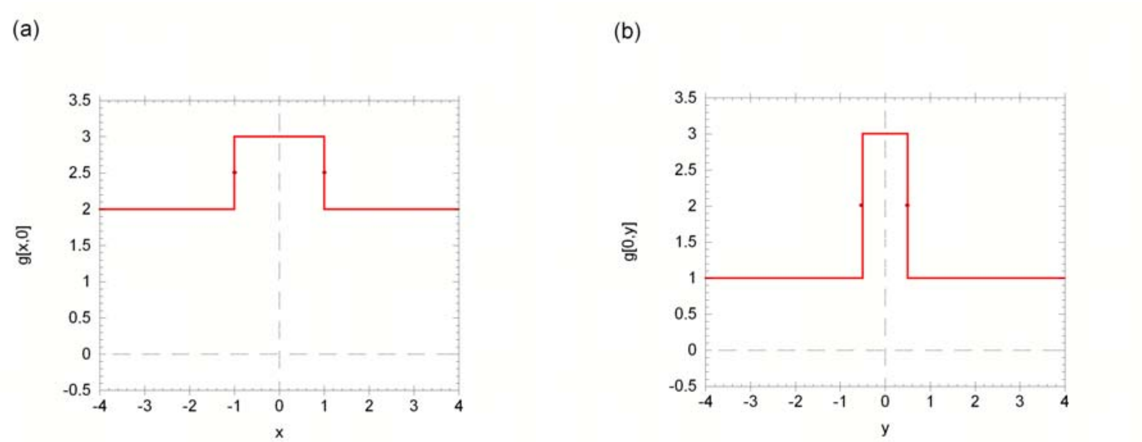
9. Evaluate the 2-D convolution $RECT\left[\frac{x}{2}, y\right] * (\delta[x] \cdot 1[y] + 1[x] \cdot \delta[y])$ and sketch the profiles along the x and y axes.

$$RECT\left[\frac{x}{2}, y\right] = RECT\left[\frac{x}{2}\right] \cdot RECT[y]$$

$$\begin{aligned}
g[x,y] &= \left(RECT\left[\frac{x}{2}\right] \cdot RECT[y]\right) * (\delta[x] \cdot 1[y] + 1[x] \cdot \delta[y]) \\
&= \left(RECT\left[\frac{x}{2}\right] \cdot RECT[y]\right) * (\delta[x] \cdot 1[y]) + \left(RECT\left[\frac{x}{2}\right] \cdot RECT[y]\right) * (1[x] \cdot \delta[y]) \\
&= \left(RECT\left[\frac{x}{2}\right] * \delta[x]\right) \cdot (RECT[y] * 1[y]) + \left(RECT\left[\frac{x}{2}\right] * 1[x]\right) \cdot (RECT[y] * \delta[y]) \\
&= RECT\left[\frac{x}{2}\right] \cdot 1[y] + (2 \cdot 1[x]) \cdot RECT[y]
\end{aligned}$$

$$\begin{aligned}
g[x,0] &= RECT\left[\frac{x}{2}\right] \cdot 1[y=0] + (2 \cdot 1[x]) \cdot RECT[y=0] \\
&= RECT\left[\frac{x}{2}\right] \cdot 1 + (2 \cdot 1[x]) \cdot 1 \\
&= 2 + RECT\left[\frac{x}{2}\right]
\end{aligned}$$

