28 OCTOBER 2009

MAKEUP CLASS F 10/30 4PM–6PM

HW #7 HAS BEEN POSTED (HW #8 MAY NOT BE FAR BEHIND)

2-D FOURIER

(1) SEPARABLE
(2) CIRCULARLY SYMMETRIC (EVEN)
→ (3) RELEVANT TO MANY IMAGING SYSTEMS. (RADON TRANSFORM)

$$F[\xi,\eta] = \iint_{-\infty}^{+\infty} f(x,y)\left(e^{\underbrace{+2\pi i(\xi x+\eta y)}}\right)^{*} dx\, dy$$

"PROJECTING $f(x,y)$ ONTO BASIS FUNCTIONS OF CONVOLUTION"
EIGENFUNCTIONS

$$\text{WHAT IF} \quad F[\xi,0] = \int_{-\infty}^{+\infty} dx\; e^{-2\pi i \xi x}\left(\underbrace{\int_{-\infty}^{+\infty} f(x,y)\cdot 1[x,y]\, dy}_{\lambda_f(x)}\right)$$

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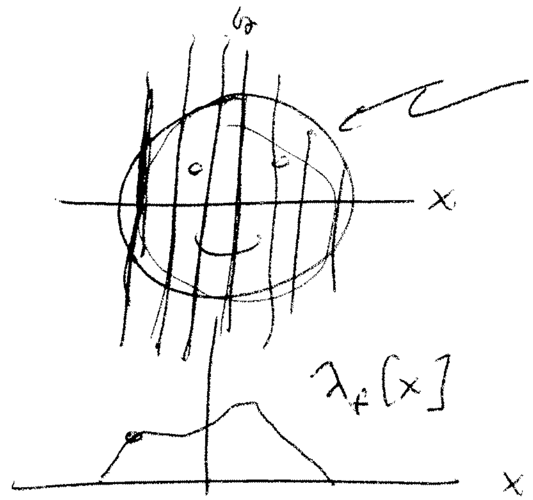
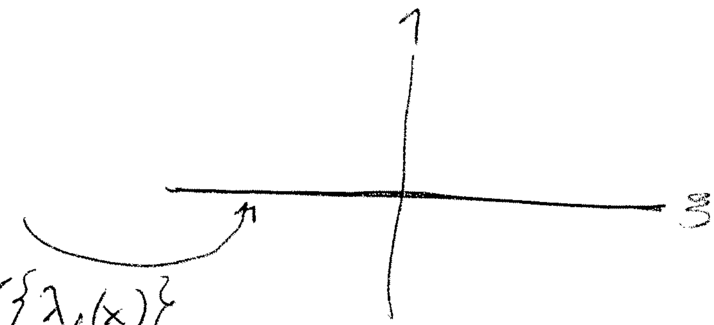


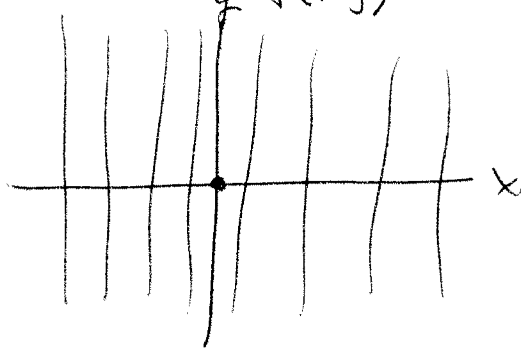
$$\xrightarrow[x\to\xi]{\mathcal{F}}$$



$$\mathcal{F}_{x\to\xi}\{\lambda_f(x)\}$$

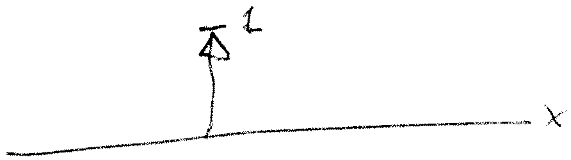
$$= \Lambda_f(\xi)$$

e.g. $f(x,y) = \delta(x,y)$



$$\int_{-\infty}^{+\infty} \delta[x,y]\,dy = \int_{-\infty}^{+\infty} \delta(x]\,\delta(y)\,dy$$

