Scalings

Shifting

\[ f(x) \Rightarrow F(\xi) \]

\[ f(x - x_0) \Rightarrow F(\xi - \xi x_0) \]

\[ F(\xi) e^{-i2\pi \xi x_0} \]

\[ \Phi = -2\pi \xi x_0 \]

\[ x_0 \propto \xi x_0 \]

\[ f(x) = R(\xi, x) \]

\[ F(\xi) = S(\xi) \]

\[ \tilde{g}(\xi) = R(\xi, x-1) \]

\[ g(1) = S(\xi) e^{-i2\pi \xi x_0} \]

\[ |\tilde{g}(\xi)| = |S(\xi)| |L(1)| \]

\[ |g(1)| = e^{i2\pi \xi x_0} |L(1)| \]
Complex Conjugate

\[ f(x) \rightarrow F^* \]

\[ f^*(x) = \left( f(x) \right)^* \rightarrow ? \]

\[ \frac{d}{dx} \left\{ f^*(x) \right\} = \int f^*(x) e^{-ix \xi} \, dx \]

\[ = \int f^*(x) \left( e^{ix \xi} \right)^* \, dx \]

\[ = \int \left( f(x) e^{ix \xi} \right)^* \, dx \]

\[ = \left( \int f(x) e^{ix \xi} \, dx \right)^* \]

\[ = F\left( -\xi \right) \]

\[ \frac{d}{dx} \left\{ f^*(-x) \right\} = \frac{d}{dx} \left\{ f^*(x) \right\} = F^* \left( \frac{x}{\xi} \right) \]
F.T. of \( f(x) \star m(x) = f(x) \star m^*(x) \)

\[
\mathcal{F}\left\{ f(x) \star m(x) \right\} = F(\xi) \cdot M^*(\xi)
\]

**Autocorrelation**

\[
\mathcal{F}\left\{ f(x) \star f(x) \right\} = \mathcal{F}\left\{ f(x) \star f^*(x) \right\}
\]

\[
= F(\xi) \cdot F^*(\xi)
\]

\[
= \left| F(\xi) \right|^2
\]

**Wiener - Khinchin Theorem**

\[
f(x) \star f(x) = f(x) \cdot f^*(x)
\]

\[
= \int_{-\infty}^{\infty} f(a) f^*(a-x) \, da
\]

\[
\mathcal{F}\left\{ f(x) \star f(x) \right\} = \left| F(\xi) \right|^2
\]

\[
\mathcal{F}\left\{ f(x) \star m(x) \right\} = F(\xi) \cdot M^*(\xi)
\]
FT of Periodic Function

Discrete ("Sample") Function

\[ f(x) \text{ Periodic over } X_0 \]

\[ w(x) = f(x) \cdot \text{Rect} \left( \frac{x}{X_0} \right) \]

\[ f(x) = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x}{X_0} - n \right) = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x - nX_0}{X_0} \right) \]

\[ \text{comb} \left( \frac{x}{X_0} \right) = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x}{X_0} - n \right) = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x - nX_0}{X_0} \right) \]

\[ f(x) = w(x) \cdot \sqrt{\frac{X_0}{2}} \text{comb} \left( \frac{x}{X_0} \right) \]

\[ = W(\xi) \cdot \sqrt{\frac{X_0}{2}} \text{comb} \left( \frac{X_0}{X_0} \right) \]

\[ = W(\xi) \cdot \sum_{k=-\infty}^{\infty} \delta \left( \frac{\xi}{X_0} - k \right) \]

\[ = W(\xi) \cdot \sum_{k=-\infty}^{\infty} \delta \left( \xi - \frac{kX_0}{X_0} \right) \]

\[ = \sum_{n=-\infty}^{\infty} W(\xi) \delta \left( \xi - \frac{nX_0}{X_0} \right) \]

\[ = \frac{1}{X_0} \sum_{n=-\infty}^{\infty} W(\xi) \delta \left( \xi - \frac{nX_0}{X_0} \right) \]

\[ \text{Periodic } f(x) \rightarrow \text{Discrete } F(\xi) \]

\[ \cos(2\pi \xi_0 x) \rightarrow \frac{1}{2} \delta(\xi + \xi_0) + \frac{1}{2} \delta(\xi - \xi_0) \]
\[ f(x) \cdot \frac{1}{\Delta x} \text{comB}\left(\frac{x}{\Delta x}\right) = f(x) \cdot \sum \delta(x - n\cdot\Delta x) = \sum f(n\cdot\Delta x) \delta(x - n\cdot\Delta x) \]

\[ = f(x) \cdot \frac{1}{\Delta x} \text{comB}\left(\frac{x}{\Delta x}\right) \]

\[ \rightarrow F(\xi) \times \frac{1}{\Delta x} \text{comB}\left(\Delta x \cdot \xi\right) \]