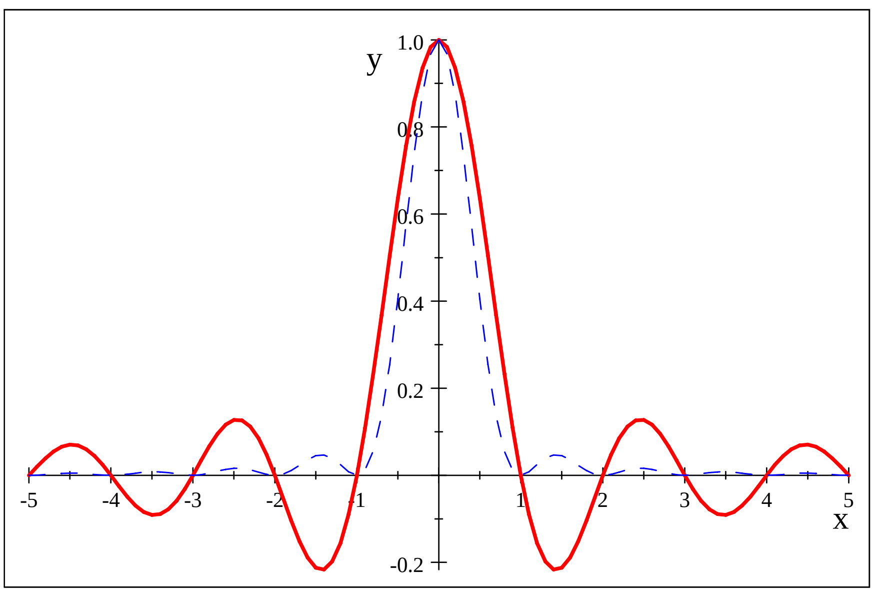
$$\begin{aligned}
g[0,y] &= RECT\left[\frac{0}{2}\right] \cdot 1[y] + (2 \cdot 1[x=0]) \cdot RECT[y] \\
&= 1[y] + 2 \cdot RECT[y] \\
&= 1 + 2 \cdot RECT[y]
\end{aligned}$$



10. For $s[x] = COMB[x]$, $h[x] = SINC[x]$, and $f_n[x] = SINC^2\left[\frac{x}{n}\right]$ $(n = 1, 2, 3)$, find the forms of and sketch the three functions $g_n[x] = (f_n[x] \cdot s[x]) * h[x]$

$$\begin{aligned} g_1[x] &= (f_1[x] \cdot s[x]) * h[x] = \left(SINC^2\left[\frac{x}{1}\right] \cdot COMB[x]\right) * SINC[x] \\ SINC^2[x] \cdot COMB[x] &= SINC^2[x] \cdot \sum_{n=-\infty}^{+\infty} \delta[x-n] = \sum_{n=-\infty}^{+\infty} SINC^2[x] \cdot \delta[x-n] \\ &= \sum_{n=-\infty}^{+\infty} SINC^2[n] \cdot \delta[x-n] \end{aligned}$$

$$\begin{aligned} &= \cdots + SINC^2[-1] \cdot \delta[x-(-1)] + SINC^2[0] \cdot \delta[x-0] + SINC^2[+1] \cdot \delta[x-(+1)] + \cdots \\ &= \delta[x] \\ \Longrightarrow\ & g_1[x] = (f_1[x] \cdot s[x]) * h[x] = \delta[x] * SINC[x] = SINC[x] \\ & \boxed{g_1[x] = SINC[x] \neq SINC^2[x] = f_1[x]} \end{aligned}$$



$g_1[x]$ (red solid) and $f_1[x] = SINC^2[x]$ (blue dashed), showing the the input is not recovered correctly.

$$g_2[x] = (f_2[x] \cdot s[x]) * h[x] = \left(SINC^2\left[\frac{x}{2}\right] \cdot COMB[x]\right) * SINC[x]$$

$$SINC^2\left[\frac{x}{2}\right] \cdot COMB[x] \sum_{n=-\infty}^{+\infty} SINC^2\left[\frac{x}{2}\right] \cdot \delta[x-n]$$

$$= \sum_{n=-\infty}^{+\infty} SINC^2\left[\frac{n}{2}\right] \cdot \delta[x-n]$$

