26 OCTOBER 2009

$\boxed{\text{MAKEUP CLASS} \quad \text{F} \quad 10/30 \quad 4\text{PM} - 6\text{PM}}$

CIRCULARLY SYMMETRIC FUNCTIONS, TRANSFORMS OF

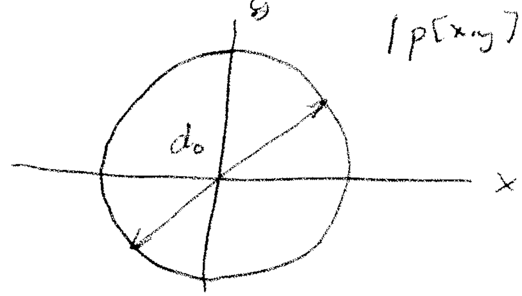
WHY? SYSTEMS THAT ARE CIRCULARLY SYMMETRIC

$$\text{LSI: } \mathcal{O}\{f[x,y]\} = f(x,y) \underbrace{* \; h(x,y)}$$

e.g., LENSES w/ CIRCULAR CROSS-SECTION

OPTICAL IMAGING SYSTEMS

$$|p[x,y]|\, e^{i\Phi[x,y]}$$



$$|p(x,y)| = \mathrm{CYL}\left(\frac{r}{d_0}\right)$$

$$f(x,y) \longrightarrow \underset{1(\theta)}{f_r\left(\sqrt{x^2+y^2}\right)} = \underset{1(\theta)}{f_r(r)}$$

$$\underset{\text{ORDER}}{\mathcal{H}_0}\left\{ f_r(r) \right\} = \int_{\theta=-\pi}^{\theta=+\pi} \int_{r=0}^{r=\infty} \left(f_r(r)\, 1(\theta) \right) e^{-2\pi i |\underline{r}| |\underline{\rho}| \cos\theta}\, r\, dr\, d\theta$$

$$= \int_{\theta=-\pi}^{+\pi} \int_{r=0}^{r=\infty} f_r(r)\, e^{-2\pi i r\rho\cos\theta}\, r\, dr\, d\theta$$

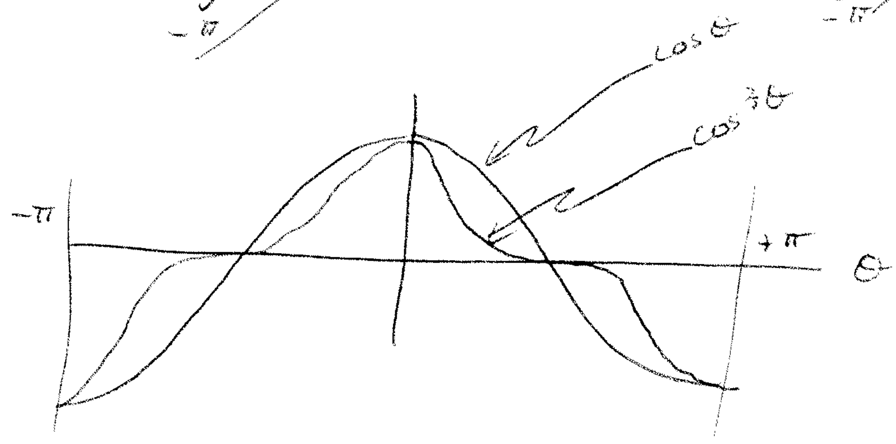
$$= \int_{r=0}^{\infty} f_r(r)\, r \left(\int_{\theta=-\pi}^{+\pi} e^{-2\pi i r\rho\cos\theta}\, d\theta \right) dr$$

$$\int_{-\pi}^{+\pi} \cos\left(2\pi r\rho\cos\theta\right) d\theta - i \int_{-\pi}^{+\pi} \sin\left(2\pi r\rho\cos\theta\right) d\theta$$

$$\int_{-\pi}^{+\pi} \sin(2\pi r\rho\cos\theta)\,d\theta = \int_{-\pi}^{+\pi}\left((2\pi r\rho\cos\theta) - \frac{(2\pi r\rho\cos\theta)^3}{3!} + \cdots\right)d\theta$$

