4 November 2009

Phase Filters

Computing Fourier Transform via Quad. Phase Operations

"Chirp Fourier Transform"

$$\int_{-\infty}^{+\infty} f(x)\, \underbrace{e^{-2\pi i \xi x}}\, dx$$

$$e^{+i\pi(-2\xi x)}$$

$$-2\xi x = (\xi - x)^2 - \xi^2 - x^2$$

$$-2\xi x = \left(\alpha\xi - \frac{x}{\alpha}\right)^2 - (\alpha\xi)^2 - \left(\frac{x}{\alpha}\right)^2$$

$$F(\xi) = \int_{-\infty}^{+\infty} \underline{f(x)}\, e^{+i\pi\left(\alpha\xi - \frac{x}{\alpha}\right)^2}\, e^{-i\pi(\alpha\xi)^2}\, \underline{e^{-i\pi\left(\frac{x}{\alpha}\right)^2}}\, dx$$

$$F(\xi) = e^{-i\pi(\alpha\xi)^2} \int_{-\infty}^{+\infty} \Big(\underbrace{f(x)\, e^{-i\pi\left(\frac{x}{\alpha}\right)^2}}_{M} \Big)\, e^{+i\pi\left(\alpha\xi - \frac{x}{\alpha}\right)^2} dx$$

$$h\left(\alpha\xi - \frac{x}{\alpha}\right)$$

WHERE $h(u) = e^{+i\pi u^2}$

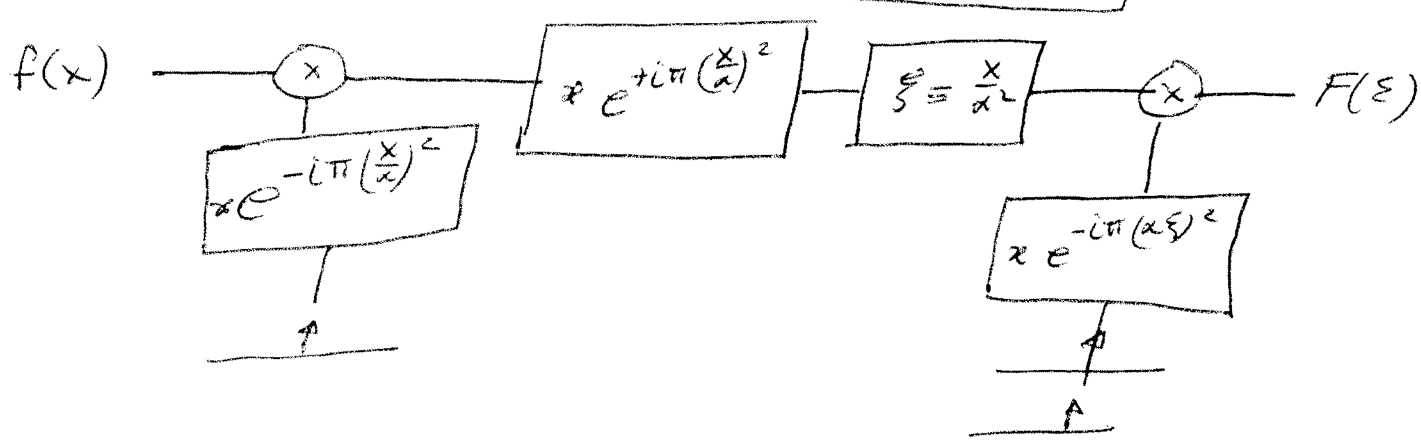
CONVOLUTION WITH $h(u)$

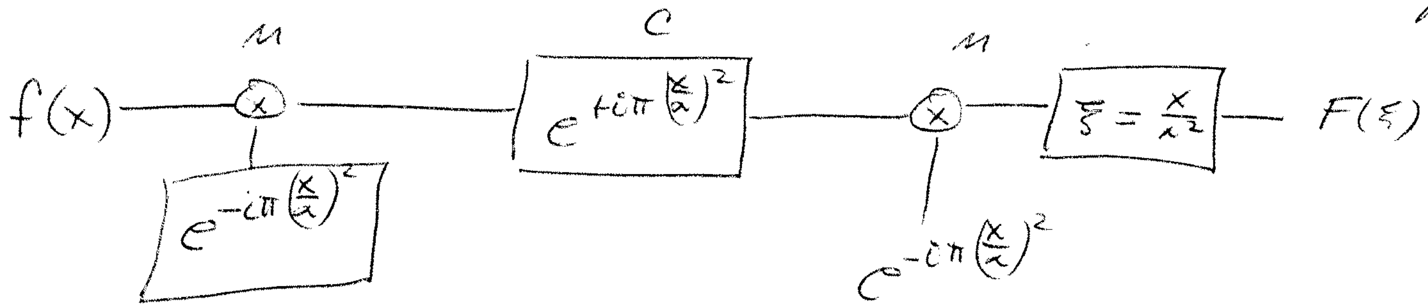
EVALUATED WHERE $\alpha\xi = \frac{x}{\alpha}$