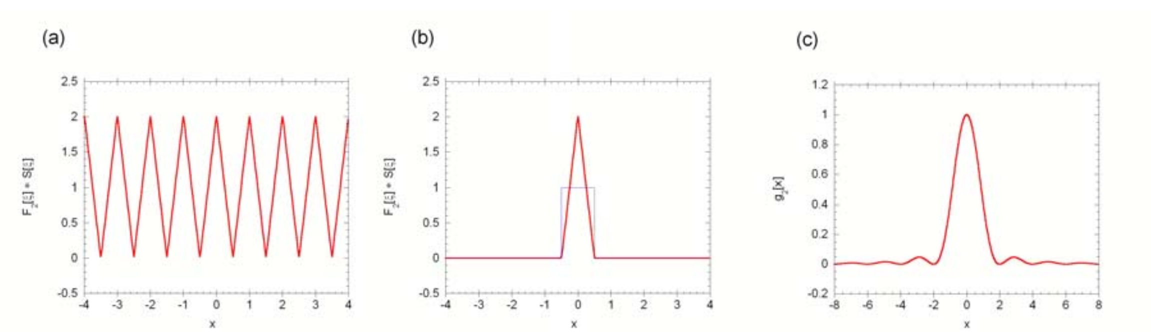
$$= \cdots + SINC^2\left[-\frac{1}{2}\right] \cdot \delta[x-(-1)] + SINC^2[0] \cdot \delta[x-0] + SINC^2\left[+\frac{1}{2}\right] \cdot \delta[x-(+1)] + \cdots$$

$$\Longrightarrow \left(\sum_{n=-\infty}^{+\infty} SINC^2\left[\frac{n}{2}\right] \cdot \delta[x-n]\right) * SINC[x]$$

which isn't much help, so let's evaluate in the frequency domain

$$\begin{aligned}
G_2[\xi] &= (\mathcal{F}_1\{f_2[x]\} * \mathcal{F}_1\{s[x]\}) \cdot \mathcal{F}_1\{h[x]\} = (F_2[\xi] * S[\xi]) \cdot H[\xi] \\
F_2[\xi] &= \mathcal{F}_1\{f_2[x]\} = \mathcal{F}_1\left\{SINC^2\left[\frac{x}{2}\right]\right\} = 2 \cdot TRI[2\xi] = 2 \cdot TRI\left[\frac{\xi}{0.5}\right] \\
S[\xi] &= \mathcal{F}_1\{s[x]\} = \mathcal{F}_1\{COMB[x]\} = COMB[\xi] \\
H[\xi] &= \mathcal{F}_1\{h[x]\} = RECT[\xi] \\
G_2[\xi] &= \left(2 \cdot TRI\left[\frac{\xi}{0.5}\right] * COMB[\xi]\right) \cdot RECT[\xi]
\end{aligned}$$

Sketch them:

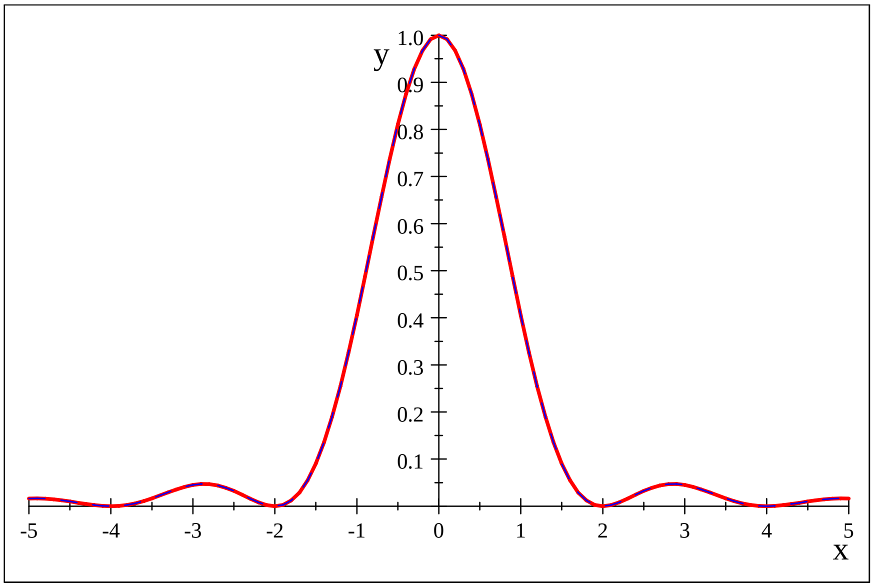


(a) $F_2[\xi] * S[\xi] = 2 \cdot TRI\left[\frac{\xi}{0.5}\right] * COMB[\xi]$, *(b)* $RECT[\xi]$ *(blue) and* $(F_2[\xi] * S[\xi]) \cdot RECT[\xi] = 2 \cdot TRI\left[\frac{\xi}{0.5}\right]$ *(red); (c)* $\mathcal{F}_1^{-1}\left\{2 \cdot TRI\left[\frac{\xi}{0.5}\right]\right\} = SINC^2\left[\frac{x}{2}\right]$