$$= \delta(x)\int_{-\infty}^{+\infty} \delta(y)\,dy = \delta(x)\cdot 1$$



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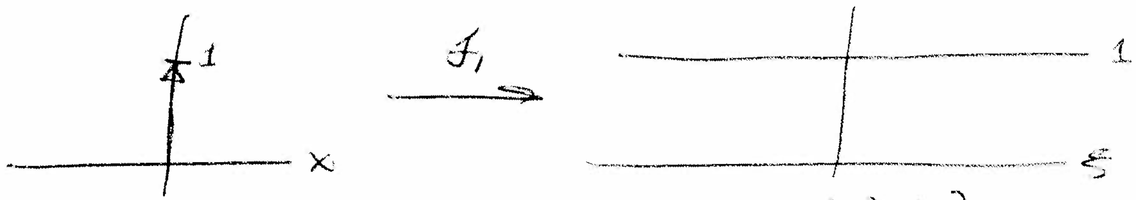
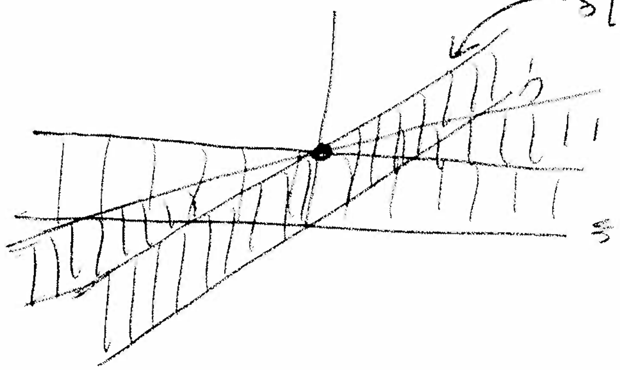


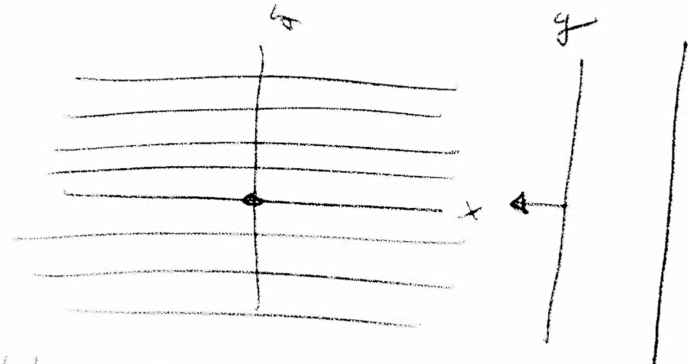


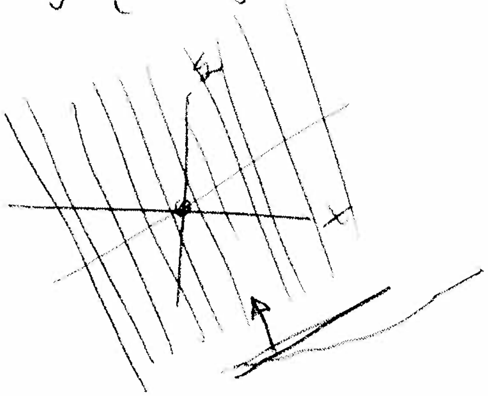
$\delta[\xi]\,1(\eta)$

$1[\xi]\,\delta[\eta]$

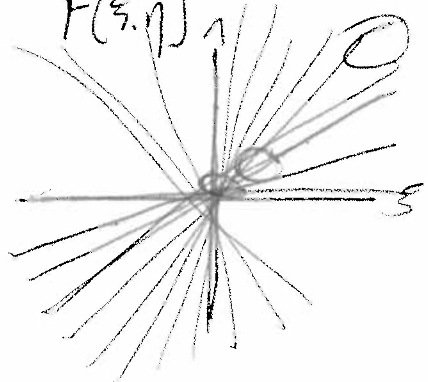
$\mathcal{F}_1\{\lambda_f(x)\} \Rightarrow F(\xi, 0]$