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$$

$$= 2\pi r\rho \overbrace{\int_{-\pi}^{+\pi} \cos\theta\,d\theta}^{0} - \frac{(2\pi r\rho)^3}{3!}\overbrace{\int_{-\pi}^{+\pi} \cos^3\theta\,d\theta}^{0} + \cdots$$



$$\Rightarrow \int_{-\pi}^{+\pi} \sin(2\pi r\rho\cos\theta)\,d\theta = 0$$

$$\int_{-\pi}^{+\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta = \int_{-\pi}^{+\pi}\cos(2\pi r\rho\cos\theta)\,d\theta$$

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots$$

$$\int_{-\pi}^{+\pi}\left(1 - \frac{(2\pi r\rho)^2}{2!}\cos^2\theta + \frac{(2\pi r\rho)^4}{4!}\cos^4\theta + \cdots\right)d\theta$$

$$2\pi - \cdots = 2\pi\left(1 + \frac{(2\pi r\rho)^2}{4} + \frac{(2\pi r\rho)^4}{64}\,\frac{1}{2304}\cdots\right)$$

$$= 2\pi\left(J_0(2\pi r\rho)\right)$$

$$\mathcal{H}_0\{f_r(r)\} = 2\pi\int_{r=0}^{\infty} r\, f_r(r)\, J_0(2\pi r\rho)\,dr$$

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$$\mathcal{H}_0\{f_r(r)\} = \mathcal{F}_2\{f_r(r)\,1(\theta)\} = \int_{r=0}^{r=\infty} f_r(r)\cdot\Big(\underbrace{2\pi r\, J_0(2\pi r\rho)}_{e^{-2\pi i(\xi x+\eta y)}}\Big)dr$$

CONVERTED 2-D INTEGRAL WITH PERIODIC "KERNEL"
"MASK"
"REFERENCE"

TO 1-D RADIAL INTEGRAL WITH KERNEL $2\pi r\, J_0(2\pi r\rho)$

$$r = \sqrt{x^2+y^2}$$

$$\rho = \sqrt{\xi^2+\eta^2}$$

2-D CIRCULAR SYMMETRY IN BOTH DOMAINS

$$\mathcal{F}_2\left\{f_r\left(\sqrt{x^2+y^2}\right)1(\theta)\right\} \longrightarrow F_r\left(\sqrt{\xi^2+\eta^2}\right)\cdot 1(\varphi)$$

$$\mathcal{H}_0\left\{f_r\left(\sqrt{x^2+y^2}\right)\right\} = F_r\left(\sqrt{\xi^2+\eta^2}\right)$$

$2\pi r \, J_0(2\pi r \rho)$ IS NOT PERIODIC $\Rightarrow$ CHALLENGING CALCULATION

2-D FOURIER $\Rightarrow$ FAST $\underbrace{\text{DISCRETE}}_{\text{SAMPLED}}$ CALCULATION (BECAUSE OF PERIODICITY)

1-D HANKEL $\Rightarrow$ "NO" FAST DISCRETE CALCULATION

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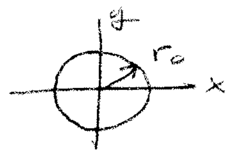
(1) $\delta(x)\delta(y) = \delta[x,y] = \dfrac{\delta(r)}{\pi r} \longrightarrow 1(\xi,\eta) = 1\left(\sqrt{\xi^2+\eta^2}\right) = 1(\rho)$