which shows that

$$G_2[\xi] = 2 \cdot TRI\left[\frac{\xi}{0.5}\right]$$

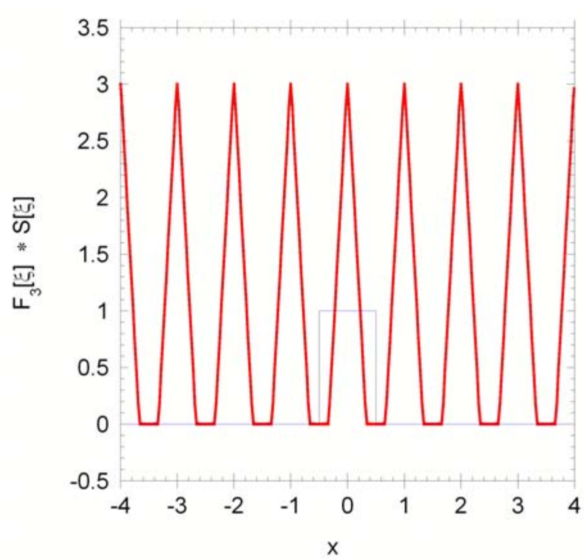
$$\boxed{g_2[x] = SINC^2\left[\frac{x}{2}\right] = f_2[x]}$$



$g_2[x]$ (red solid) and $f_2[x] = SINC^2\left[\frac{x}{2}\right]$ (blue dashed), showing the the input IS recovered correctly.

$$
\begin{aligned}
g_3[x] &= (f_3[x] \cdot s[x]) * h[x] = \left(SINC^2\left[\frac{x}{3}\right] \cdot COMB[x]\right) * SINC[x] \\
F_3[\xi] &= \mathcal{F}_1\{f_3[x]\} = 3 \cdot TRI[3\xi] = 3 \cdot TRI\left[\frac{\xi}{\left(\frac{1}{3}\right)}\right] \\
S[\xi] &= \mathcal{F}_1\{COMB[\xi]\} = COMB[\xi] \\
F_3[\xi] * COMB[\xi] &= 3 \cdot TRI\left[\frac{\xi}{\left(\frac{1}{3}\right)}\right] * COMB[\xi]
\end{aligned}
$$

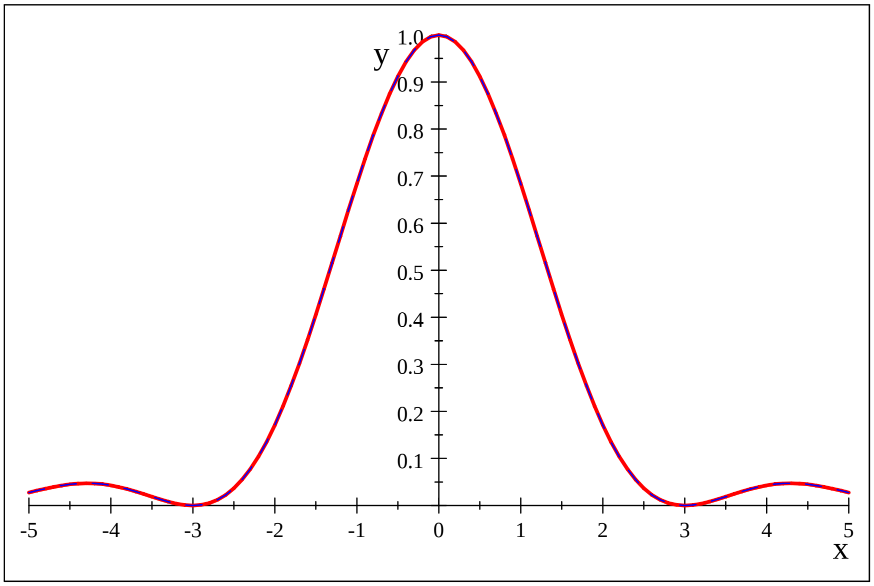
Graph it:



$F_3[\xi] * S[\xi]$ (red) and $(F_3[\xi] * S[\xi]) \cdot H[\xi]$ (blue)