$\mathcal{F}_{x\to\xi}\{\delta(x)\} = 1(\xi, 0]$

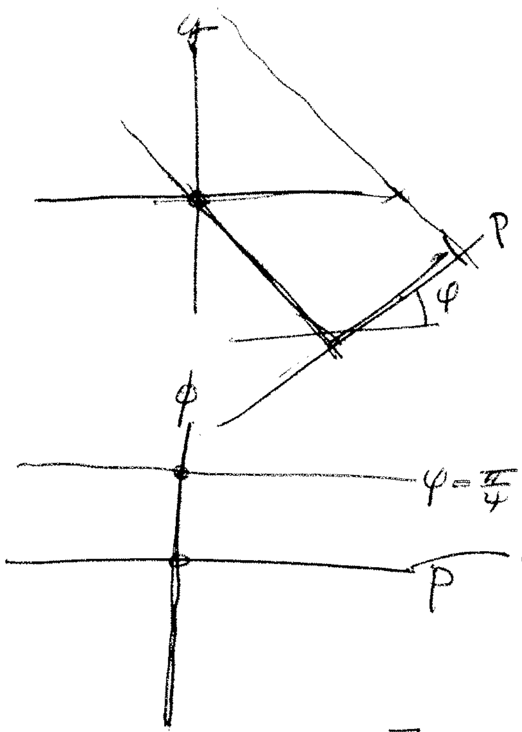




$F[\xi, \eta]$



ADD THE 1-D SPECTRA
FROM EACH PROJECTION



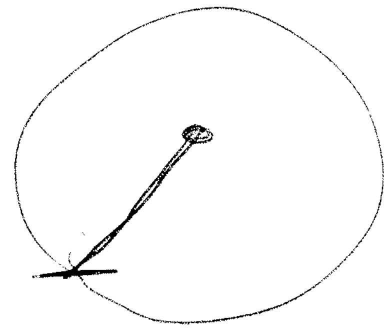
$\lambda_f(x)$

$\lambda_f(y)$

PEDAL VECTOR

$\lambda_f(p, \varphi)$

x, y



$S[x, y]$ $\qquad [p, \varphi] \Rightarrow$ SINOGRAM

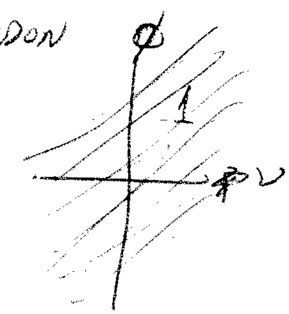
LINE-INTEGRAL

1 PROJECTION ONTO ALL DIRECTIONS $\Rightarrow$ RADON TRANSFORM

1917 JOHAN RADON

$$f(x, y) \xrightarrow{R_2} \lambda_f(p, \varphi) \xrightarrow[p \to \nu]{\mathcal{F}} \Lambda_f(\nu, \varphi)$$

$$S(x, y) \longrightarrow S[p]\, 1(\varphi) \longrightarrow 1(\nu) \cdot 1(\varphi)$$



