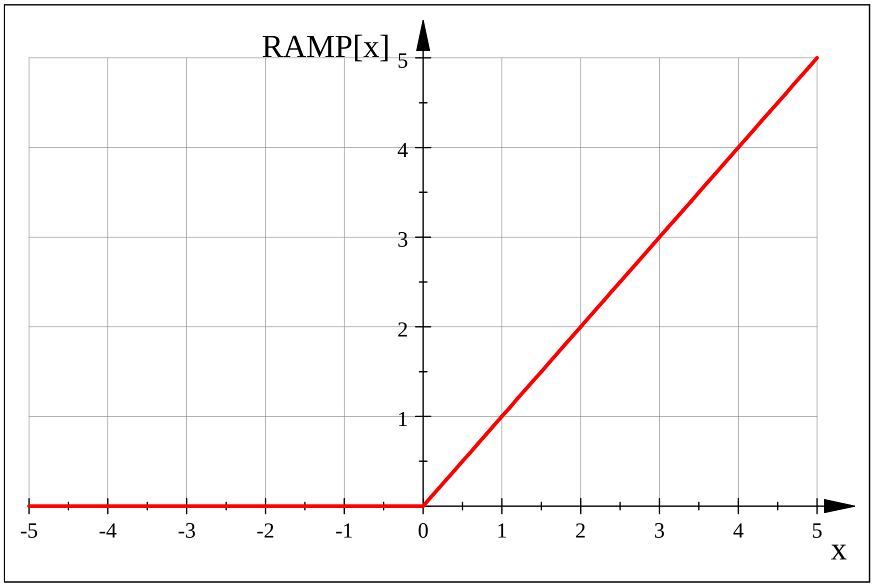
1051-716-20091 Solutions to Midterm Exam

1. The "ramp" function may be defined:

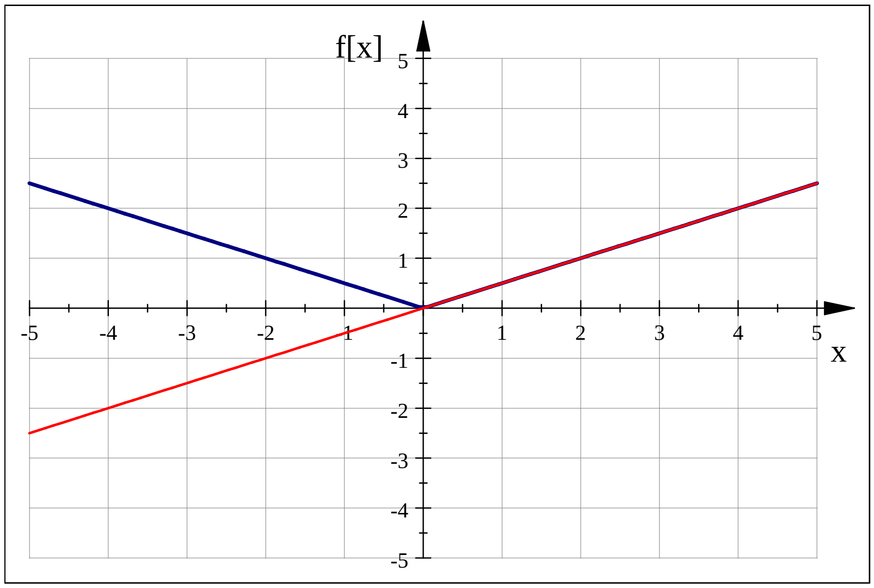
$$f[x] = RAMP[x] \equiv x \cdot STEP[x]$$

   (a) Sketch $f[x] = RAMP[x]$



(b) Find expressions for the even and odd parts of $f[x]$ AND sketch them.

$$\begin{aligned}
f[x] &= f_{even}[x] + f_{odd}[x] \\
f_{even}[x] &= \frac{1}{2}(f[x] + f[-x]) = \frac{1}{2}(x \cdot STEP[x] + (-x \cdot STEP[-x])) \\
&= \frac{1}{2} \cdot \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases} = \boxed{f_{even}[x] = \frac{1}{2}|x|} \\
f_{odd}[x] &= \frac{1}{2}(f[x] - f[-x]) = \frac{1}{2}(x \cdot STEP[x] - (-x \cdot STEP[-x])) \\
&= \frac{1}{2} \cdot \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x < 0 \end{cases} = \boxed{f_{odd}[x] = \frac{1}{2}x}
\end{aligned}$$



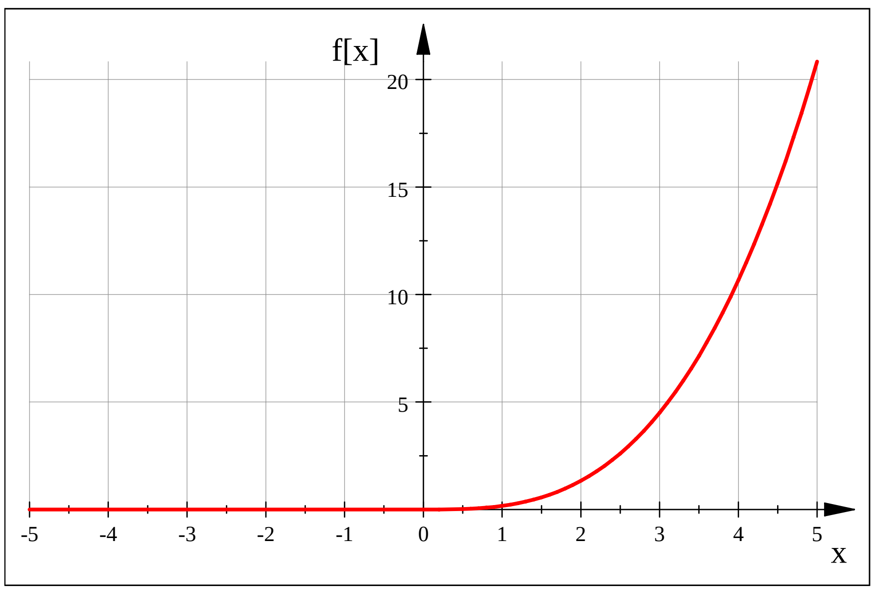
EVEN part $\frac{1}{2}|x|$ *in BLUE, ODD part* $\frac{1}{2}x$ *in RED; note that they are identical for* $x \geq 0$.

Several students did not recognize which was even and which was odd from the graphs! This should be obvious: the even function is symmetric ($f[-x] = +f[x]$) *and the odd function is antisymmetric* ($f[-x] = -f[+x]$).

(c) Evaluate $f[x] * f[x]$ AND sketch it.

This was a problem from HW#4

$$\begin{aligned}
RAMP[x] * RAMP[x] &= \int_{-\infty}^{+\infty} (\alpha \cdot STEP[\alpha]) \cdot ((x-\alpha) \cdot STEP[x-\alpha])\, d\alpha \\
&= \int_{0}^{+\infty} (\alpha \cdot (x-\alpha)) \cdot STEP[x-\alpha]\, d\alpha \\
&= \begin{cases} \displaystyle\int_0^x (\alpha \cdot (x-\alpha))\, d\alpha & \text{if } \quad x > 0 \\ 0 & \text{if } \quad x \leq 0 \end{cases} \\
&= STEP[x] \cdot \int_0^x \left(\alpha x - \alpha^2\right)\, d\alpha \\
&= STEP[x] \cdot \left(\frac{\alpha^2}{2} x - \frac{\alpha^3}{3}\right)\Bigg|_{\alpha=0}^{\alpha=x} \\
&= STEP[x] \cdot \left(\frac{x^3}{2} - \frac{x^3}{3}\right) \\
&= STEP[x] \cdot \frac{x^3}{6}
\end{aligned}$$



(d) Evaluate $f[x] \bigstar f[x]$.

$$
\begin{aligned}
RAMP[x] \bigstar RAMP[x] &= RAMP[x] * (RAMP[-x])^* \\
&= RAMP[x] * RAMP[-x] \;\; \textit{because } RAMP[x] \;\; \textit{is real valued} \\
&= \int_{-\infty}^{+\infty} RAMP[\alpha] \cdot RAMP[-(x-\alpha)] \;\; d\alpha \\
&= \int_{-\infty}^{+\infty} RAMP[\alpha] \cdot RAMP[\alpha - x] \;\; d\alpha \\
& \int_{-\infty}^{+\infty} (\alpha \cdot STEP[\alpha]) \cdot ((\alpha - x) \cdot STEP[\alpha - x]) \, d\alpha \\
&= \int_{0}^{+\infty} (\alpha \cdot (\alpha - x)) \cdot STEP[\alpha - x] \, d\alpha \\
&= \begin{cases} \displaystyle\int_{x}^{+\infty} (\alpha \cdot (x - \alpha)) \, d\alpha & \text{if} \quad x > 0 \\ \displaystyle\int_{0}^{+\infty} (\alpha \cdot (x - \alpha)) \, d\alpha & \text{if} \quad x \leq 0 \end{cases} = \infty \cdot 1[x]
\end{aligned}
$$

So the autocorrelation evaluates to infinity (i.e., is not defined) at all coordinates.

2. Evaluate the integral and sketch the output(s):

$$\int_{-\infty}^{+\infty} \cos[2\pi\xi_0 x + \phi_0] \cdot (\cos[2\pi\xi x] + \sin[2\pi\xi x])\ dx$$

where ξ_0 and ϕ_0 are real-valued numerical constants.