(2) $\delta(r - r_0)$



(3) $\mathrm{CYL}\left(\dfrac{r}{d_0}\right)$

(4) $\dfrac{1}{r} = \dfrac{1}{\sqrt{x^2+y^2}} \longrightarrow \dfrac{1}{\rho}$

(5) $e^{-\pi r^2} \longrightarrow e^{-\pi \rho^2}$

(6) $e^{\pm i\pi r^2}$

(7) $r^2 e^{-\pi r^2} = r^2 \,\mathrm{GAUS}(r)$

(1) $\mathcal{F}_2\left\{ \delta(x)\delta(y) \right\} = 1(\rho)$

$$\delta[x,y] = \frac{\delta(r)}{\pi r} \xrightarrow{\mathcal{H}_0} \int_{r=0}^{\infty} \frac{\delta(r)}{\cancel{\pi r}}\, 2\cancel{\pi r}\, J_0(2\pi r\rho)\, dr$$

$\downarrow$

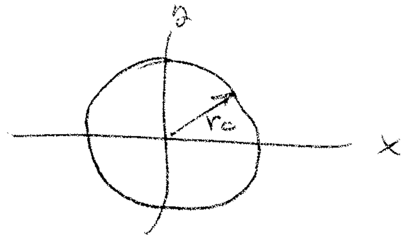
$1(\xi,\eta) = 1(\rho)$

$$= 2\int_{r=0}^{\infty} \underbrace{\delta(r)\, J_0(2\pi r\rho)}_{\text{even}}\, dr$$

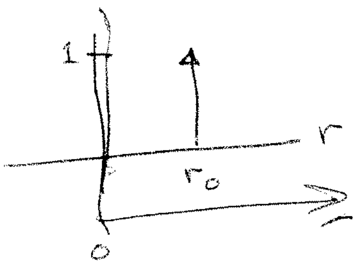
$$= \cancel{2}\,\tfrac{1}{\cancel{2}} \int_{r=-\infty}^{+\infty} \delta(r)\, J_0(2\pi r\rho)\, dr \qquad \left[f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0) \right]$$

$$\int_{-\infty}^{+\infty} \delta(r)\, J_0(2\pi\cdot 0\cdot \rho)\, dr = \int_{r=-\infty}^{+\infty} \delta(r)\cdot 1\, dr = 1(\rho)$$

(2) $\delta(r - r_0)$



$$\mathcal{H}_0\{\delta(r-r_0)\} = \int_{r=0}^{\infty} \delta(r-r_0)\cdot\left(2\pi r\, J_0(2\pi r\rho)\right) dr$$



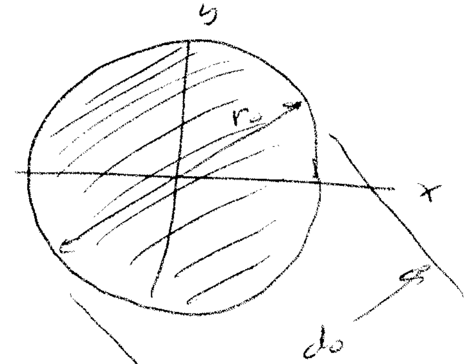
$$= \int_{r=0}^{\infty} \delta(r-r_0)\cdot\left(2\pi r_0\, J_0(2\pi r_0\rho)\right) dr$$

$$= 2\pi r_0\, J_0(2\pi r_0\rho) \underbrace{\int_{r=0}^{\infty} \delta(r-r_0)\, dr}_{1}$$

$$\boxed{\mathcal{H}_0\{\delta(r-r_0)\} = 2\pi r_0\, J_0(2\pi r_0\rho)}$$

$$\mathcal{F}_2\{\delta(r-r_0)\, 1(\theta)\} = 2\pi r_0\, J_0\left(2\pi r_0 \sqrt{\xi^2+\eta^2}\right) 1(\varphi)$$