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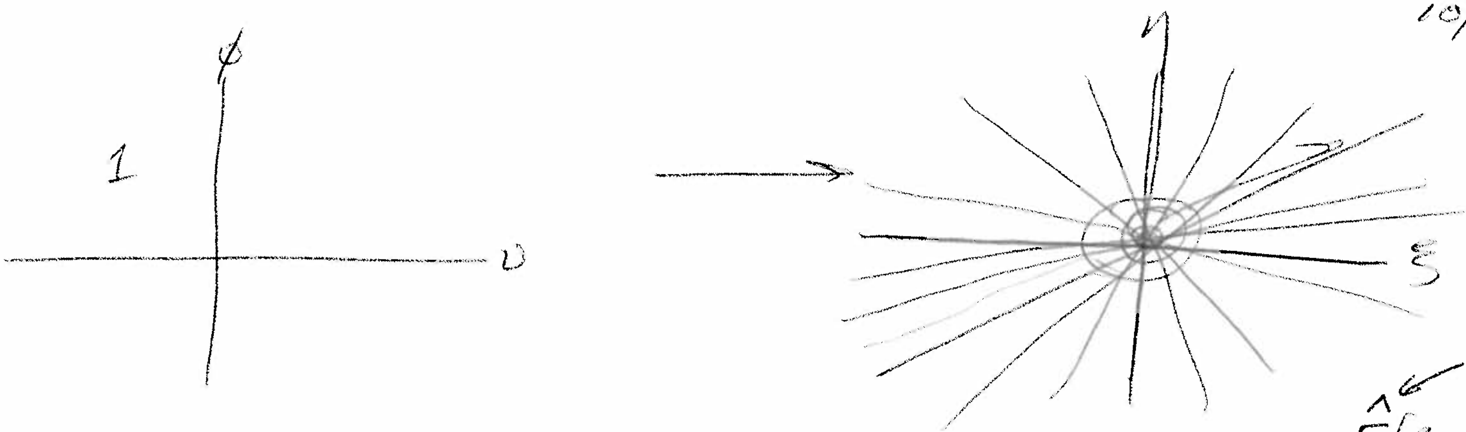
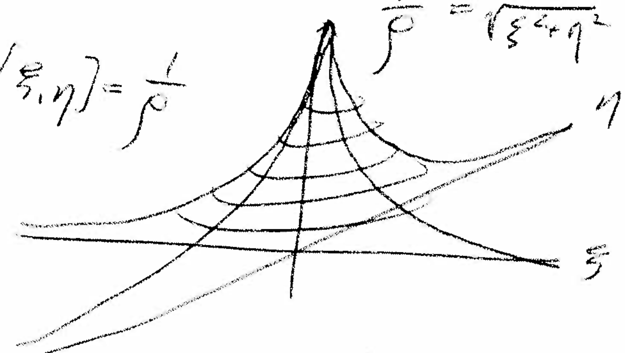


ESTIMATE → $\hat{F}[\xi,\eta]$

SUM 1-D SPECTRA OVER AZIMUTH ANGLE

$$\frac{1}{\rho} = \frac{1}{\sqrt{\xi^2+\eta^2}}$$

$$\hat{F}[\xi,\eta] = \frac{1}{\rho}$$



$$\hat{F}[\xi,\eta] = \frac{1}{\sqrt{\xi^2+\eta^2}}$$ ← ESTIMATE OF CONSTANT

$$F[\xi,\eta] = 1[\xi,\eta]$$

$$\mathcal{F}_2^{-1}\left\{\frac{1}{\rho}\right\} = \frac{1}{r}$$

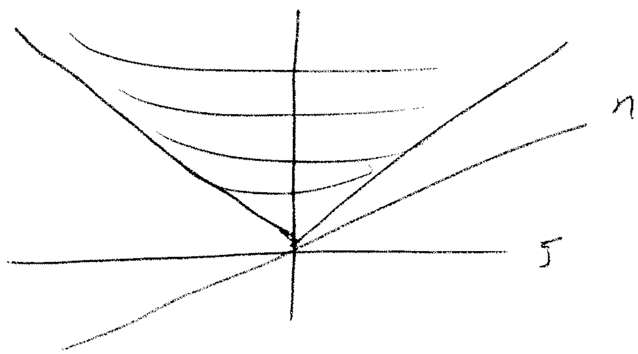
$$\hat{F}[\xi,\eta] \xrightarrow{\mathcal{F}_2^{-1}} \frac{1}{r} \neq \delta(x,y] = \frac{\delta(r)}{\pi r}$$

$$\frac{1}{r} = \hat{\delta}[x,y] =$$

$$\hat{F}[\xi,\eta] = \frac{1}{\rho} \longrightarrow \hat{F}[\xi,\eta]\cdot H[\xi,\eta] = \hat{F}[\xi,\eta]\left(\sqrt{\xi^2+\eta^2}\right)$$

$$F[\xi,\eta] = 1[\xi,\eta] \qquad H[\xi,\eta] = \sqrt{\xi^2+\eta^2} = \rho$$

"RHO FILTER" ("HIGH-BOOST" FILTER)