This is the Hartley transform of a real-valued sinusoid with arbitrary initial phase. First, I suggest that you break the input cosine into its even and odd parts:

$$\cos[2\pi\xi_0 x + \phi_0] = \cos[\phi_0]\cos[2\pi\xi_0 x] + (-\sin[\phi_0])\sin[2\pi\xi_0 x]$$

This gives four integrals:

$$\begin{aligned} I_1[\xi] &\equiv \cos[\phi_0]\int_{-\infty}^{+\infty} \cos[2\pi\xi_0 x] \cdot \cos[2\pi\xi x]\ dx \\ I_2[\xi] &\equiv (-\sin[\phi_0])\int_{-\infty}^{+\infty} \sin[2\pi\xi_0 x] \cdot \cos[2\pi\xi x]\ dx \\ I_3[\xi] &\equiv \cos[\phi_0]\int_{-\infty}^{+\infty} \cos[2\pi\xi_0 x] \cdot \sin[2\pi\xi x]\ dx \\ I_4[\xi] &\equiv (-\sin[\phi_0])\int_{-\infty}^{+\infty} \sin[2\pi\xi_0 x] \cdot \sin[2\pi\xi x]\ dx \end{aligned}$$

Note that the integrands of I_2 and I_3 *are the products of one even and one odd function, and that the limits are symmetric. Together, these mean that the integrals are evaluating the areas of odd functions, which must be zero:*

$$I_2[\xi] = I_3[\xi] = 0$$

The integral I_1 *may be evaluated via:*

$$\cos A\cos B = \frac{1}{2}\cos[A-B] + \frac{1}{2}\cos[A+B]$$

$$I_1[\xi] = \cos[\phi_0]\cdot\frac{1}{2}\int_{-\infty}^{+\infty}\cos[2\pi(\xi_0+\xi)x]\,dx + \cos[\phi_0]\cdot\frac{1}{2}\int_{-\infty}^{+\infty}\cos[2\pi(\xi_0-\xi)x]\,dx$$

We have already seen that the integrals evaluate to zero if the arguments of the cosines are nonzero, that they are infinite if the arguments are zero, and that they obey the requirements for Dirac delta functions:

$$\int_{-\infty}^{+\infty}\cos[2\pi(\xi_0+\xi)x]\ dx = \delta[\xi_0+\xi] = \delta[\xi+\xi_0]$$

$$\Longrightarrow I_1[\xi] = \frac{1}{2}\cos[\phi_0]\cdot(\delta[\xi_0+\xi] + \delta[\xi_0-\xi])$$

Similarly, we can evaluate $I_4[\xi]$ *via:*

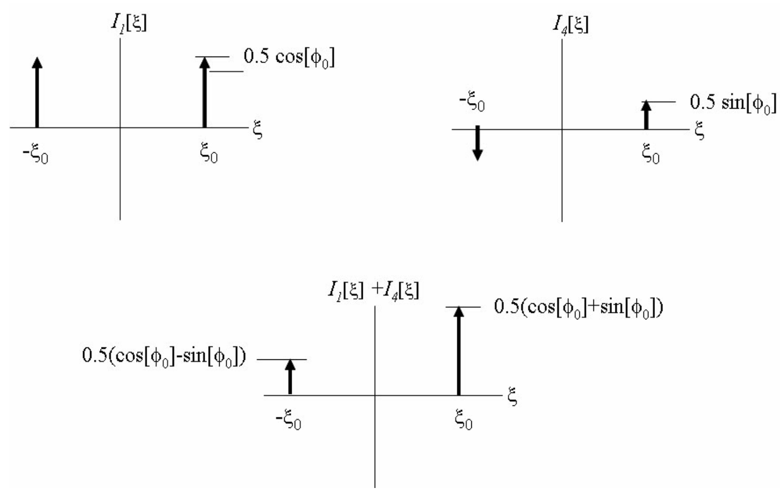
$$\sin A\sin B = \frac{1}{2}\cos[A-B] - \frac{1}{2}\cos[A+B]$$

$$
\begin{aligned}
I_4[\xi] &= (-\sin[\phi_0]) \int_{-\infty}^{+\infty} \sin[2\pi\xi_0 x] \cdot \sin[2\pi\xi x] \; dx \\
&= (-\sin[\phi_0]) \cdot \frac{1}{2} \int_{-\infty}^{+\infty} \cos[2\pi(\xi_0+\xi)x] \, dx - (-\sin[\phi_0]) \cdot \frac{1}{2} \int_{-\infty}^{+\infty} \cos[2\pi(\xi_0-\xi)x] \, dx \\
&= \left(\frac{1}{2}\sin[\phi_0]\right) \cdot (-\delta[\xi+\xi_0] + \delta[\xi-\xi_0])
\end{aligned}
$$

So the final output is the sum of I_1 *and* I_4*:*

$$
\begin{aligned}
I_1[\xi] + I_4[\xi] &= \frac{1}{2}\cos[\phi_0] \cdot (\delta[\xi_0+\xi] + \delta[\xi_0-\xi]) + \left(\frac{1}{2}\sin[\phi_0]\right) \cdot (-\delta[\xi+\xi_0] + \delta[\xi-\xi_0]) \\
&= \left(\frac{1}{2}(\cos[\phi_0] - \sin[\phi_0])\right) \cdot \delta[\xi_0+\xi] + \left(\frac{1}{2}(\cos[\phi_0] + \sin[\phi_0])\right) \cdot \delta[\xi-\xi_0]
\end{aligned}
$$

which depends on the value of ϕ_0!



Sample values of the output:

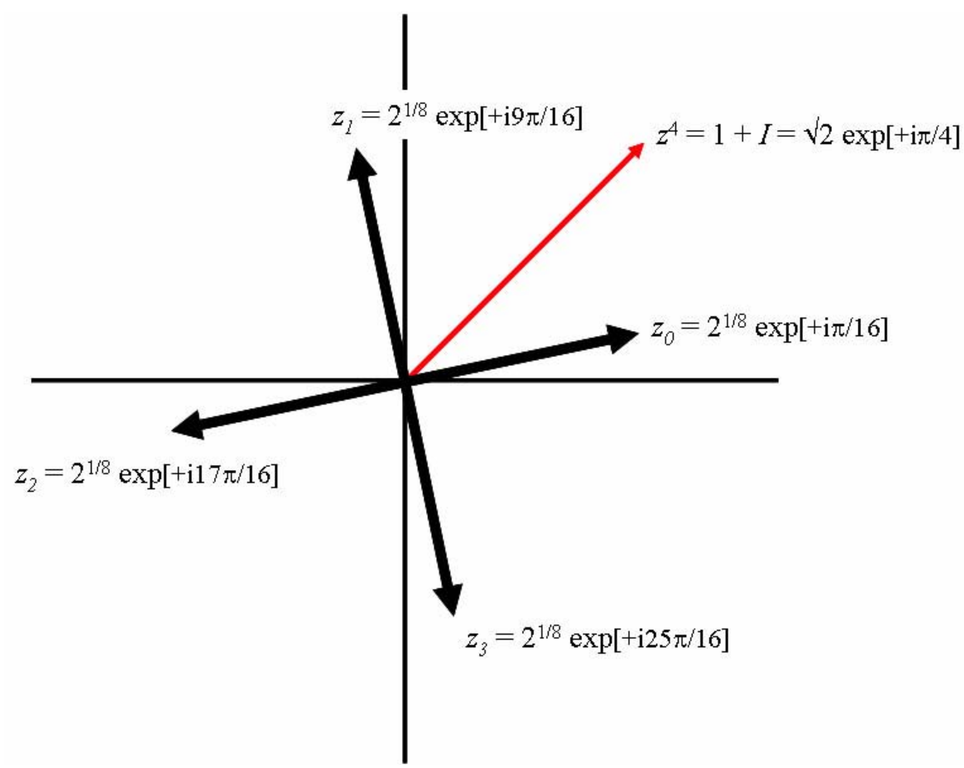
$$
\begin{aligned}
\phi_0 &= 0 \implies I_1 + I_4 = \frac{1}{2} \cdot \delta[\xi_0+\xi] + \frac{1}{2} \cdot \delta[\xi-\xi_0] \\
\phi_0 &= -\frac{\pi}{2} \implies I_1 + I_4 = -\frac{1}{2} \cdot \delta[\xi_0+\xi] + \frac{1}{2} \cdot \delta[\xi-\xi_0]
\end{aligned}
$$

Note that the output is ALWAYS real valued.

3. For TWO (2) of the following three expressions (your choice), find <u>and</u> sketch the Argand diagram of the sets of complex numbers z that satisfy these sets of conditions independently:

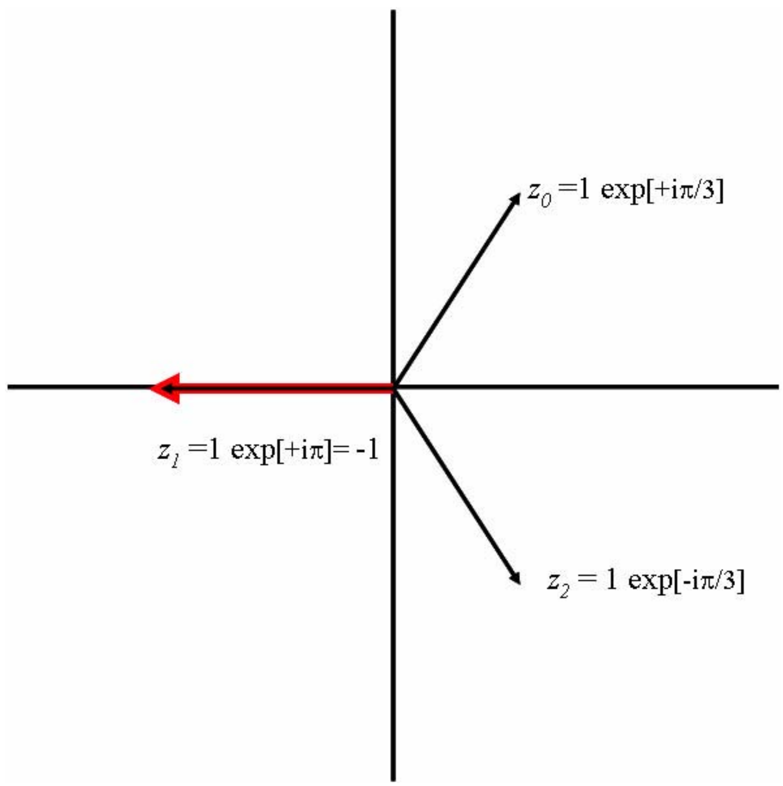
(a) $z^4 = 1 + i$

$$
\begin{aligned}
z &= |z| \exp\left[+i\phi\right] \implies z^4 = |z|^4 \exp\left[+i4\phi\right] \\
1 + i &= \sqrt{2} \exp\left[+i\left(\frac{\pi}{4} + 2\pi\ell\right)\right] \\
|z|^4 &= \sqrt{2} \implies |z| = 2^{\frac{1}{8}} \\
\exp\left[+i\left(\frac{\pi}{4} + 2\pi\ell\right)\right] &= \exp\left[+i4\phi_0\right] \implies \frac{\pi}{4} + 2\pi\ell = 4\phi \\
\phi &= \frac{\pi}{16} + \frac{\pi}{2}\ell \textit{ where } \ell = 0, 1, 2, 3 \\
\phi_0 &= \frac{\pi}{16} \implies z_0 = 2^{\frac{1}{8}} \cdot \exp\left[+i \cdot \frac{\pi}{16}\right] \\
\phi_1 &= \frac{9\pi}{16} \implies z_1 = 2^{\frac{1}{8}} \cdot \exp\left[+i \cdot \frac{9\pi}{16}\right] \\
\phi_2 &= \frac{17\pi}{16} = -\frac{15\pi}{16} \implies z_2 = 2^{\frac{1}{8}} \cdot \exp\left[+i \cdot \frac{17\pi}{16}\right] \\
\phi_3 &= \frac{25\pi}{16} = -\frac{7\pi}{16} \implies z_3 = 2^{\frac{1}{8}} \cdot \exp\left[+i \cdot \frac{25\pi}{16}\right]
\end{aligned}
$$



(b) $z^3 = -8$

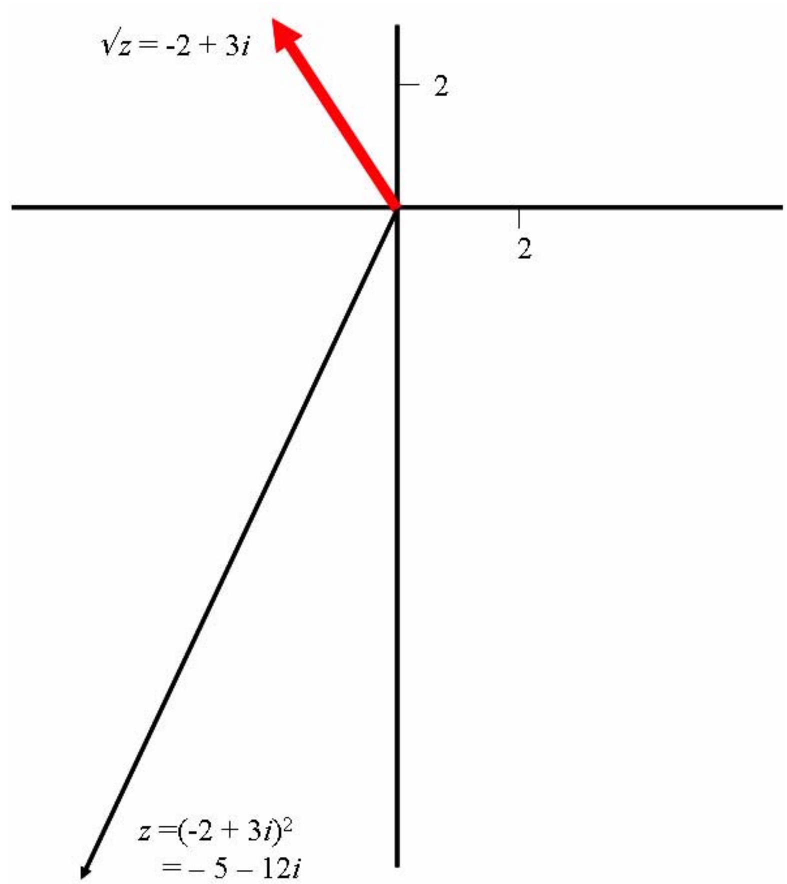
$$
\begin{aligned}
z &= |z| \exp\left[+i\phi\right] \implies z^3 = |z|^3 \exp\left[+i3\phi\right] \\
-8 &= -8 + 0i = |8| \exp\left[\pm i\left(\pi + 2\pi\ell\right)\right] \\
z &= |8|^{\frac{1}{3}} \exp\left[\pm i\frac{(\pi + 2\pi\ell)}{3}\right] = 2\exp\left[\pm i\frac{(\pi + 2\pi\ell)}{3}\right] \\
z_0 &= 2\exp\left[\pm i\frac{\pi}{3}\right] \\
z_1 &= 2\exp\left[\pm i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)\right] = 2\exp\left[\pm i\pi\right] = -2 \\
z_2 &= 2\exp\left[\pm i\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = 2\exp\left[\pm i\frac{5\pi}{3}\right] = z_0^*
\end{aligned}
$$



(c) $z^{\frac{1}{2}} = -2 + 3i$

$$z^{\frac{1}{2}} = -2 + 3i \implies z = (-2+3i)^2 = 4 - 9 - 6i - 6i$$

$$\boxed{z = -5 - 12i}$$



4. Evaluate THREE (3) of the following expressions; you may use any expressions derived in class without proof, but state what you use.

(a) $f[x] = \cos[2\pi\xi_0 x + \phi_0] * RECT\left[\frac{x}{b_0}\right]$ where ξ_0, b_0, and ϕ_0 are positive real numbers.

$$\begin{aligned}\cos[2\pi\xi_0 x + \phi_0] * RECT\left[\frac{x}{b_0}\right] &= \int_{-\infty}^{+\infty} \cos[2\pi\xi_0\alpha + \phi_0] \cdot RECT\left[\frac{x-\alpha}{b_0}\right] d\alpha \\ &= \int_{x-\frac{b_0}{2}}^{x+\frac{b_0}{2}} \cos[2\pi\xi_0\alpha + \phi_0]\ d\alpha\end{aligned}$$

Change variable of integration:

$$\begin{aligned}u &= 2\pi\xi_0\alpha + \phi_0 \implies du = 2\pi\xi_0\ d\alpha \implies d\alpha = \frac{1}{2\pi\xi_0} du \\ \alpha &= x \pm \frac{b_0}{2} \implies u = 2\pi\xi_0\left(x \pm \frac{b_0}{2}\right) + \phi_0 \\ &= 2\pi\xi_0 x + \phi_0 \pm \pi\xi_0 b_0\end{aligned}$$

$$\begin{aligned}\int_{x-\frac{b_0}{2}}^{x+\frac{b_0}{2}} \cos[2\pi\xi_0\alpha + \phi_0]\ d\alpha &= \frac{1}{2\pi\xi_0}\int_{2\pi\xi_0 x+\phi_0-\pi\xi_0 b_0}^{2\pi\xi_0 x+\phi_0+\pi\xi_0 b_0} \cos[u]\ du \\ &= \frac{1}{2\pi\xi_0}\left(\sin[u]\Big|_{2\pi\xi_0 x+\phi_0-\pi\xi_0 b_0}^{2\pi\xi_0 x+\phi_0+\pi\xi_0 b_0}\right) \\ &= \frac{1}{2\pi\xi_0} \cdot (\sin[2\pi\xi_0 x + \phi_0 + \pi\xi_0 b_0] - \sin[2\pi\xi_0 x + \phi_0 - \pi\xi_0 b_0])\end{aligned}$$

Recall that:

$$\begin{aligned}\sin[A \pm B] &= \sin A \cos B \pm \cos A \sin B \\ \sin[2\pi\xi_0 x + \phi_0 + \pi\xi_0 b_0] &= \sin[2\pi\xi_0 x + \phi_0]\cos[\pi\xi_0 b_0] + \cos[2\pi\xi_0 x + \phi_0]\sin[\pi\xi_0 b_0] \\ \sin[2\pi\xi_0 x + \phi_0 - \pi\xi_0 b_0] &= \sin[2\pi\xi_0 x + \phi_0]\cos[\pi\xi_0 b_0] - \cos[2\pi\xi_0 x + \phi_0]\sin[\pi\xi_0 b_0]\end{aligned}$$

$$\sin[2\pi\xi_0 x + \phi_0 + \pi\xi_0 b_0] - \sin[2\pi\xi_0 x + \phi_0 - \pi\xi_0 b_0] = 2\cos[2\pi\xi_0 x + \phi_0]\sin[\pi\xi_0 b_0]$$

So the final result is:

$$\begin{aligned}\cos[2\pi\xi_0 x + \phi_0] * RECT\left[\frac{x}{b_0}\right] &= \frac{1}{2\pi\xi_0} \cdot 2\cos[2\pi\xi_0 x + \phi_0]\sin[\pi\xi_0 b_0] \\ &= \frac{b_0}{b_0}\frac{\sin[\pi\xi_0 b_0]}{\pi\xi_0} \cdot \cos[2\pi\xi_0 x + \phi_0] \\ &= (|b_0| \cdot SINC[b_0\xi_0]) \cdot \cos[2\pi\xi_0 x + \phi_0]\end{aligned}$$

so the output is a replica of the input scaled by the factor $(|b_0| \cdot SINC[b_0\xi_0])$

(b) $g[x] = \cos[2\pi\xi_0 x + \phi_0] * TRI\left[\frac{x}{b_0}\right]$ where ξ_0, b_0, and ϕ_0 are positive real numbers.