From the graph, we see that the transfer function "cuts out" the central replica of the triangle from the "sawtooth" spectrum. Thus the image $g_3[x]$ *is:*

$$\boxed{g_3[x] = SINC^2\left[\frac{x}{3}\right] = f_3[x]}$$



$g_2[x]$ *(red solid) and* $f_3[x] = SINC^2\left[\frac{x}{3}\right]$ *(blue dashed), showing the the input IS recovered correctly.*

Note that $f_2[x]$ *and* $f_3[x]$ *were correctly "imaged" by the system, but that* $f_1[x]$ *is not.*

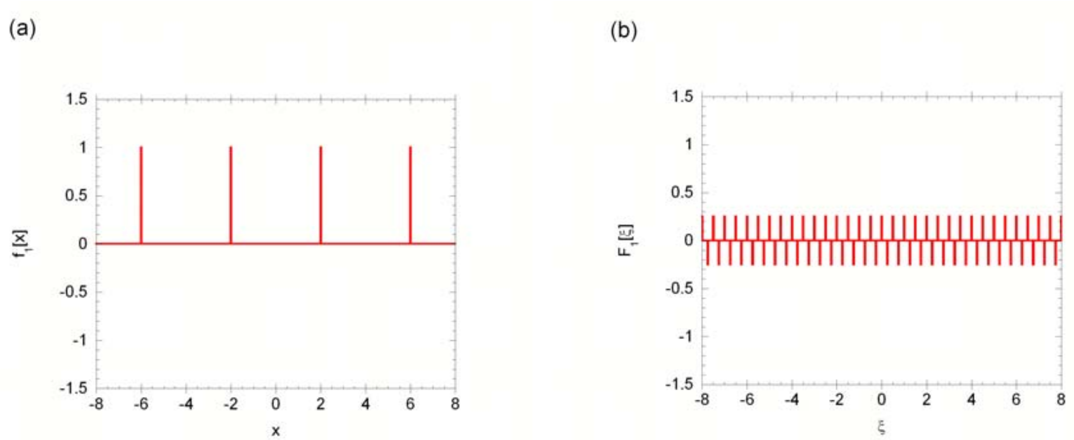
11. Consider the function $f_1[x] = \frac{1}{4} \cdot COMB\left[\frac{x-2}{4}\right]$

(a) Evaluate and sketch its Fourier transform $F_1[\xi]$.

$$
\begin{aligned}
F_1[\xi] &= \mathcal{F}_1\left\{\frac{1}{4} \cdot COMB\left[\frac{x-2}{4}\right]\right\} = \frac{1}{4} \cdot \mathcal{F}_1\left\{COMB\left[\frac{x-2}{4}\right]\right\} \\
&= \frac{1}{4} \cdot |4| \cdot COMB[4\xi] \cdot \exp[-2\pi i\xi \cdot 2] \\
&= COMB[4\xi] \cdot \exp[-4\pi i\xi] \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta[4\xi - k]\right) \cdot \exp[-4\pi i\xi] \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta\left[4\left(\xi - \frac{k}{4}\right)\right]\right) \cdot \exp[-4\pi i\xi] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp[-4\pi i\xi] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp\left[-4\pi i\frac{k}{4}\right] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp[-i\pi k]
\end{aligned}
$$

$$
\boxed{F_1[\xi] = \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{1}{4}\right] \cdot (-1)^k}
$$

This is a set of Dirac delta function with weights oscillating between $\pm\frac{1}{4}$ *at multiples of* $\xi = k \cdot \frac{1}{4}$.