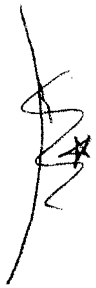


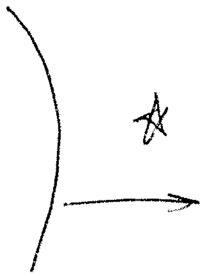
$$H(\rho) = \rho \qquad H(\xi) = \xi$$

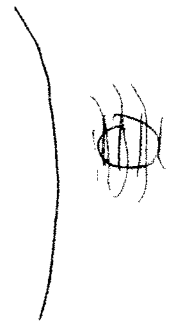
$$h(r)$$

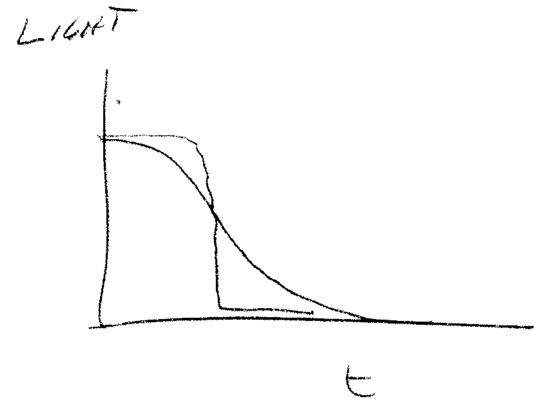
$$f[x,y] \xrightarrow{\mathcal{R}_2} \boxed{\lambda_f(p,\varphi)} \xrightarrow[p\to\nu]{\mathcal{F}_1} \Lambda_f(\nu,\varphi) \xrightarrow{\sum_\varphi} \hat{F}[\xi,\eta]$$

$$\hookrightarrow \frac{\cdot\,\rho}{\cdot\sqrt{\xi^2+\eta^2}} \rightarrow F[\xi,\eta] \xrightarrow{\mathcal{F}_2^{-1}} \boxed{f[x,y]}$$








LIGHT
t

FILTERING TO SOLVE IMAGING TASKS

1. DIRECT $\quad \mathcal{O}\{f(x)\} = g(x)$
2. INVERSE $\quad \mathcal{O}^{-1}\{g(x)\} = f(x)$ — OFTEN THE SAME
3. ANALYSIS $\quad g(x), f(x) \Rightarrow \mathcal{O}$

LSI SYSTEM

FILTERING

psf

$$\mathcal{O}\{f(x)\} = f(x) * h(x) = g(x)$$

$$\mathcal{F}_1 \downarrow \qquad \downarrow \qquad \downarrow \qquad \uparrow \mathcal{F}_1^{-1}$$

$$F(\xi) \cdot H(\xi) = G(\xi)$$

OTF

DIFFERENT CLASSES OF $h(x)$, $H(\xi)$

OTF $\quad H(\xi) = |H(\xi)|\, e^{i\Phi_H(\xi)}$

(1) MAGNITUDE FILTERS ($\Phi = 0$)

(2) PHASE FILTERS ($|H| = 1$)

(3) GENERAL CASE

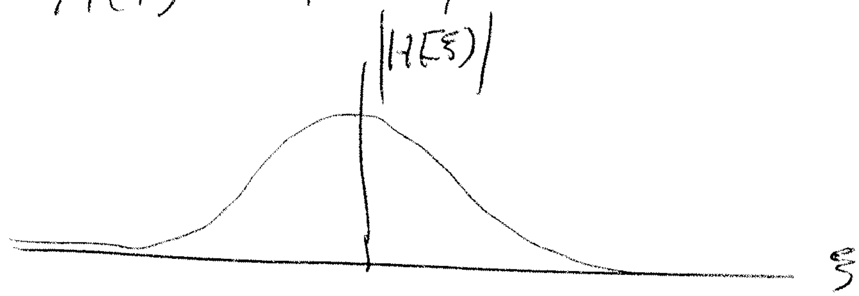
$\rightarrow$ INVERSE PROBLEM (WIENER, WIENER-HELSTROM