The purpose of this example is to see if students can assess the difficulty quickly and move on to more reasonable cases in parts (c) and (d). But the solution IS possible (though exceedingly tedious – not somthing you would do on a two-hour test) by direct integration, but is trivial if we apply the filter theorem:

$$\cos[2\pi\xi_0 x + \phi_0] * TRI\left[\frac{x}{b_0}\right] = \mathcal{F}_1^{-1}\left\{\mathcal{F}_1\{\cos[2\pi\xi_0 x + \phi_0]\} \cdot \mathcal{F}_1\left\{TRI\left[\frac{x}{b_0}\right]\right\}\right\}$$

$$\begin{aligned}\mathcal{F}_1\{\cos[2\pi\xi_0 x + \phi_0]\} &= \mathcal{F}_1\{\cos[2\pi\xi_0 x]\cos[\phi_0] - \sin[2\pi\xi_0 x]\sin[\phi_0]\}\\ &= \cos[\phi_0]\cdot\frac{1}{2}(\delta[\xi+\xi_0] + \delta[\xi-\xi_0]) - \sin[\phi_0]\cdot\frac{i}{2}(\delta[\xi+\xi_0] - \delta[\xi-\xi_0])\\ &= \left(\frac{\cos[\phi_0] - i\ \sin[\phi_0]}{2}\right)\cdot\delta[\xi+\xi_0] + \left(\frac{\cos[\phi_0] + i\ \sin[\phi_0]}{2}\right)\cdot\delta[\xi-\xi_0]\\ &= \frac{1}{2}\exp[-i\phi_0]\cdot\delta[\xi+\xi_0] + \frac{1}{2}\exp[+i\phi_0]\cdot\delta[\xi-\xi_0]\end{aligned}$$

$$\mathcal{F}_1\left\{TRI\left[\frac{x}{b_0}\right]\right\} = |b_0|\cdot SINC^2[b_0\xi]$$

$$\begin{aligned}&\mathcal{F}_1\{\cos[2\pi\xi_0 x + \phi_0]\}\cdot\mathcal{F}_1\left\{TRI\left[\frac{x}{b_0}\right]\right\}\\ &= \left(\frac{1}{2}\exp[-i\phi_0]\cdot\delta[\xi+\xi_0] + \frac{1}{2}\exp[+i\phi_0]\cdot\delta[\xi-\xi_0]\right)\cdot|b_0|\cdot SINC^2[b_0\xi]\\ &= \frac{|b_0|}{2}\exp[-i\phi_0]\,SINC^2[-b_0\xi_0]\cdot\delta[\xi+\xi_0] + \frac{|b_0|}{2}\exp[+i\phi_0]\,SINC^2[+b_0\xi_0]\cdot\delta[\xi-\xi_0]\\ &= \frac{|b_0|}{2}\cdot SINC^2[+b_0\xi_0](\exp[-i\phi_0]\cdot\delta[\xi+\xi_0] + \exp[+i\phi_0]\cdot\delta[\xi-\xi_0])\end{aligned}$$

$$\begin{aligned}g[x] &= \mathcal{F}_1^{-1}\left\{\frac{|b_0|}{2}\cdot SINC^2[+b_0\xi_0](\exp[-i\phi_0]\cdot\delta[\xi+\xi_0] + \exp[+i\phi_0]\cdot\delta[\xi-\xi_0])\right\}\\ &= \frac{|b_0|}{2}\cdot SINC^2[+b_0\xi_0]\cdot\left(\mathcal{F}_1^{-1}\{\exp[-i\phi_0]\cdot\delta[\xi+\xi_0]\} + \mathcal{F}_1^{-1}\{\exp[+i\phi_0]\cdot\delta[\xi-\xi_0]\}\right)\\ &= \frac{|b_0|}{2}\cdot SINC^2[+b_0\xi_0]\cdot(\exp[-i(2\pi\xi_0 x + \phi_0)] + \exp[+i(2\pi\xi_0 x + \phi_0)])\\ &= \left(|b_0|\cdot SINC^2[+b_0\xi_0]\right)\cdot\cos[2\pi\xi_0 x + \phi_0]\end{aligned}$$

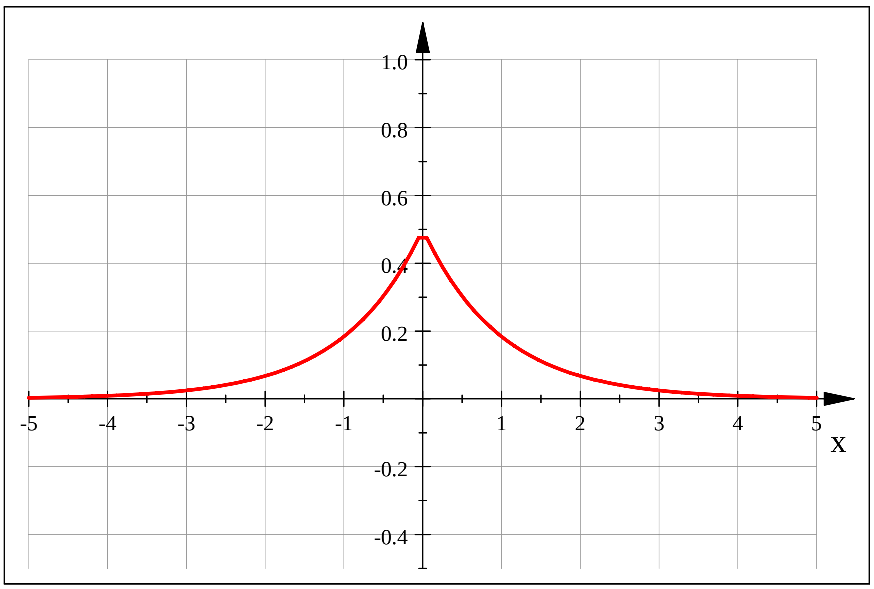
which (again) is a replica of the input function $\cos[2\pi\xi_0 x + \phi_0]$ *modulated by the Fourier transform of the impulse response* $TRI\left[\frac{x}{b_0}\right]$ *evaluated at the frequencies of the input function. In fact, not the similarity with the answer to part (a) – the only difference is that the cosine function in (a) is multiplied by* $SINC[b_0\xi_0]$ *and that in (b) is multiplied by* $SINC^2[b_0\xi_0]$

(c) $r[x] = (e^{-x} \cdot STEP[x]) \bigstar (e^{-x} \cdot STEP[x])$

$$f[x] \bigstar f[x] = f[x] * f^*[-x] \to f[x] * f[-x] \;\; \textit{because } f[x] \textit{ is real valued}$$

$$\begin{aligned}\left(e^{-x} \cdot STEP[x]\right) \bigstar \left(e^{-x} \cdot STEP[x]\right) &= \left(e^{-x} \cdot STEP[x]\right) * \left(e^{-(-x)} \cdot STEP[-x]\right) \\ &= \left(e^{-x} \cdot STEP[x]\right) * \left(e^{x} \cdot STEP[-x]\right)\end{aligned}$$

$$\begin{aligned}&= \int_{-\infty}^{+\infty} \left(e^{-\alpha} \cdot STEP[\alpha]\right) \cdot \left(e^{x-\alpha} \cdot STEP[-(x-\alpha)]\right) \, d\alpha \\ &= \int_{0}^{+\infty} \left(e^{-\alpha}\right) \cdot e^{x-\alpha} \cdot STEP[\alpha - x] \, d\alpha \\ &= \left\{ \begin{array}{ll} \int_{x}^{+\infty} \left(e^{-\alpha}\right) \cdot e^{x-\alpha} \, d\alpha & \text{if } \; x \geq 0 \\ \int_{0}^{+\infty} \left(e^{-\alpha}\right) \cdot e^{x-\alpha} \, d\alpha & \text{if } \; x < 0 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \frac{1}{2} e^{-x} & \text{if } \; x \geq 0 \\ \frac{1}{2} e^{x} & \text{if } \; x < 0 \end{array} \right\} = \frac{1}{2} e^{-|x|}\end{aligned}$$

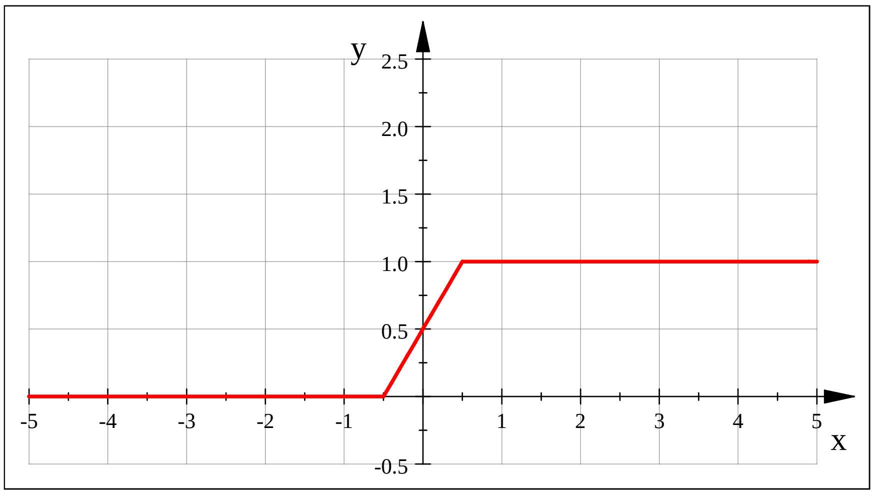


(d) $s[x] = (RECT[x+1] + RECT[x-1]) * STEP[-x]$

$$\begin{aligned} & STEP[-x] * (RECT[x+2] + RECT[x-2]) \\ = \;& (STEP[-x] * RECT[x+2]) + (STEP[-x] * RECT[x-2]) \\ = \;& (STEP[-x] * (RECT[x] * \delta[x+2])) + (STEP[-x] * (RECT[x] * \delta[x-2])) \\ = \;& (STEP[-x] * RECT[x]) * \delta[x+2] + (STEP[-x] * RECT[x]) * \delta[x-2] \end{aligned}$$