(a) weights applied to the Dirac delta functions in $f_1[x]$*; (b) weights applied to the Dirac delta functions in* $F_1[\xi]$

(b) Evaluate and sketch the Fourier transform of $f_2[x] = \frac{1}{4} \cdot COMB\left[\frac{x+2}{4}\right]$

$$
\begin{aligned}
F_2[\xi] &= \mathcal{F}_1\left\{\frac{1}{4} \cdot COMB\left[\frac{x+2}{4}\right]\right\} = \frac{1}{4} \cdot \mathcal{F}_1\left\{COMB\left[\frac{x+2}{4}\right]\right\} \\
&= \frac{1}{4} \cdot |4| \cdot COMB[4\xi] \cdot \exp[+2\pi i\xi \cdot 2] \\
&= COMB[4\xi] \cdot \exp[+4\pi i\xi] \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta[4\xi - k]\right) \cdot \exp[+4\pi i\xi] \\
&= \left(\sum_{k=-\infty}^{+\infty} \delta\left[4\left(\xi - \frac{k}{4}\right)\right]\right) \cdot \exp[+4\pi i\xi] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp[+4\pi i\xi] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp\left[+4\pi i\frac{k}{4}\right] \\
&= \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{k}{4}\right] \cdot \exp[+i\pi k]
\end{aligned}
$$

$$
\boxed{F_2[\xi] = \frac{1}{4} \cdot \sum_{k=-\infty}^{+\infty} \delta\left[\xi - \frac{1}{4}\right] \cdot (-1)^k}
$$

This is the same set of Dirac delta functions as in part (a), so the plots are the same too.

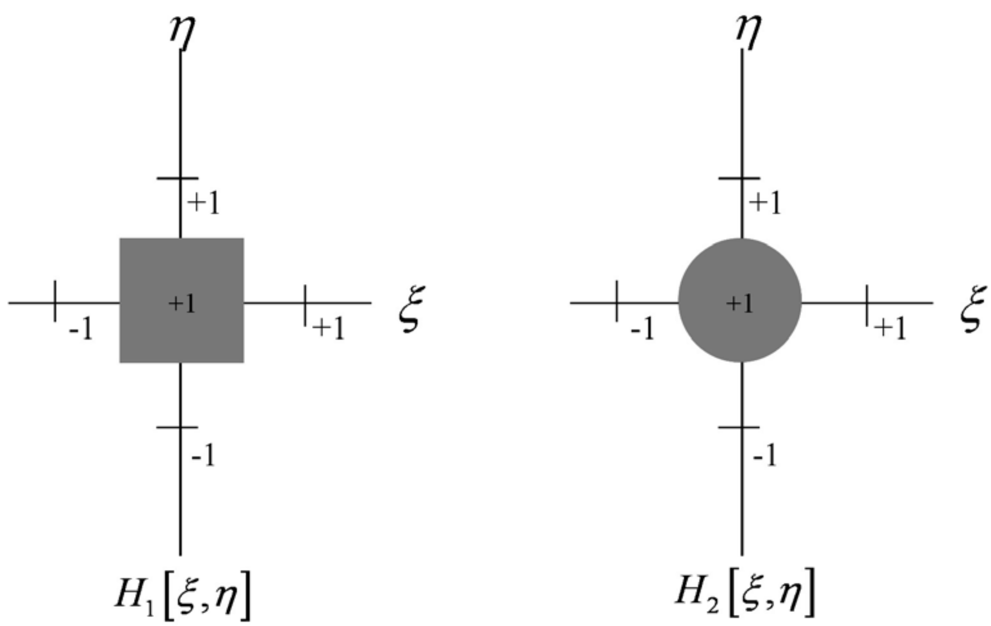
(c) Compare the sketches of the transforms $F_1[\xi]$ and $F_2[\xi]$; what does the comparison tell you about $f_1[x]$ and $f_2[x]$?

The two spectra are identical, which means that $f_1[x] = f_2[x]$;

the two expressions describe the same function.