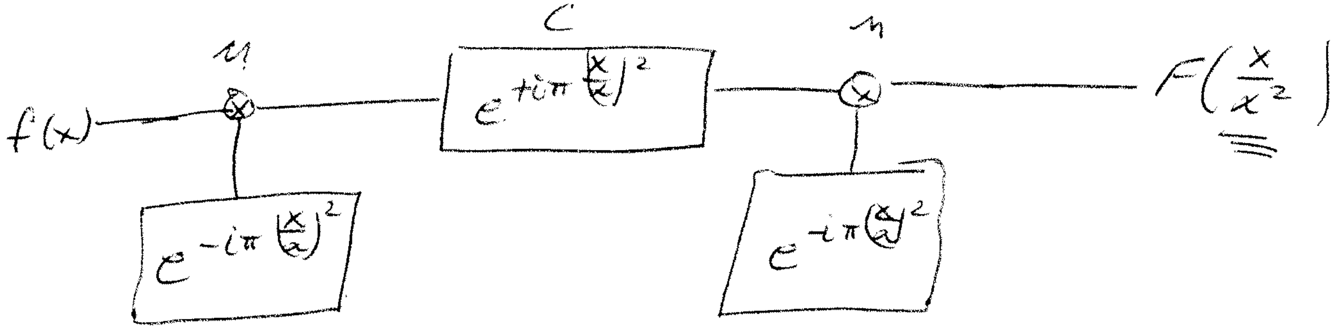
$$\xi = \frac{x}{\alpha^2}$$

$$x = \alpha^2 \xi$$

$$\boxed{F(\xi) = \left(\Big(\underbrace{f(x) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}}_{M} \Big) * \underbrace{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}}_{C} \right)\Bigg|_{\xi = \frac{x}{\alpha^2}} \cdot \underbrace{e^{-i\pi(\alpha\xi)^2}}_{M}}$$







ALL IN SPACE DOMAIN

$$\left(\left(f(x)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right)\cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = F\left(\frac{x}{\alpha^2}\right)$$

$$F\left(-\frac{x}{\alpha^2}\right) \xrightarrow{\mathcal{F}_1} \alpha^2 f\left(-\left(-\alpha^2\xi\right)\right) = \alpha^2 f\left(\alpha^2\xi\right)\Big|_{\xi=\frac{x}{\alpha^2}}$$

$$= \alpha^2 f(x)$$

$$\left(\left(F\left[-\frac{x}{\alpha^2}\right] \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2} = \underline{\alpha^2 f(x)}$$

$$\left(\left(\alpha^2 f(\alpha^2\xi) * \cancel{e}\, |\alpha|\, e^{-i\frac{\pi}{4}} e^{+i\pi(\alpha\xi)^2}\right) \cdot |\alpha|\, e^{+i\frac{\pi}{4}} e^{-i\pi(\alpha\xi)^2}\right) * |\alpha| e^{-i\frac{\pi}{4}}\, e^{+i\pi(\alpha\xi)^2}$$

$$\cancel{|\alpha|^3}\, e^{-i\frac{\pi}{4}}\left[\left(\left(f(\alpha^2\xi) * e^{+i\pi(\alpha\xi)^2}\right) \cdot e^{-i\pi(\alpha\xi)^2}\right) * e^{+i\pi(\alpha\xi)^2}\right] = \alpha^2 F(\xi) = \cancel{\alpha^2} F(\xi)$$

$|\alpha|^3$

$$f(\alpha^2\xi) * e^{+i\pi \cancel{(\alpha\xi)^2} \frac{(\alpha^2\xi)^2}{\alpha^2}}$$

IF $f(x) * h(x) = g(x)$ THEN $f\left(\frac{x}{b}\right) * h\left(\frac{x}{b}\right) = |b|\, g\left(\frac{x}{b}\right)$

$\alpha^2\xi = x$

SKIPPING STEPS

$$\underline{\underline{\frac{1}{|\alpha|} e^{-i\frac{\pi}{4}}}}\left[\left(\left(f(\cancel{\alpha^2\xi}\; x) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\alpha}\right)^2}\right) * e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right] = F\left(\frac{x}{\alpha^2}\right)$$

C M C

$$\left(\left(\left(f(x) * e^{+i\pi\left(\frac{x}{\lambda}\right)^2}\right) \cdot e^{-i\pi\left(\frac{x}{\lambda}\right)^2}\right) * e^{+i\pi\left(\frac{x}{\lambda}\right)^2}\right) \cdot \frac{1}{|\alpha|} e^{-i\frac{\pi}{4}} = F\left(\frac{x}{\alpha^2}\right)$$

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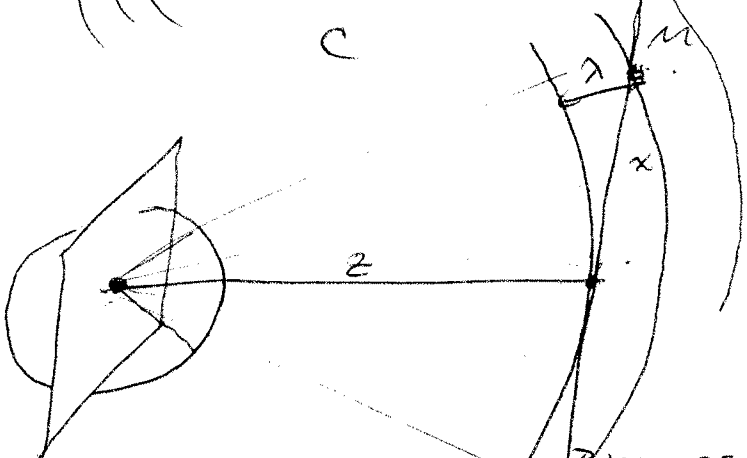

$$x^2 + y^2 + z^2 = R^2$$

$$\cancel{x^2+y^2} \quad z^2\left(1 + \frac{x^2+y^2}{z^2}\right) = R^2$$

SURFACE OF CONSTANT PHASE OF WAVE IS A SPHERE

CHRISTIAAN HUYGENS

$$z^2\left(1 + \left(\frac{x^2+y^2}{z^2}\right)\right) \simeq z^2\left(1 + \frac{x^2+y^2}{z^2}\right)$$