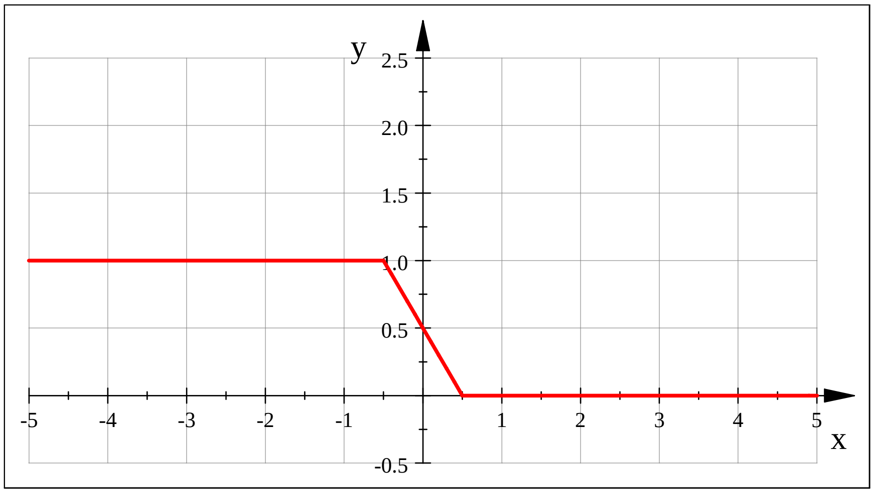
The easiest way to solve the convolution is to look at the individual pieces. Recall that:

$$\begin{aligned} STEP[x] * RECT[x] &= \int_{-\infty}^{+\infty} STEP[\alpha] \;\; RECT[x-\alpha] \;\; d\alpha \\ &= \int_{0}^{+\infty} RECT[x-\alpha] \;\; d\alpha \\ &= \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ 1 & \text{if} \quad x > +\frac{1}{2} \\ x+\frac{1}{2} & \text{if} \quad -\frac{1}{2} \le x \le +\frac{1}{2} \end{cases} \end{aligned}$$



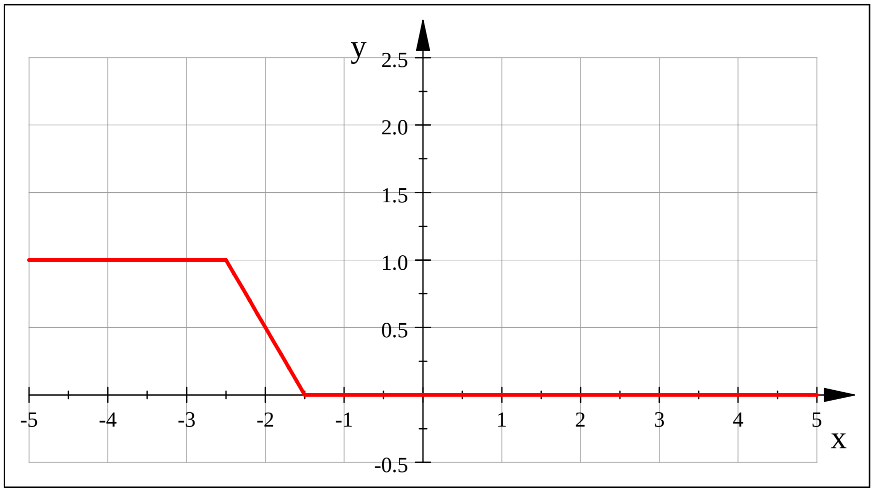
The substitution of the reversed STEP function reverses the convolution:

$$\begin{aligned} STEP[-x] * RECT[x] &= \int_{-\infty}^{+\infty} STEP[-\alpha] \;\; RECT[x-\alpha] \;\; d\alpha \\ &= \int_{-\infty}^{0} RECT[x-\alpha] \;\; d\alpha \\ &= \begin{cases} 1 & \text{if} \quad x < -\frac{1}{2} \\ 0 & \text{if} \quad x > +\frac{1}{2} \\ -x+\frac{1}{2} & \text{if} \quad -\frac{1}{2} \le x \le +\frac{1}{2} \end{cases} \end{aligned}$$

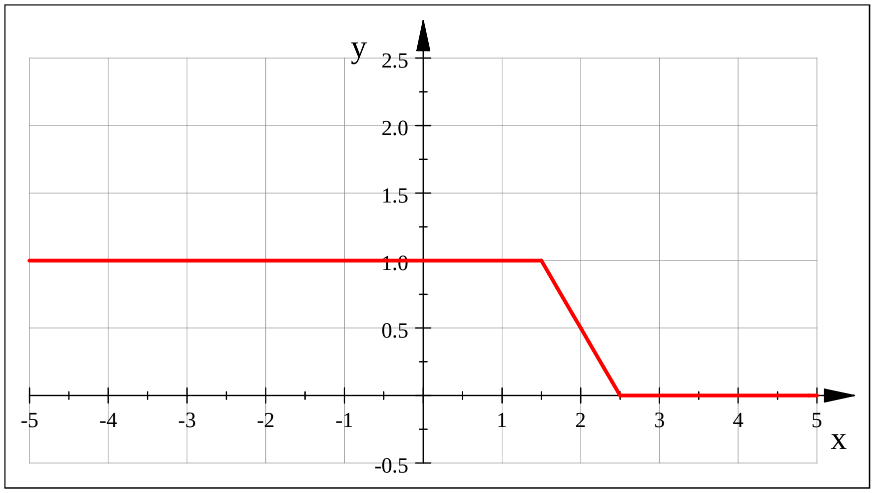


Now we have to add two translated replicas:

$$(STEP\,[-x] * RECT\,[x]) * \delta\,[x+2] = \begin{cases} 1 & \text{if} \quad x < -\frac{1}{2} - 2 \\ 0 & \text{if} \quad x > +\frac{1}{2} - 2 \\ -x - 2 + \frac{1}{2} & \text{if} \quad -\frac{1}{2} - 2 \leq x \leq +\frac{1}{2} - 2 \end{cases}$$
$$= \begin{cases} 1 & \text{if} \quad x < -\frac{5}{2} \\ 0 & \text{if} \quad x > -\frac{3}{2} \\ -x - \frac{3}{2} & \text{if} \quad -\frac{5}{2} \leq x \leq -\frac{3}{2} \end{cases}$$

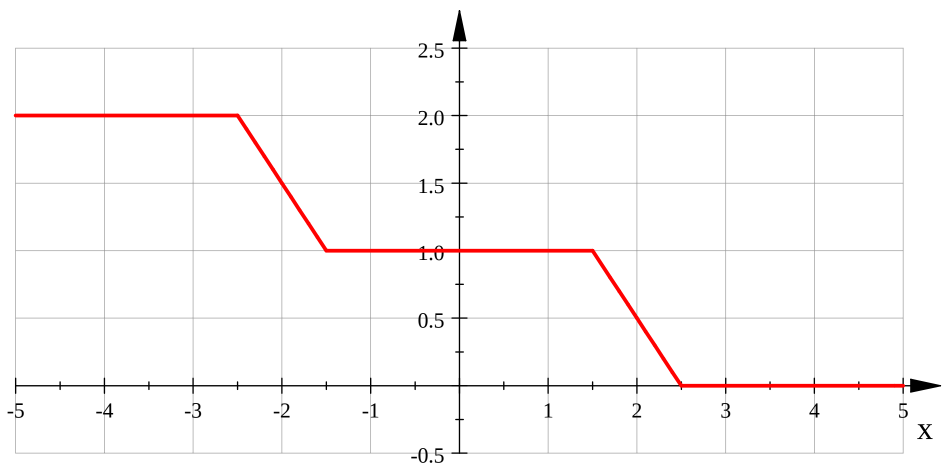


$$(STEP\,[-x] * RECT\,[x]) * \delta\,[x-2] = \begin{cases} 1 & \text{if} \quad x - 2 < -\frac{1}{2} \\ 0 & \text{if} \quad x - 2 > +\frac{1}{2} \\ -(x-2) + \frac{1}{2} & \text{if} \quad -\frac{1}{2} \leq x - 2 \leq +\frac{1}{2} \end{cases}$$
$$= \begin{cases} 1 & \text{if} \quad x < +\frac{3}{2} \\ 0 & \text{if} \quad x > +\frac{5}{2} \\ -x + \frac{5}{2} & \text{if} \quad +\frac{3}{2} \leq x \leq +\frac{5}{2} \end{cases}$$