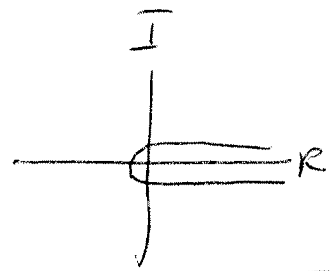
MATE MATCHED

MAGNITUDE - ONLY FILTERS

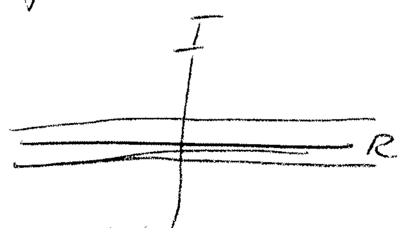
$$\Phi_H(\xi) = 0(\xi)$$

$$H(\xi) = |H(\xi)| \geq 0 \text{ AND REAL}$$

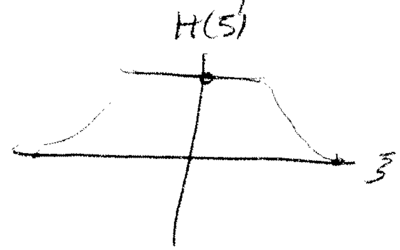




LOOSEN CONSTRAINT $\Rightarrow$ $H(\xi)$ IS REAL

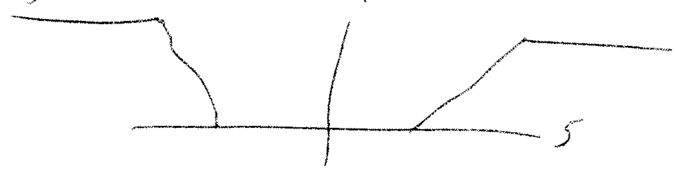


$$\Phi = 0, \pm\pi$$



LOWPASS $\Rightarrow$ $\cancel{H(\xi) =}$ $H(\xi=0) = 1$

$$H(\xi = \pm\infty) = 0$$



HIGH PASS $\quad H(\xi=0) = 0$

$$H(\xi = \pm\infty) = 1$$

BANDPASS, BANDSTOP, BANDBOOST, HIGHBOOST, ...

PHASE - ONLY FILTERS (ALLPASS) 10/28 ⑩

$$H(\xi) = \underline{\underline{1(\xi)}}\, e^{+i\Phi_H(\xi)}$$

$$G(\xi) = F(\xi) \cdot H(\xi) \implies \begin{aligned} |G(\xi)| &= |F(\xi)| \cdot |H(\xi)| \\ \Phi_G(\xi) &= \Phi_F(\xi) + \Phi_H(\xi) \end{aligned}$$

$$|G(\xi)| = |F(\xi)|$$

$$\Phi_G(\xi) = \Phi_F(\xi) + \Phi_H(\xi)$$

NONLINEAR PHASES ARE $\underline{\underline{\text{BAD}}}$

POWER TRANSMISSION OF FILTERS

PARSEVAL'S THEOREM

$$\int_{-\infty}^{+\infty} |f[x]|^2 \, dx = \int_{-\infty}^{+\infty} |F(\xi)|^2 \, d\xi$$

$$G(\xi) = F(\xi) \cdot H(\xi)$$

$$\int_{-\infty}^{+\infty} |f(x) * h(x)|^2 \, dx = \int_{-\infty}^{+\infty} |G(\xi)|^2 \, d\xi = \underline{\int |F(\xi) \cdot H(\xi)|^2 \, d\xi}$$

SCHWARZ'S INEQUALITY $\quad |\underset{\sim}{x} \circ \underset{\sim}{y}| \leq |\underset{\sim}{x}| \cdot |\underset{\sim}{y}|$

10/28 -12

$$\int_{-\infty}^{+\infty} |F(\xi)\, H(\xi)|^2 \, d\xi \le \int |F(\xi)|^2 |H(\xi)|^2 \, d\xi$$

$$\int_{-\infty}^{+\infty} |g(x)|^2 \, dx \le \int |F(\xi)|^2 |H(\xi)|^2 \, d\xi$$

IF $|H(\xi)| \le 1$

(DOES NOT AMPLIFY ANY FREQ. COMPONENTS)