IF z IS LARGE,

SPHERE IS APPROXIMATED BY PARABOLOID ⇒ QUADRATIC PHASE

⇒ FRESNEL APPROXIMATION

IF $f(x) = \delta(x)$, $g(x) = e^{+i\pi \frac{x^2}{\alpha^2}}$ ← $\lambda_0 z$

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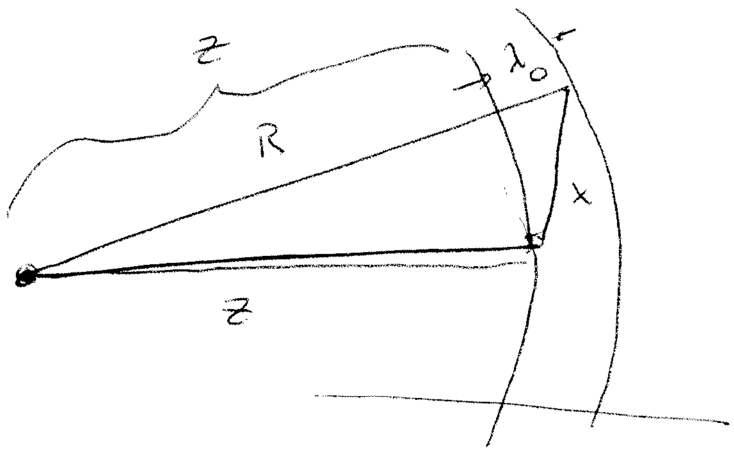


$$z^2 + x^2 = R^2 \Rightarrow x = \sqrt{R^2 - z^2}$$

$$R = z + \lambda_0 \Rightarrow R^2 = z^2 + \lambda_0^2 + 2\lambda_0 z$$

$$R^2 - z^2 = 2\lambda_0 z + \cancel{\lambda_0^2} = x^2$$

$$\xrightarrow{} 0$$

$x^2 = 2\lambda_0 z$ IF PHASE CHANGE IS 2π

$x = \sqrt{2\lambda_0 z}$ FOR $\Delta\phi = 2\pi$

$$e^{+i\pi\left(\frac{x}{\alpha}\right)^2}$$

α IS DISTANCE WHERE $\Delta\phi = \pi$

$$\alpha = \frac{\sqrt{2\lambda_0 z}}{\sqrt{2}} = \sqrt{\lambda_0 z}$$

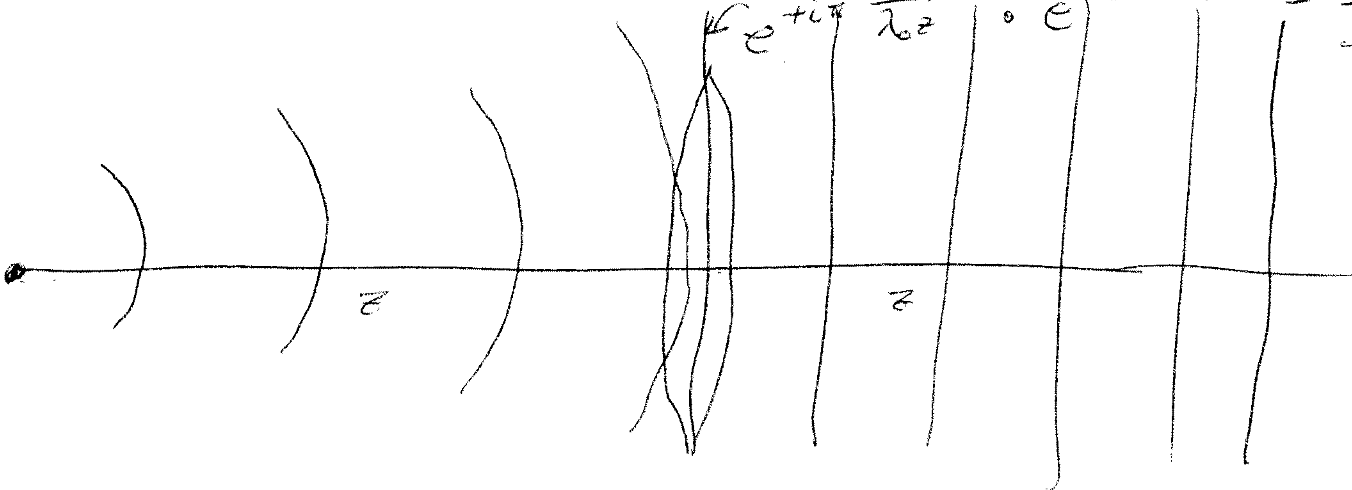
$$h[x; z_0, \lambda_0] \propto e^{+i\pi \left(\frac{x}{\sqrt{\lambda_0 z}}\right)^2} = e^{+i\pi \frac{x^2}{\lambda_0 z}}$$