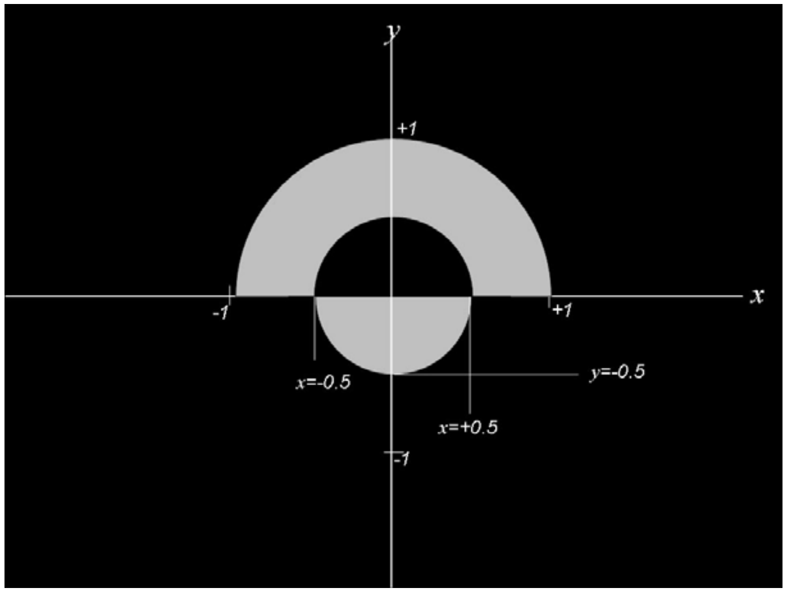
Now add these two together:

$$s[x] = \begin{cases} 0 & \text{if} & x > \frac{5}{2} \\ -x + \frac{5}{2} & \text{if} & \frac{3}{2} < x < \frac{5}{2} \\ 1 & \text{if} & \frac{3}{2} > x > -\frac{3}{2} \\ -x - \frac{1}{2} & \text{if} & -\frac{5}{2} \leq x \leq -\frac{3}{2} \\ 2 & \text{if} & x \leq -\frac{5}{2} \end{cases}$$



$STEP[-x] * (RECT[x+2] + RECT[x-2])$

5. For the real-valued 2-D function shown below, white corresponds to amplitude $\text{Re}\{f[x,y]\} = 1$, gray to amplitude $\text{Re}\{f[x,y]\} = 0.5$ and black to $\text{Re}\{f[x,y]\} = 0$.



(a) Determine the limits of the region of support of the autocorrelation.

We know that the autocorrelation is Hermitian (even real part, odd imaginary part). Since $f[x,y]$ is real, the autocorrelation is even. The function clearly includes a term with $CYL\left(\frac{r}{2}\right)$, so the autocorrelation must be bounded by the autocorrelation of this cylinder $CYL\left(\frac{r}{2}\right) \bigstar CYL\left(\frac{r}{2}\right)$, which is bounded by a cylinder of diameter 4. Since the vertical extent is $+1-\left(-\frac{1}{2}\right) = \frac{3}{2}$, the vertical extent is $+3$. The horizontal extent of the autocorrelation is $+4$. The overall support is elliptical.

(b) Evaluate the amplitude of the autocorrelation at $[x,y] = [0,0]$.

This is the volume of the square of the function. Since the amplitudes are "all" half-unity, this is easy to evaluate. The amplitude of the square of the function is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and the volumes of the individual pieces are obtained by taking half of the volume of the cylinders:

$$\begin{aligned} V &= \frac{1}{4}\cdot\left(\frac{1}{2}\cdot \textit{volume of } CYL\left(\frac{r}{2}\right) - CYL(r)\right) + \left(\frac{1}{2}\right)^2\cdot\frac{1}{2}\cdot \textit{volume of } CYL(r) \\ &= \frac{1}{4}\cdot\frac{1}{2}\cdot\left(\pi\cdot 1^2 - \pi\cdot\left(\frac{1}{2}\right)^2\right) + \left(\frac{1}{2}\right)^3\cdot 1\cdot\pi\cdot\left(\frac{1}{2}\right)^2 \end{aligned}$$

$$\boxed{\textit{volume} = \frac{\pi}{8}}$$

(c) Determine the answers to (a) and (b) if $f[x,y]$ were purely imaginary, i.e., if $\text{Re}\{f[x,y]\} = 0[x,y]$ and the imaginary part is as shown.

nothing changes: support is the same and amplitude at origin is the same.

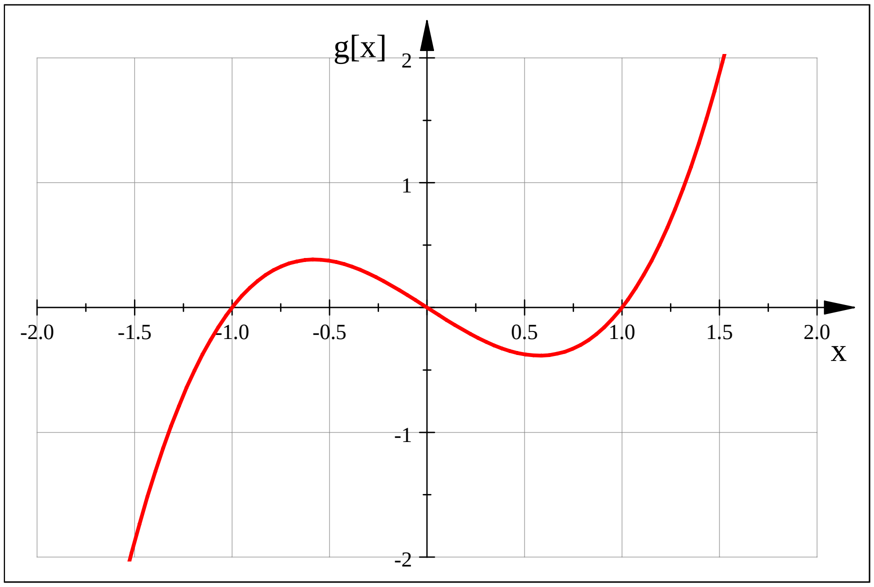
(d) OPTIONAL, EXTRA CREDIT: write down the functional form of the real-valued function $f[x,y]$ as the SUM OF TWO (and only two) terms, where each term may be the product of two (or more) special functions.

$$\boxed{f[x,y] = CYL\left(\frac{r}{2}\right) \cdot STEP[y] + CYL(r) \cdot SGN[-y]}$$

6. For ALL of the following, sketch the functions and determine their areas or volumes as appropriate:

(a) $\delta\left[x^3 - x\right]$

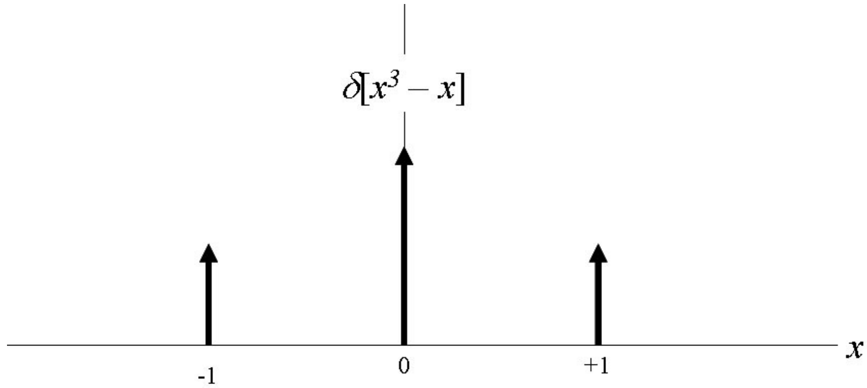
$$\begin{aligned} x^3 - x &= x\left(x^2-1\right) = 0 \implies x = 0, \pm 1 \\ \delta\left[x^3 - x\right] &= \delta\left[x \cdot (x-1) \cdot (x+1)\right] = \delta\left[g\left[x\right]\right] \end{aligned}$$



$$\begin{aligned} \delta\left[g\left[x\right]\right] &= \sum_{\text{zeros}} \frac{\delta\left[x - x_n\right]}{\left|\frac{\partial g}{\partial x}\right|_{x=x_n}} \\ g\left[x\right] &= 0 \;\; at \;\; x = 0, \pm 1 \\ \frac{dg}{dx} &= 3x^2 - 1 \\ \left|\frac{\partial g}{\partial x}\right|_{x=-1} &= \left|3x^2 - 1\right|_{x=-1} = \left|+2\right| = +2 \\ \left|\frac{\partial g}{\partial x}\right|_{x=0} &= \left|3x^2 - 1\right|_{x=0} = \left|-1\right| = +1 \\ \left|\frac{\partial g}{\partial x}\right|_{x=-1} &= \left|3x^2 - 1\right|_{x=+1} = \left|+2\right| = +2 \end{aligned}$$

$$\delta\left[x^3 - x\right] = \frac{1}{2}\delta\left[x+1\right] + \frac{1}{\left|-1\right|}\delta\left[x\right] + \frac{1}{2}\delta\left[x+1\right]$$

$$\boxed{\delta\left[x^3 - x\right] = \frac{1}{2}\delta\left[x+1\right] + \delta\left[x\right] + \frac{1}{2}\delta\left[x+1\right]}$$

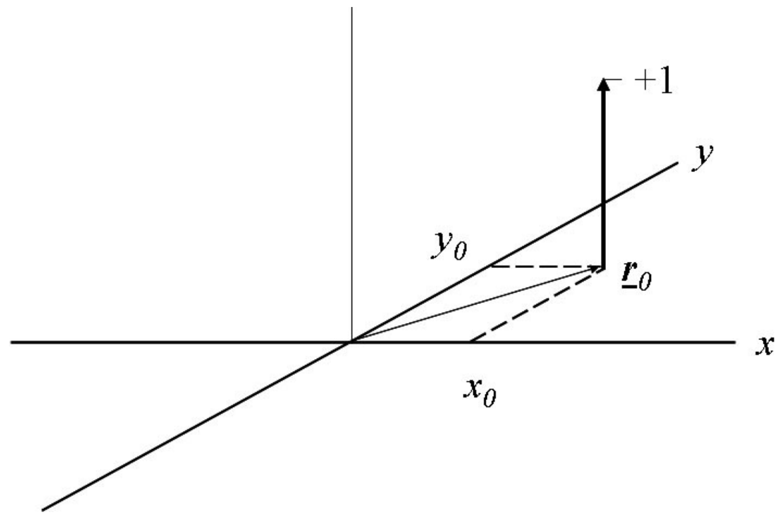


$$Area = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\delta\left[x+1\right] + \delta\left[x\right] + \frac{1}{2}\delta\left[x+1\right] \right) dx = \frac{1}{2} + 1 + \frac{1}{2} = +2$$