(PASSIVE FILTER)

$$\int |g(x)|^2 \, dx \le \int |F(\xi)|^2 \, d\xi = \int |f(x)|^2 \, dx$$

$$\boxed{\int_{-\infty}^{+\infty} |g(x)|^2 \, dx \le \int_{-\infty}^{+\infty} |f(x)|^2 \, dx}$$

$$\cancel{|f(x)|^2 \le |g(x)|^2}$$

INTEGRATED "BRIGHTNESS" OF OUTPUT $\le$ INTEGRATED BRIGHTNESS OF INPUT

$f(x) = \delta(x) = 1 \cdot \delta(x)$

$|f(x)|^2 = \delta(x)$

$H(\xi) = \cancel{e^{2\pi\xi}}$

$= \underline{1(\xi)\, e^{+2\pi i \xi}}$

$h(x) = \delta(x+1)$

$g(x) = \delta(x) * \delta(x+1) = \delta(x+1)$

$|g(x)|^2 = \delta(x+1) \neq |f(x)|^2$

$|f(x)|^2 \neq |g(x)|^2$

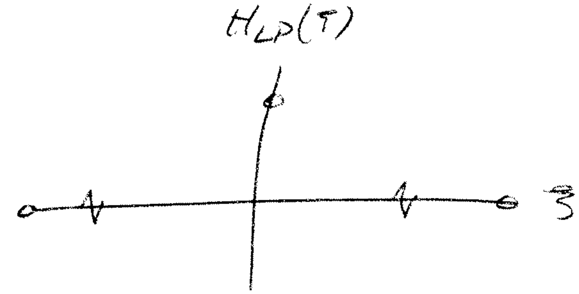
$\int_{-\infty}^{+\infty} |g(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} |f(x)|^2 dx$

IF $H(\xi) = 1(\xi)\, e^{\underline{\underline{+i\Phi_H(\xi)}}} \Longrightarrow \int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |g(x)|^2 dx$

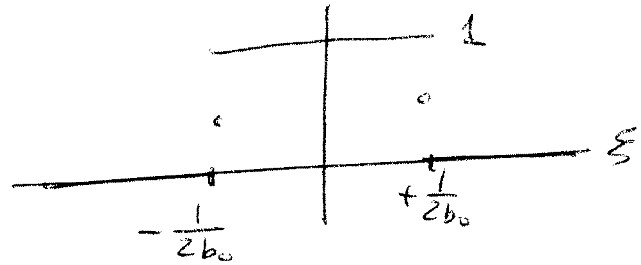
"MAGNITUDE" FILTERS $\quad H(\xi)$ IS REAL-VALUED

LOWPASS FILTER

$$H(0) = 1$$
$$H(\xi = \pm\infty) = 0$$



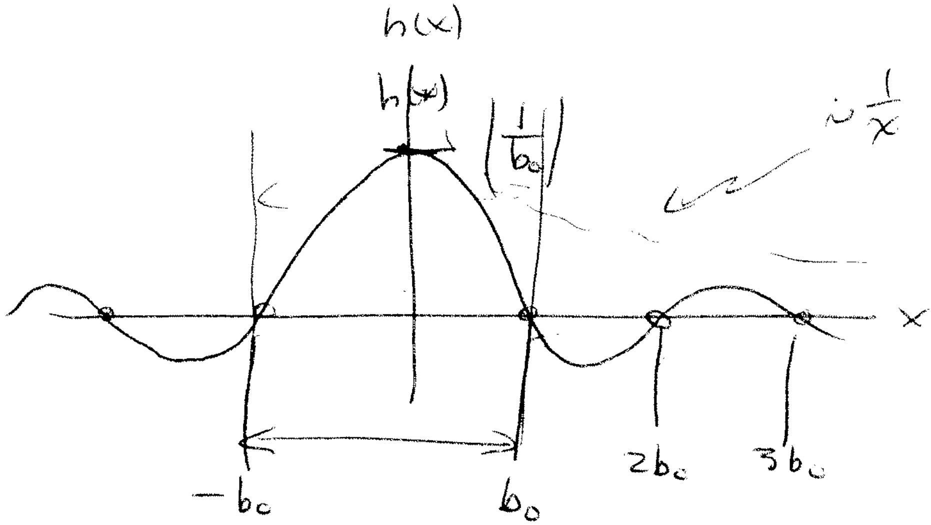
$$H_1(\xi) = \mathrm{RECT}(b_0 \xi) = \mathrm{RECT}\left(\frac{\xi}{b_0^{-1}}\right)$$