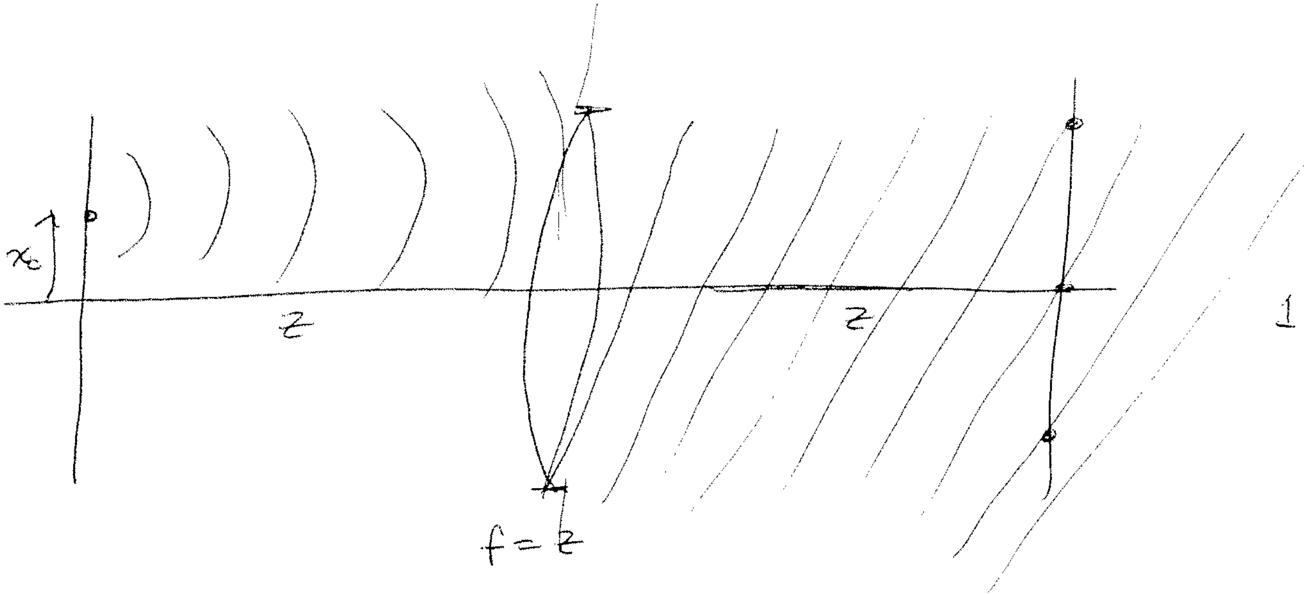
(APPROXIMATE)
IMPULSE RESPONSE OF LIGHT PROPAGATION OVER DISTANCE z WITH λ_0

$$h(x, y, z, \lambda_0] \propto e^{+i\pi \frac{x^2+y^2}{\lambda_0 z}}$$

$$g(x,y; z, \lambda_0] = f(x,y) * h(x,y; z, \lambda_0]$$

$$e^{+i\pi \frac{x^2+y^2}{\lambda_0 z}} \cdot e^{-i\pi \frac{x^2+y^2}{\lambda_0 z}} = \underline{1(x,y]}$$

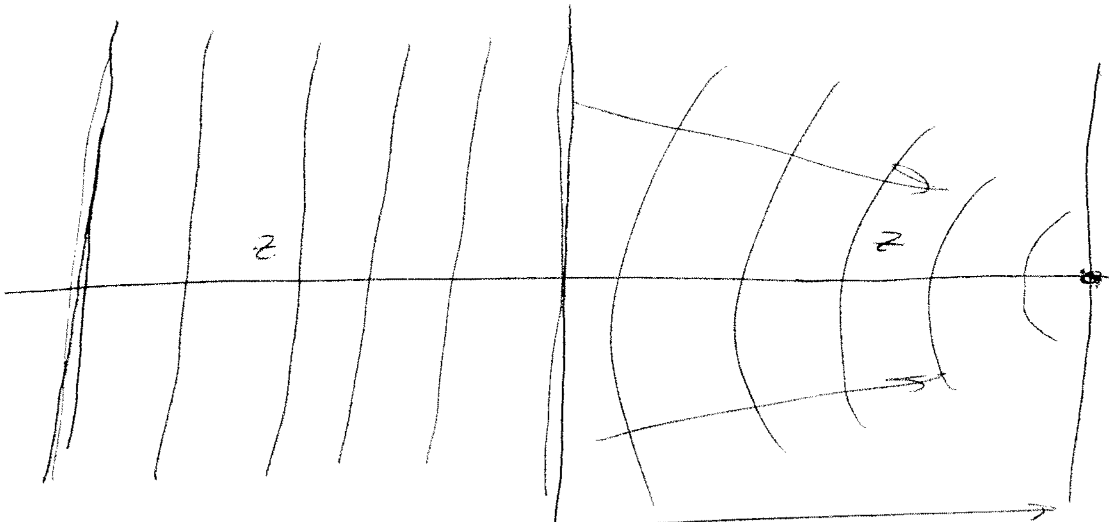




$1(x)\, e^{+2\pi i x_0 \xi}$

$\uparrow$

$\frac{x}{\lambda z}$



$g(x,y) \propto \delta(x,y)$

$f(x,y) = 1(x,y)$ $f = z$ z

$\propto e^{+i\pi \frac{x^2+y^2}{\lambda_0 z}}$ $\uparrow$ $\propto e^{+i\pi \frac{x^2+y^2}{\lambda_0 z}}$ $\longrightarrow$ $g(x,y) \propto F\left\{\underbrace{\frac{x}{\lambda_0 z}}_{\xi}, \frac{y}{\lambda_0 z}\right\}$