(b) $\delta\left[\underline{\mathbf{r}}-\underline{\mathbf{r}}_0\right]$

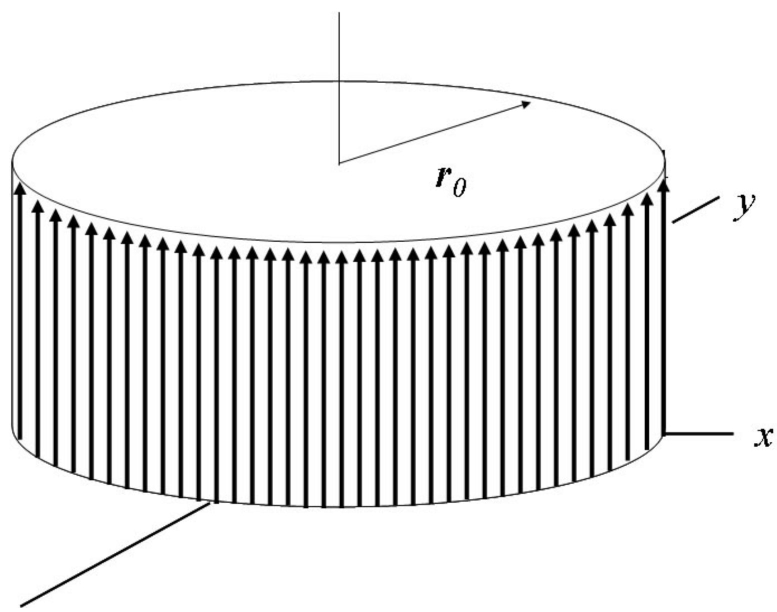
$$\boxed{\delta\left[\underline{\mathbf{r}}-\underline{\mathbf{r}}_0\right]=\delta\left[x-x_0,y-y_0\right]=\delta\left[x-x_0\right]\cdot\delta\left[y-y_0\right]}$$

$$Volume\ =1\cdot 1=1$$



(c) $\delta(r - r_0)$

$$
\begin{aligned}
\delta(r - r_0) &= \delta(r - r_0) \cdot 1(\theta) \\
Volume &= \int_{\theta=-\pi}^{\theta=+\pi} \int_{r=0}^{r=\infty} \delta(r - r_0) \cdot 1(\theta) \cdot r \cdot dr \cdot d\theta \\
&= 2\pi \cdot \int_{r=0}^{r=\infty} \delta(r - r_0) \cdot r \cdot dr \\
&= 2\pi \cdot \int_{r=0}^{r=\infty} \delta(r - r_0) \cdot r_0 \cdot dr \\
&= 2\pi r_0 \cdot \int_{r=0}^{r=\infty} \delta(r - r_0) \cdot dr \\
&= 2\pi r_0 \cdot 1 = 2\pi r_0
\end{aligned}
$$



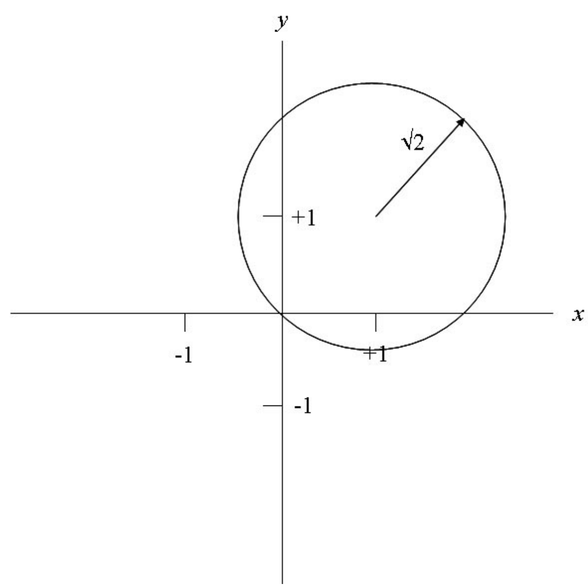
7. Sketch the following functions and find expressions for their even and odd parts. (OPTIONAL EXTRA CREDIT: sketch the even and odd parts).

(a) $f[x,y] = \delta[x-1,y-1] * \delta(r-\sqrt{2})$

We know (or should know) that convolution of some function $f[x.y]$ with a Dirac delta function yields a replica of $f[x,y]$ centered at the location of the Dirac delta function. In this example, we could consider either of these to be $f[x,y]$ but since the second is a ring delta function, it is the obvious choice for $f[x,y]$, which makes this really a snap to solve:

$$\begin{aligned}\delta[x-1,y-1] * \delta\left(r-\sqrt{2}\right) &= \delta[x-1,y-1] * \delta\left(\sqrt{x^2+y^2}-\sqrt{2}\right) \\ &= \delta\left(\sqrt{(x-1)^2+(y-1)^2}-\sqrt{2}\right)\end{aligned}$$

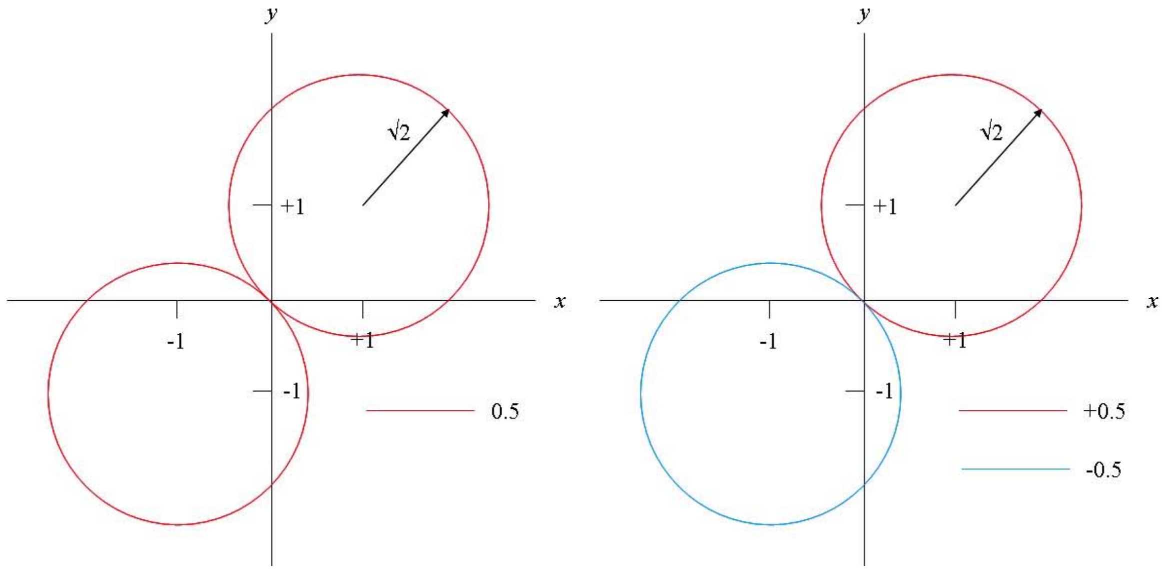
This is a ring delta function centered at $[x,y] = [+1,+1]$ with radius $r_0 = \sqrt{2}$, which means that the ring just touches the origin of coordinates.



The even and odd parts are:

$$\begin{aligned}f_{even}[x] &= \frac{1}{2}\left(\delta[x-1,y-1] * \delta\left(r-\sqrt{2}\right) + \delta[x+1,y+1] * \delta\left(r-\sqrt{2}\right)\right) \\ f_{odd}[x] &= \frac{1}{2}\left(\delta[x-1,y-1] * \delta\left(r-\sqrt{2}\right) - \delta[x+1,y+1] * \delta\left(r-\sqrt{2}\right)\right)\end{aligned}$$

and their sketches are:



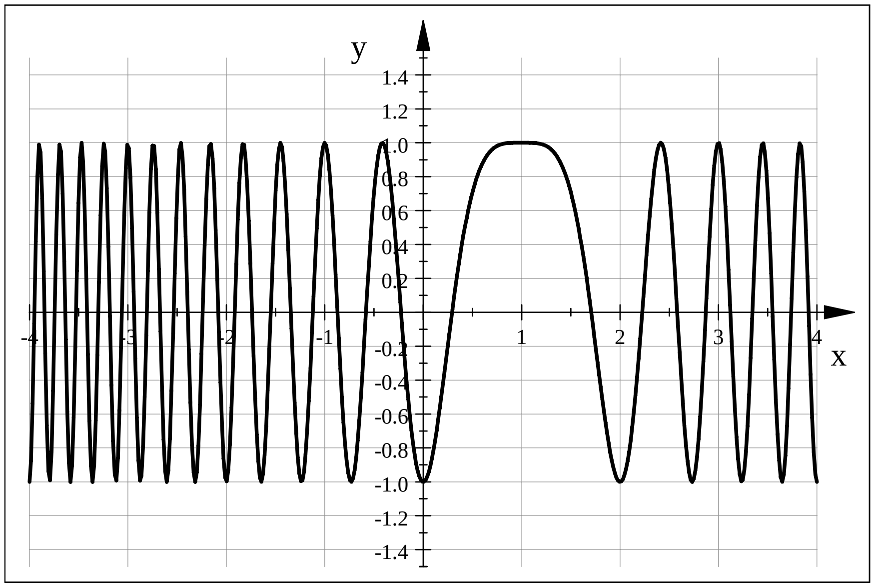
Even part of $f[x,y]$ *(LEFT) and odd part (RIGHT). Note that the sum of these yields* $f[x,y]$.