\[ = F(\xi) \times \sum \delta(\Delta x \cdot \xi - k) \]

\[ = F(\xi) \times \sum \delta(\xi - \frac{k}{\Delta x}) \]

\[ = \frac{1}{\Delta x} \sum F(\xi - \frac{k}{\Delta x}) \] \text{Periodic}

\[ \text{Period} = \frac{1}{\frac{1}{\Delta x}} = \frac{1}{\Delta x} \]

\[ \rightarrow \text{SOURCE OF ALIASING} \]
\[
\begin{align*}
    f(x) & \rightarrow F(\xi) \\
\text{LSI} & \quad f(x) \star h(x) \Rightarrow F(\xi) \cdot H(\xi) = G(\xi) \\
    F(\xi_0) &= 0 \quad \Rightarrow \quad G(\xi_0) = 0 \\
    H(\xi_0) &= 0 \quad \Rightarrow \quad G(\xi_0) = 0 \quad \text{even} \quad \Rightarrow \quad F(\xi) \neq 0
\end{align*}
\]

\[
\begin{align*}
    \mathcal{O}\left\{ f(x) \right\} &= g(x) \\
    g(x) &= \left( f(x) \right)^2 \\
    G(\xi) &= \mathcal{F}\left\{ f(x) \right\} = \frac{1}{2} \int \{ f(x) \cdot f(x) \} = \mathcal{F}(F(\xi) \cdot F(\xi)) \\
    f(x) &= \text{SINC}(x) \quad \Rightarrow \quad F(\xi) = \text{RELT}(\xi) \\
    g(x) &= \text{SINC}^2(x) \quad \Rightarrow \quad G(\xi) = \text{TRF}(\xi)
\end{align*}
\]

\[
\begin{align*}
    f(x) &= \cos(2\pi f_0 x) \quad \Rightarrow \quad F(\xi) = \frac{1}{2} \delta(\xi + f_0) + \frac{1}{2} \delta(\xi - f_0) \\
    g(x) &= \cos^2(2\pi f_3 x) \quad \Rightarrow \quad G(\xi) = \frac{1}{4} \delta(\xi + 2f_3) + \frac{1}{2} \delta(\xi) + \frac{1}{4} \delta(\xi - 2f_3)
\end{align*}
\]
Thresholding Operator

\[
F(x) = \frac{1}{2} \delta(x - \frac{\pi}{2} \xi_0) + \frac{1}{2} \delta(x + \frac{\pi}{2} \xi_0)
\]

\[
G(x) = 2 \text{Re} \{ \frac{x}{\xi_0} \} \ast \frac{1}{\xi_0} \text{comb} \left( \frac{x}{\xi_0} \right) - 1(x)
\]

\[
g(x) = 2 \frac{\text{Re} \{ \frac{x}{\xi_0} \}}{\xi_0} \ast \frac{1}{\xi_0} \text{comb} \left( \frac{x}{\xi_0} \right) - 1(x)
\]

\[
G(x) = \frac{2}{\xi_0} \text{sinc} \left( \frac{\xi_0 x}{2} \right) - \frac{1}{\xi_0} \text{comb} \left( \frac{\xi_0 x}{2} \right) - g(x)
\]

\[
G(x) = \frac{1}{\xi_0} \text{comb} \left( \frac{\xi_0 x}{2} \right) - \frac{1}{\xi_0} \text{comb} \left( \frac{\xi_0 x}{2} \right) - g(x)
\]
Power Law  \[ g(x) = (f(x))^\theta \]

\[ f(x) = e^{i \theta x} \Rightarrow F(\xi) = \delta(\xi - \xi_0) \]
\[ g(x) = \left( e^{i \theta x} \right)^\theta = e^{i \theta (\xi x)} \Rightarrow \zeta(\xi) = \delta(\xi - \xi_0) \]

\[ \left( \phi(x) \right)^{\frac{1}{2}} \Rightarrow \zeta(\xi) = \delta(\xi - \xi_0) \]

\[ A(x) = \alpha_1 e^{i \theta x} + \alpha_2 e^{i \theta x} \]
\[ A(\xi) = \alpha_1 \delta(\xi - \xi_1) + \alpha_2 \delta(\xi - \xi_2) \]
\[ \alpha_1 < \alpha_2 \Rightarrow \alpha_1 e^{i \theta (\xi - \xi_1)} \left[ 1 + \frac{\alpha_2}{\alpha_1} e^{i \theta (\xi_2 - \xi_1)} x \right] \]

\[ g(x) = \left( \delta(\xi - \xi_1) \right)^\theta \left[ 1 + \frac{\alpha_2}{\alpha_1} e^{i \theta (\xi_2 - \xi_1)} x \right] \]

\[ (1 + \eta)^n = 1 + \frac{\eta}{1!} + \frac{n(n-1)}{2!} \eta^2 + |\eta|^3 + \ldots \]