$e^{-i\pi \frac{x^2+y^2}{\lambda_0 z}}$

$F\{\xi, \eta\}$

MAGNITUDE & PHASE FILTERS

$$H[\xi] = |H(\xi)|\, e^{+i\,\Phi\{H(\xi)\}}$$

$$g(x) = \int_{-\infty}^{x} f(u)\,du$$

3 FILTERS

INTEGRATION $\quad h(x) = \mathrm{STEP}(x) \Rightarrow H(\xi) = \left(\frac{1}{2\pi i \xi} + \frac{1}{2}\delta(\xi)\right)$

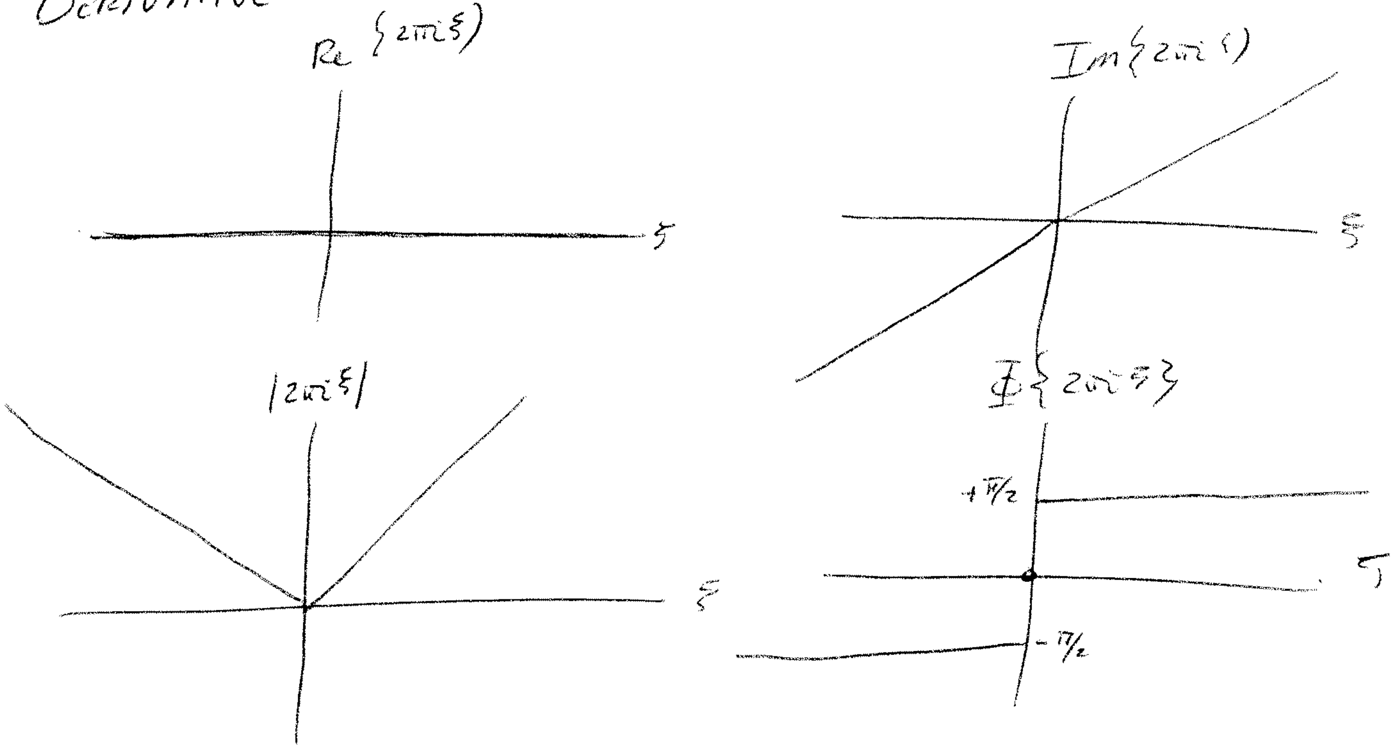
IDENTITY OPERATOR $\quad h(x) = \delta(x) \Rightarrow H(\xi) = 1(\xi) \Rightarrow g(x) = f(x)$

DERIVATIVE $\quad h(x) = \delta'(x) = \frac{d\delta}{dx} \Rightarrow H(\xi) = (+2\pi i \xi) \Rightarrow g(x) = \frac{df}{dx}$