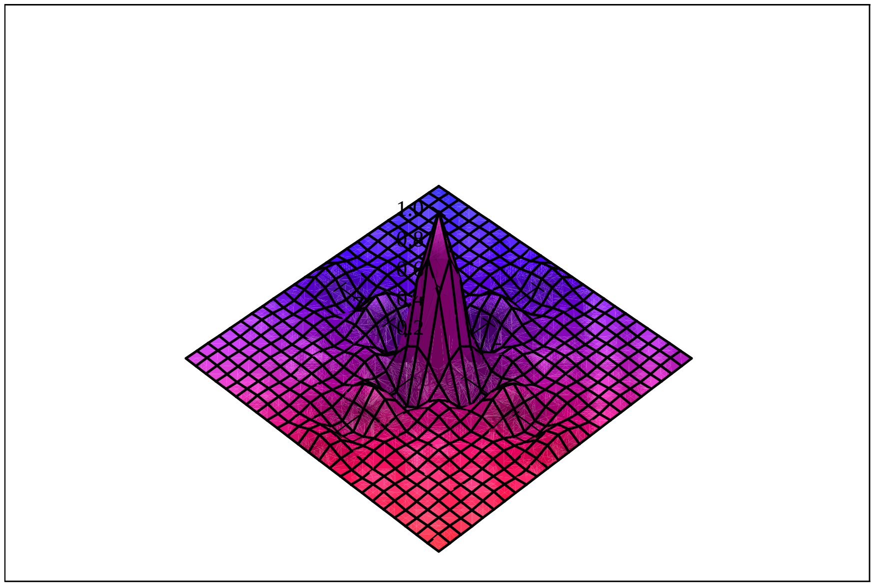
12. Two imaging systems have respective transfer functions $H_1[\xi,\eta] = RECT[\xi,\eta]$ and $H_2[\xi,\eta] = CYL\left(\sqrt{\xi^2+\eta^2}\right)$.

(a) Sketch the transfer functions

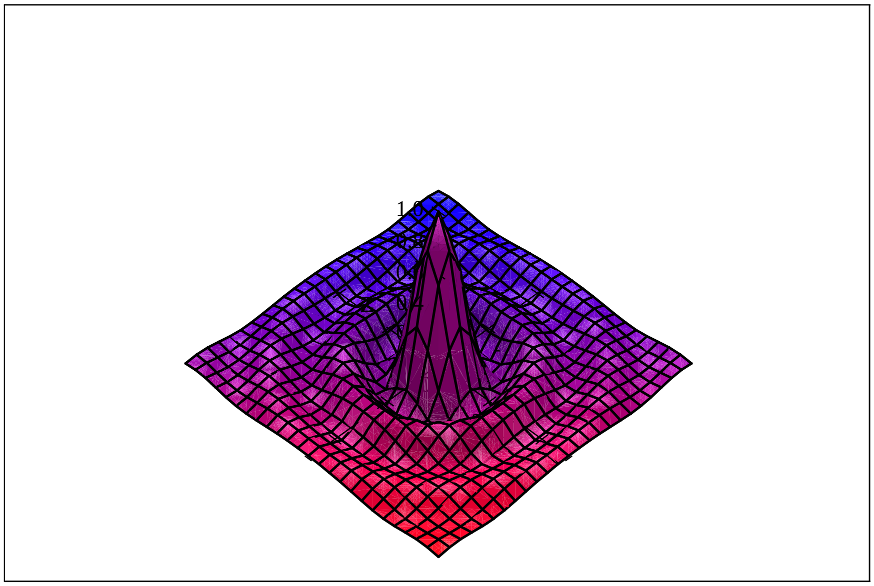


(b) Evaluate the impulse responses $h_1[x,y]$ and $h_2[x,y]$.

$$
\begin{aligned}
H_1[\xi,\eta] &= RECT[\xi,\eta] = RECT[\xi] \cdot RECT[\eta] \\
h_1[x,y] &= SINC[x] \cdot SINC[y] = SINC[x,y] = \frac{\sin[\pi x]}{\pi x} \cdot \frac{\sin[\pi y]}{\pi y}
\end{aligned}
$$