(b) $g[x] = \cos[\pi(x^2 - 2x + 1)]$ (Hint: simplify before sketching)

We can easily see that:

$$\begin{aligned}\left(x^2 - 2x + 1\right) &= (x-1)^2 \\ g[x] &= \cos\left[\pi(x-1)^2\right] \\ &= \cos\left[\pi x^2\right] * \delta[x-1]\end{aligned}$$

so $g[x]$ is a 1-D chirp with unit chirp rate centered about $x = +1$:



The even part is the average of the function and its reversed replica:

$$\begin{aligned}g_{even}[x] &= \frac{1}{2}\left(\cos\left[\pi(x-1)^2\right] + \cos\left[\pi(-x-1)^2\right]\right) \\ &= \frac{1}{2}\left(\cos\left[\pi(x-1)^2\right] + \cos\left[\pi(x+1)^2\right]\right) \\ &= \frac{1}{2}\left(\cos\left[\pi\left(\left(x^2+1\right) - 2x\right)\right] + \cos\left[\pi\left(\left(x^2+1\right) + 2x\right)\right]\right)\end{aligned}$$

Now recall that:

$$\cos[A \pm B] = \cos A \cos B \mp \sin A \sin B$$
$$\Longrightarrow \cos[A+B] + \cos[A-B] = 2\cos A \cos B$$

so that:

$$\begin{aligned}\frac{1}{2}\left(\cos\left[\pi\left(\left(x^2+1\right) - 2x\right)\right] + \cos\left[\pi\left(\left(x^2+1\right) + 2x\right)\right]\right) &= \cos\left[\pi\left(x^2+1\right)\right] \cdot \cos[2\pi x] \\ &= \cos[2\pi x] \cdot \cos\left[\pi x^2 + \pi\right] \\ & \cos[2\pi x] \cdot - \cos\left[\pi x^2\right]\end{aligned}$$

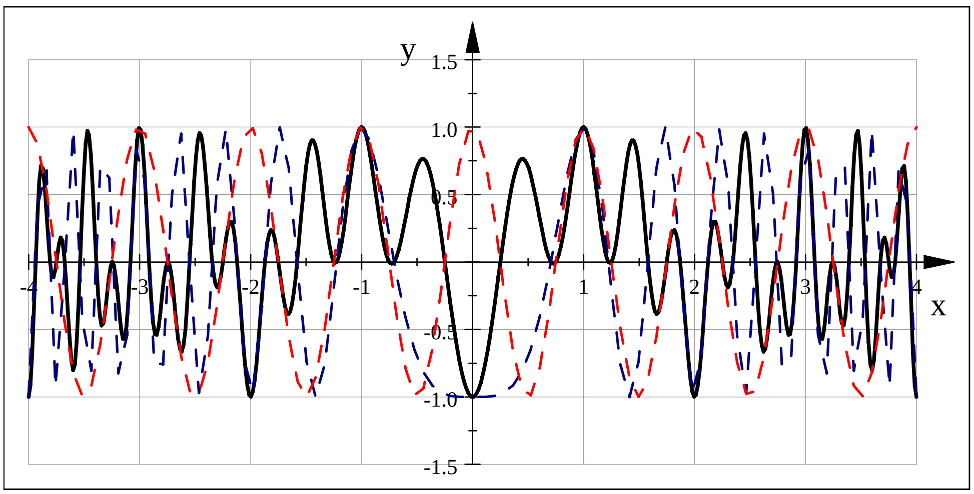
which is a cosine with unit period modulated by a "negative" chirp function with unit chirp rate.

By the same procedure, find the part to be:

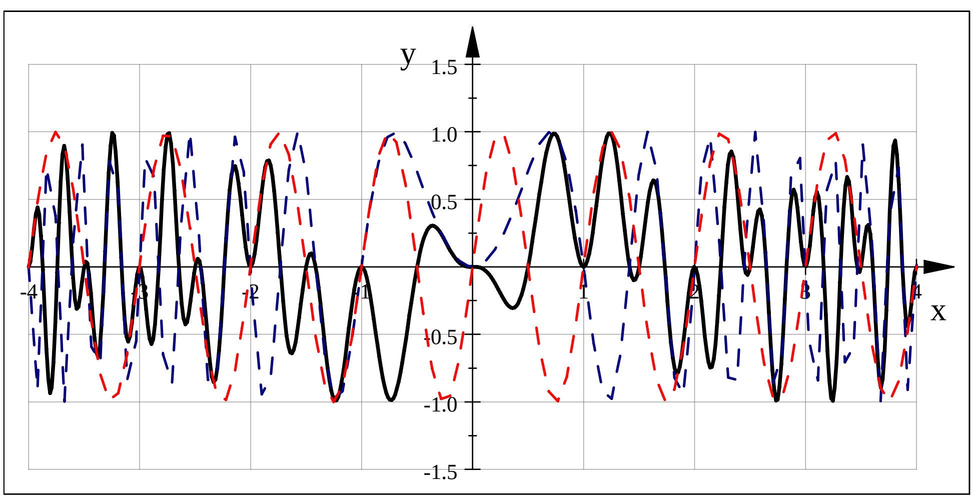
$$\begin{aligned} g_{odd}[x] &= \frac{1}{2}\left(\cos\left[\pi(x-1)^2\right] - \cos\left[\pi(-x-1)^2\right]\right) \\ &= \frac{1}{2}\left(\cos\left[\pi(x-1)^2\right] - \cos\left[\pi(x+1)^2\right]\right) \end{aligned}$$

$$\cos[A+B] - \cos[A-B] = -2\sin A \sin B$$
$$\Longrightarrow g_{odd}[x] = -\sin\left[\pi\left(x^2+1\right)\right] \cdot \sin[2\pi x]$$
$$= -\sin\left[\pi x^2 + \pi\right] \cdot \sin[2\pi x]$$
$$= +\sin[2\pi x] \cdot \sin\left[\pi x^2\right]$$

which is a sine with unit period modulated by a sine chirp with unit chirp rate. FYI, the graphs of the even and odd parts.



$-\cos[\pi x^2]$ *(blue dash),* $\cos[2\pi x]$ *(red dash), and* $g_{even}[x]$ *(solid black)*



$\sin[\pi x^2]$ *(blue dash),* $\sin[2\pi x]$ *(red dash), and* $g_{even}[x]$ *(solid black)*

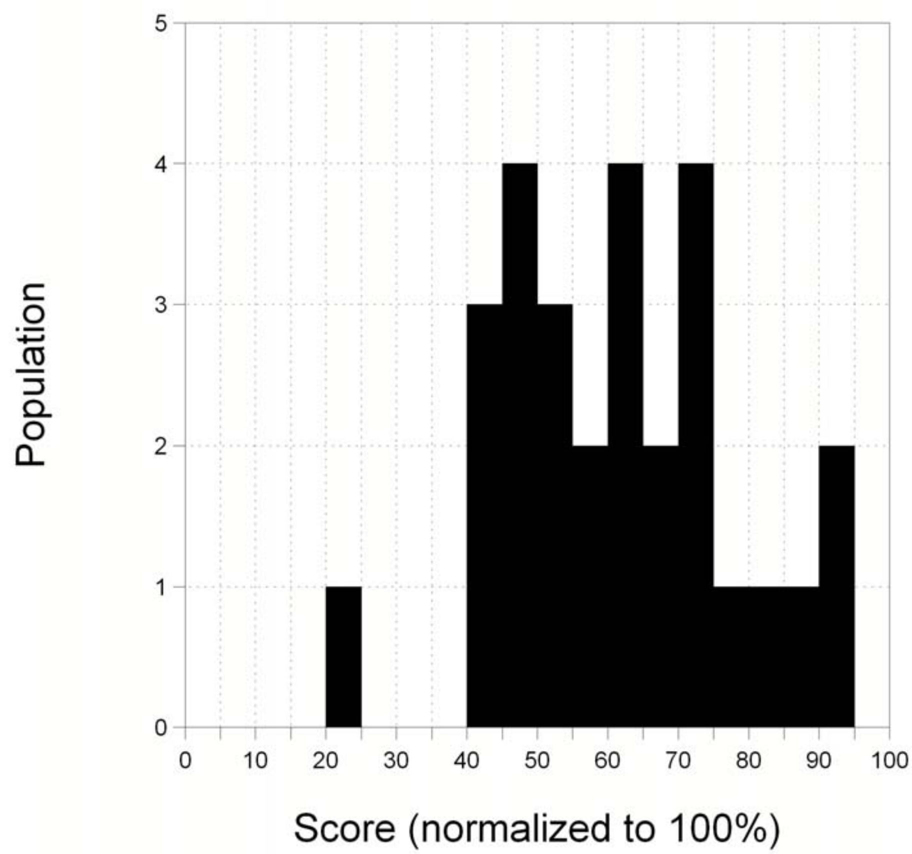
Exam Statistics: (normalized to 100%)

$\mu = 61.3$

$\sigma = 16.7$

$\max = 91.1$

$\min = 20$



Comments: the mean score on this exam were worse than usual over the years – the midterm average generally is in the range of 70% – though there were three grades ($\frac{1}{8}$ of the total) over 85%, which is typical. There were lots of simple errors (e.g., people who could graph the even and odd parts of $RAMP[x] \equiv x \cdot STEP[x]$ correctly, but misidentified which was even and which was odd, which may be the simplest concept we've considered). My recommendation is to devote more effort to the task. Even if you are doing well, you should be spending something like 12+ hours per week to the course outside of class, including readings other than the course notes.