PASSES SINUSOIDS w/o CHANGE UP TO CUTOFF FREQ

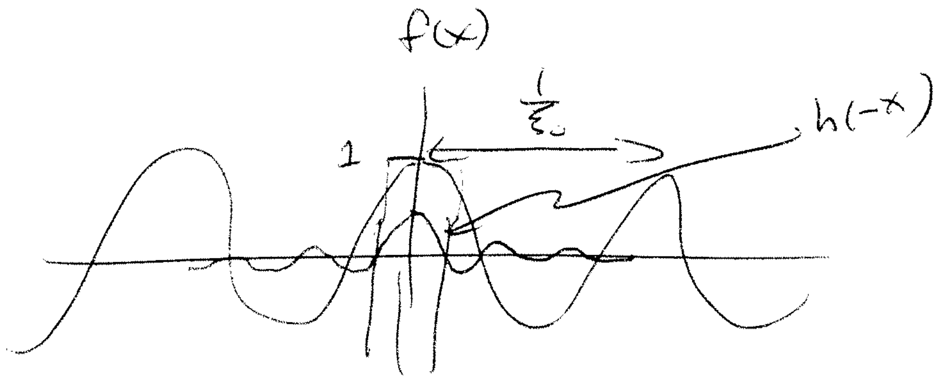
$$\xi_{CUTOFF} = \frac{1}{2b_0}$$

$$h_1(x) = \frac{1}{|b_0|}\,\mathrm{SINC}\left(\frac{x}{b_0}\right) \Longrightarrow \int_{-\infty}^{+\infty} h(x)\,dx = H(0) = 1$$



INFINITE SUPPORT

For $|x| \simeq 0$ (NEAR ORIGIN), AMPLITUDE OF $h(x)$ IS POSITIVE



IN VICINITY OF ORIGIN
ALL VALUES OF $h(x)$ HAVE
SAME SIGN $\Longrightarrow$ LOCAL WEIGHTED SUM
$\Longrightarrow$ LOCAL AVERAGE

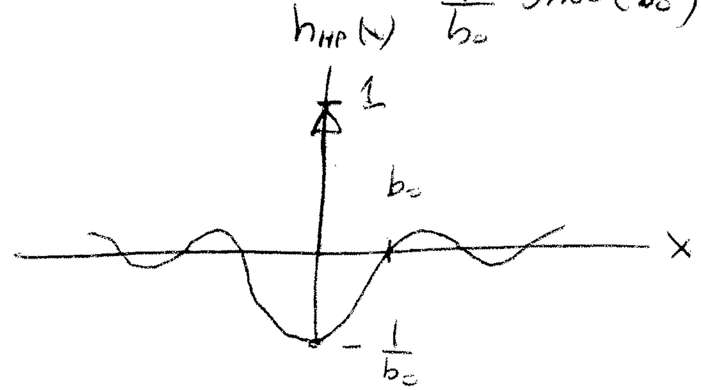
COMPLEMENT OF LOWPASS FILTER

$$H_{HP}(\xi) = 1(\xi) - H_{LP}(\xi)$$

$$h_{HP}(x) \Longrightarrow \delta(x) - h_{LP}(x)$$

$$H_{LP}(\xi) = \mathrm{RECT}\left[\frac{\xi}{b_0^{-1}}\right] \longrightarrow h_{LP}(x) = \frac{1}{b_0}\,\mathrm{SINC}\left(\frac{x}{b_0}\right)$$

$$H_{HP}(\xi) = 1 - \mathrm{RECT}\left(\frac{\xi}{b_0^{-1}}\right) \Longrightarrow h_{HP}(x) = \delta(x) - \frac{1}{b_0}\,\mathrm{SINC}\left(\frac{x}{b_0}\right)$$



LOCAL WEIGHTED DIFFERENCE

10/28 (17)

RULE OF THUMB:

LOWPASS FILTER ⟹ LOCAL WEIGHTED SUM
= LOCAL AVERAGE
⟶ REDUCE VARIABILITY OF AMPLITUDES
⟹ (PUSHES AMPLITUDES TOWARDS MEAN)

HIGH PASS FILTER ⟹ LOCAL DIFFERENCER
⟹ INCREASE VARIABILITY OF AMPLITUDES

BANDPASS FILTER ⟹ DIFFERENCES OF LOCAL AVERAGES

$h(x) = \cos(2\pi \xi_0 x)$