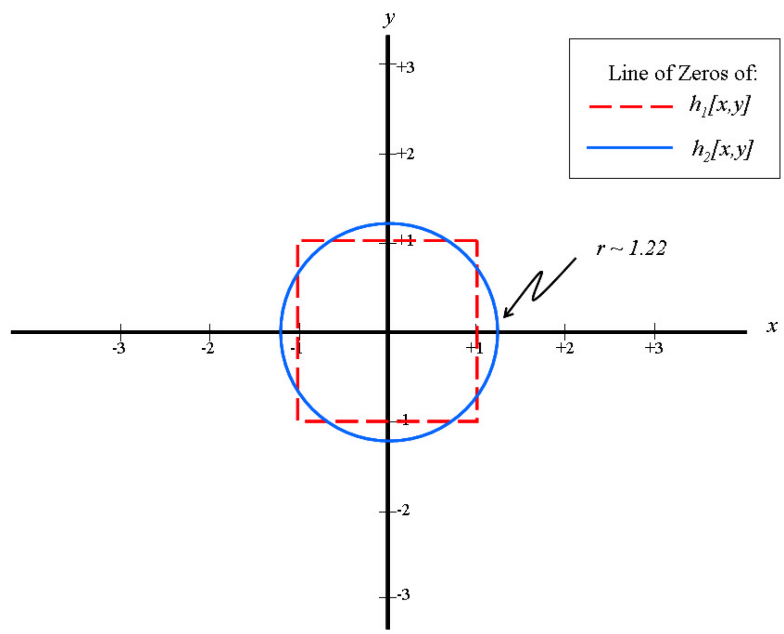
$$
\begin{aligned}
H_2[\xi,\eta] &= CYL\left(\sqrt{\xi^2+\eta^2}\right) = CYL(\rho) \\
h_2(r) &= \mathcal{H}_0^{-1}\{CYL(\rho)\} = SOMB(r) = 2\cdot\frac{J_1(\pi r)}{\pi r} \\
h_2[x,y] &= 2\cdot\frac{J_1\left(\pi\sqrt{x^2+y^2}\right)}{\pi\sqrt{x^2+y^2}}
\end{aligned}
$$



(c) Make an approximate sketch in the $[x, y]$ domain of the locations of the first zeros of the two impulse responses, i.e., draw lines in the $[x, y]$ plane that show the locations closest to the origin where $h_1[x, y] = 0$ and $h_2[x.y] = 0$. Be sure that the lines belonging to each are labeled unambiguously.

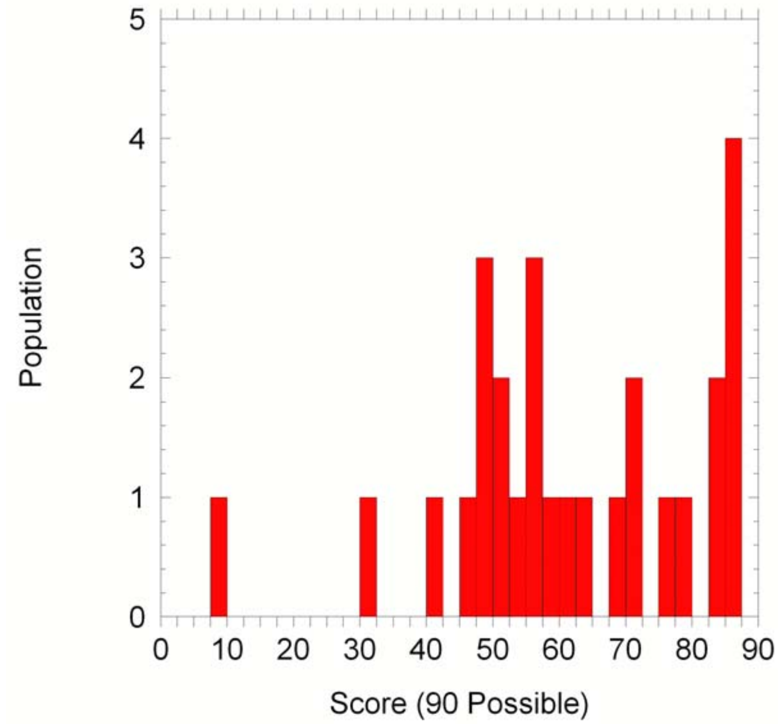
The line of the first zeros of the SINC function follow the path of the "corral" function:

$$\begin{aligned} h_1[x, y] &= 0 \text{ where } COR[x, y] \neq 0 \\ h_2[x, y] &= 0 \text{ approximately where } \delta(r - 1.22) \neq 0 \end{aligned}$$



(d) (OPTIONAL BONUS) Which of these two systems will have the better resolution? Explain.

The "wider" impulse response will "blur" the image more. The choice of wider impulse response is ambiguous, but the system with the circular impulse response is somewhat wider, so this would be one criterion for the choice.



Statistics: (90 possible points)

mean μ	61.6
σ	18.7
Max	87
Min	9
Median	58