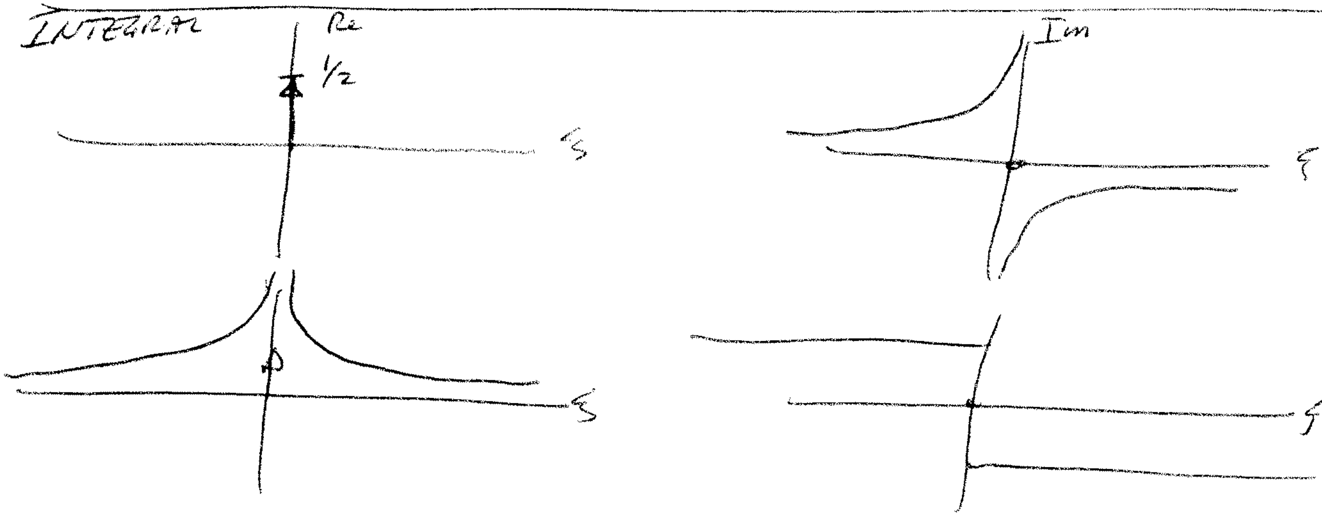
$\times 2\pi i \xi \qquad \div 2\pi i \xi$

$$h(x) = \delta''(x) \Rightarrow H(\xi) = (+2\pi i \xi)^2 \Rightarrow g(x) = \frac{d^2 f}{dx^2}$$

DERIVATIVE

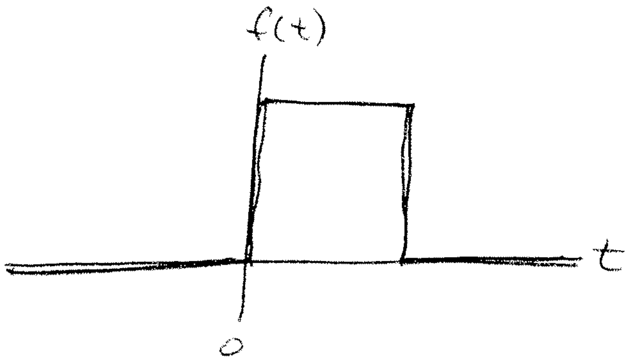


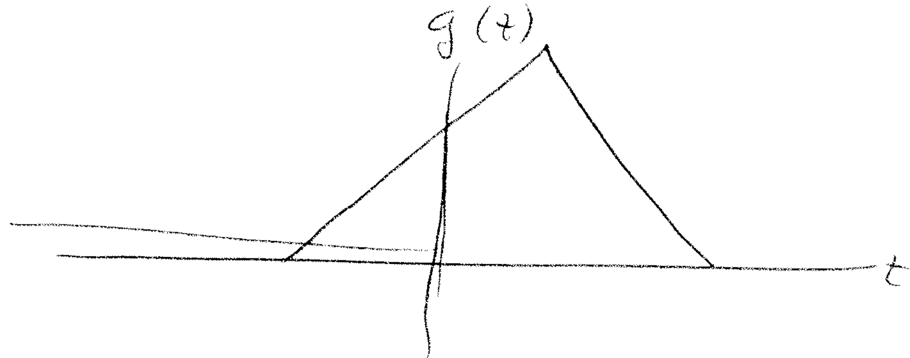
INTEGRAL



$f(x) \longrightarrow f(t)$ $\qquad$ $F(\xi) \longrightarrow F(\nu)$

$h(x) \longrightarrow h(t)$ $\qquad$ $H(\xi) \longrightarrow H(\nu)$



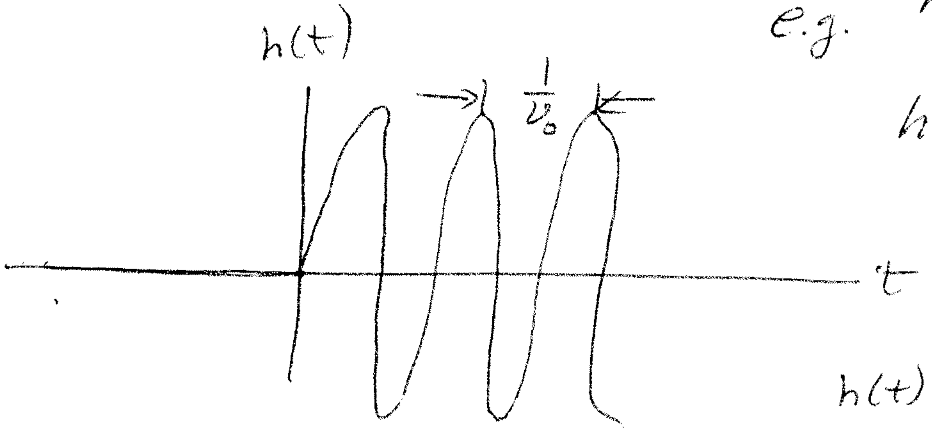


PROPERTY OF CAUSALITY $\Rightarrow$ $h(t) = 0$ IF $t < 0$

e.g. $h(t) = e^{-t}\ \mathrm{STEP}(t)$

$h(t) = \cos(2\pi\nu_0 t + \cancel{\varphi_0})\ \mathrm{STEP}(t)$

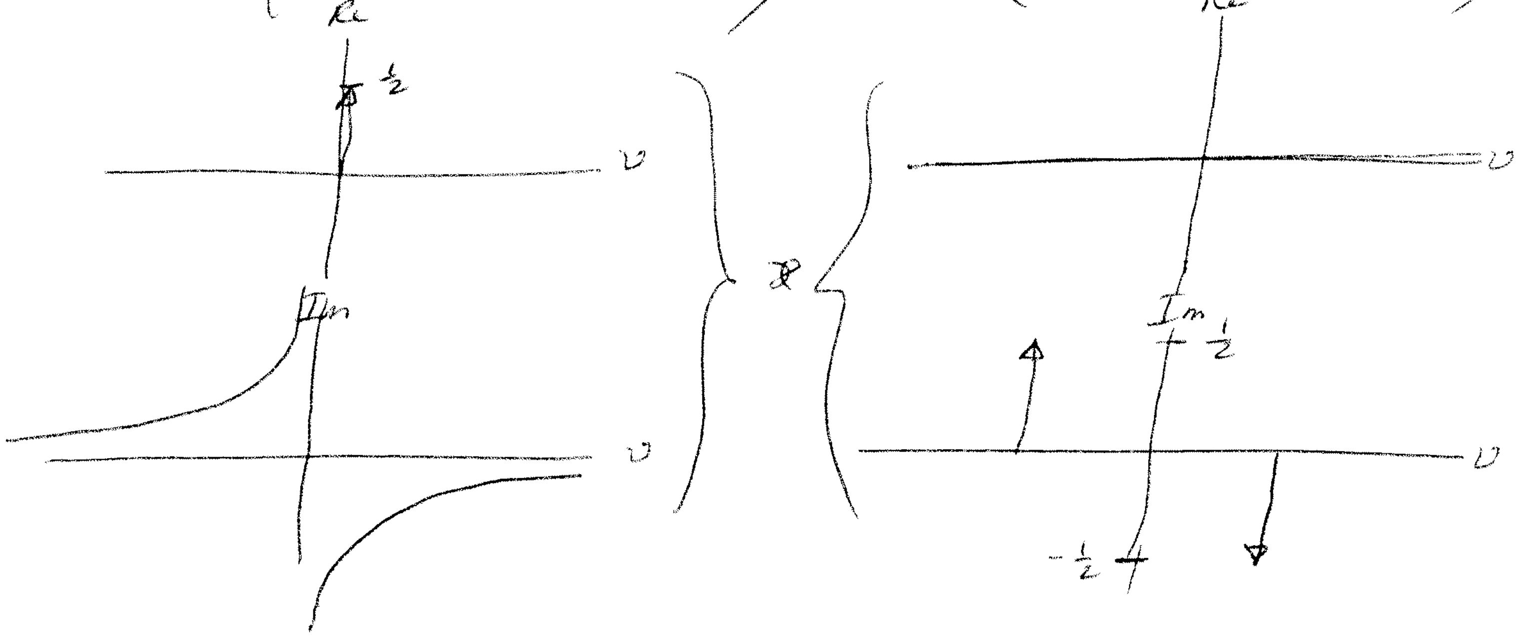
$\pm \frac{\pi}{2}$



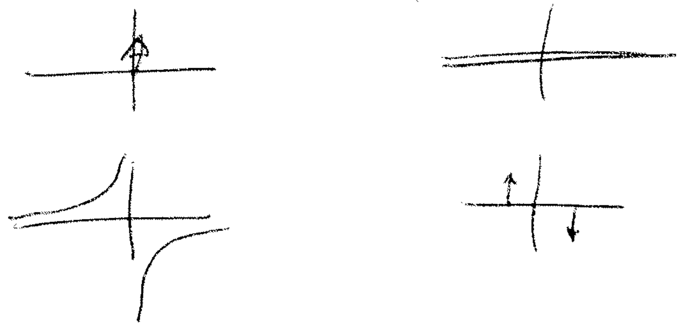
$h(t) = \mathrm{STEP}(t) \cdot \mathrm{SIN}(2\pi\nu_0 t)$

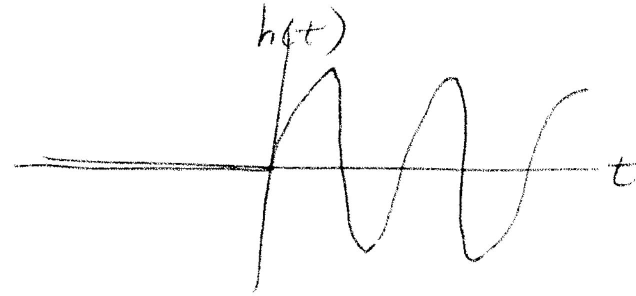
$$h(t) = \mathrm{STEP}(t)\ \sin(2\pi\nu_0 t)$$

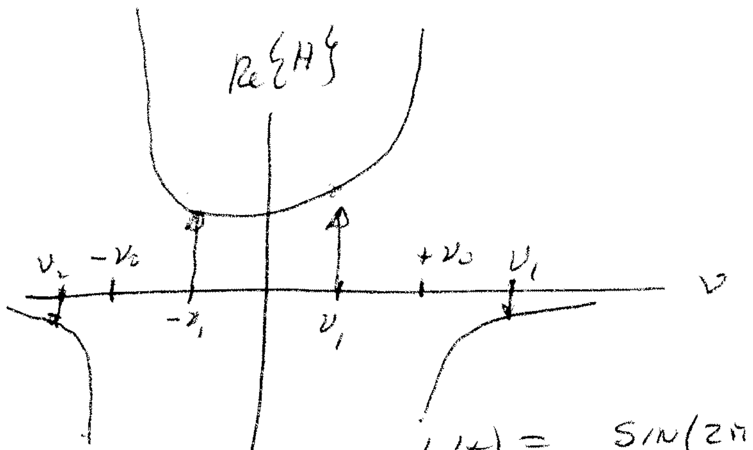
$$H(\nu) = \left(\frac{1}{2}\delta(\nu) + \frac{1}{2\pi i\nu}\right) * \frac{i}{2}\left(\delta(\nu+\nu_0) - \delta(\nu-\nu_0)\right)$$

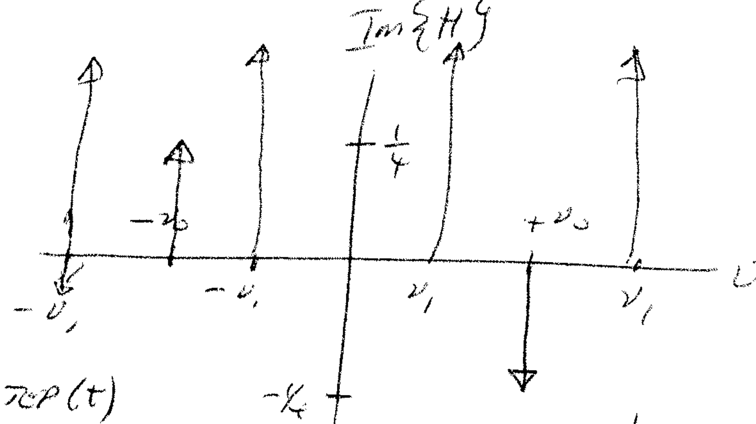


$$\mathrm{Re}\{1\} * \mathrm{Re}\{2\} + \underbrace{i\,\mathrm{Im}\{1\} * i}_{-1}\,\mathrm{Im}\{2\} + i\{\mathrm{Re}\{1\} * \mathrm{Im}\{2\} + \mathrm{Im}\{1\} * \mathrm{Re}\{2\}\}$$

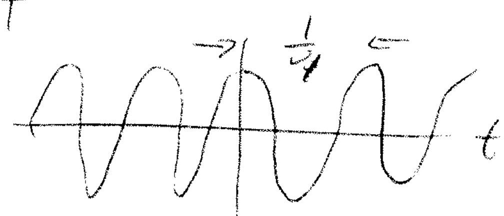








$$h(t) = \sin(2\pi\nu_0 t) \cdot \mathrm{STEP}(t)$$



WHAT IF $f(t) = \cos(2\pi\nu_1 t)$?

$$F(\nu) = \frac{1}{2}\,\delta(\nu+\nu_1) + \frac{1}{2}\,\delta(\nu-\nu_1)$$

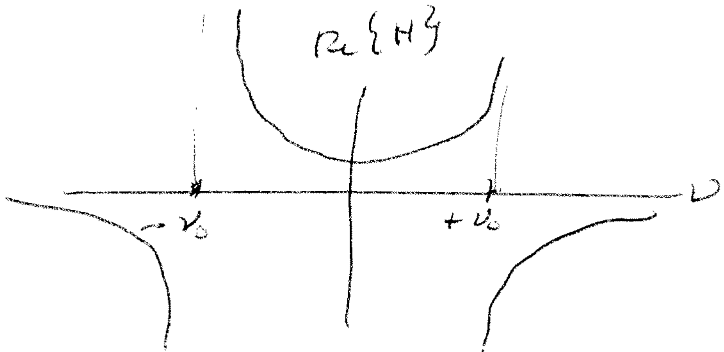
IF $\nu_1 < \nu_0 \Rightarrow H(\nu) > 0 \Rightarrow g(t) = \cos(2\pi\nu_1 t)$

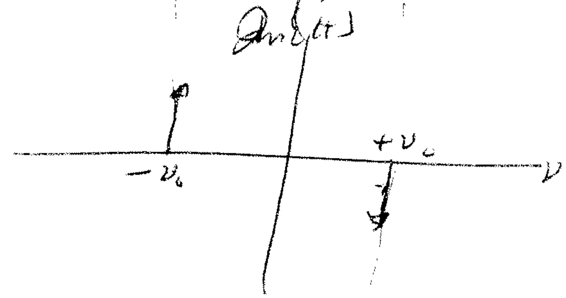
IF $\nu_1 > \nu_0$ $\quad H(\nu) < 0 \Rightarrow g(t) \propto -\cos(2\pi\nu_1 t)$

$$\underbrace{\cos(2\pi\nu_1 t)}_{f(t)} * \underbrace{\left(\sin(2\pi\nu_0 t)\ \mathrm{STEP}(t)\right)}_{h(t)}$$

DRIVEN (FORCED) HARMONIC OSCILLATOR

IF $\nu_1 = \nu_0$; DRIVING FORCE AT SAME FREQ. AS $h(t)$ (RESONANT FREQ)





TACOMA NARROWS BRIDGE