(3) $\text{CYL}\left(\frac{r}{d_0}\right) = \text{CYL}\left(\frac{r}{2r_0}\right)$



$$\mathcal{H}_0\left\{\text{CYL}\left(\frac{r}{d_0}\right)\right\} = \int_{r=0}^{\infty} \text{CYL}\left(\frac{r}{d_0}\right) 2\pi r\, J_0(2\pi r\rho)\, dr$$

$$= \int_{r=0}^{r=\frac{d_0}{2}} 1 \cdot 2\pi r\, J_0(2\pi r\rho)\, dr$$

$u \equiv 2\pi r\rho \Rightarrow r = \frac{u}{2\pi\rho}$; $r=0 \Rightarrow u=0$

$r = \frac{d_0}{2} \Rightarrow u = 2\pi \frac{d_0}{2} \cdot \rho = \pi d_0 \rho$

$dr = \frac{du}{2\pi\rho}$

$$= \int_{u=0}^{u=\pi d_0\rho} \cancel{2\pi} \frac{u}{\cancel{2\pi}\rho} J_0(u) \frac{du}{2\pi\rho} = \frac{1}{2\pi\rho^2}\int_{u=0}^{u=\pi d_0\rho} u\, J_0(u)\, du$$

$$\int_{u=0}^{u=x} u\, J_0(u)\, du = \underbrace{x\, J_1(x)}$$

$$\Rightarrow \mathcal{H}_0\left\{\mathrm{CYL}\left(\frac{r}{d_0}\right)\right\} = \frac{1}{\cancel{2\pi}\rho^2}\left[\left(\pi d_0 \rho\right) J_1\left(\pi d_0 \rho\right)\right]$$

$$= \frac{1}{2\rho}\, d_0\, J_1(\pi d_0 \rho)$$

Recall $\mathrm{SOMB}(\rho) \equiv \dfrac{2 J_1(\pi\rho)}{\pi\rho}$ (BESINC)

$$\Rightarrow J_1(\pi\rho) = \frac{\pi\rho}{2}\,\mathrm{SOMB}(\rho)$$

$$\Rightarrow J_1(\pi d_0 \rho) = \frac{\pi \rho d_0}{2}\,\mathrm{SOMB}(d_0\rho)$$

$\pi\left(\frac{d_0}{2}\right)^2$ ~~AREA~~ VOLUME OF $\mathrm{CYL}\left(\frac{r}{d_0}\right)$

$$\mathcal{H}_0\left\{\mathrm{CYL}\left(\frac{r}{d_0}\right)\right\} = \frac{d_0^2}{2\rho}\cdot\frac{\pi\rho}{2}\,\mathrm{SOMB}(d_0\rho) = \left(\frac{\pi d_0^2}{4}\right)\mathrm{SOMB}(d_0\rho)$$

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$$\mathrm{RECT}\left(\frac{x}{d_0}, \frac{y}{d_0}\right) \longrightarrow d_0^2 \,\mathrm{SINC}\left[d_0 \xi, d_0 \eta\right]$$

$$\underline{\underline{\mathrm{CYL}\left(\frac{r}{d_0}\right)}} \longrightarrow \left(\frac{\pi d_0^2}{4} \underline{\underline{\mathrm{SOMB}(d_0 \rho)}}\right)^2$$

(AIRY DISK)

$$\cancel{\delta}\ \delta(r - r_0)$$

$$\mathrm{COR}\left[\frac{x}{d_0}, \frac{y}{d_0}\right]$$

$\cancel{5}(4)$ $\quad \frac{1}{r} \qquad f_r(r) = \frac{1}{r}$

$$\mathcal{H}_0\left\{\frac{1}{r}\right\} = \int_{r=0}^{\infty} \frac{1}{\cancel{r}}\, 2\pi \cancel{r}\, J_0(2\pi r \rho)\, dr$$

$$= 2\pi \int_0^{\infty} J_0(2\pi r \rho)\, dr = \pi \int_{r=-\infty}^{+\infty} J_0(2\pi r \rho)\, dr$$

$$u \equiv 2\pi r \rho \implies r = \frac{u}{2\pi\rho} \qquad dr = \frac{du}{2\pi\rho}$$

$=$

$$2\pi \int_{r=0}^{\infty} J_0(2\pi r\rho)\, dr = \cancel{\pi} \int_{u=-\infty}^{+\infty} J_0(u)\, \frac{du}{2\cancel{\pi}\rho}$$

$$= \frac{1}{2}\,\frac{1}{\rho} \underbrace{\int_{-\infty}^{+\infty} J_0(u)\, du}_{\text{REAL-VALUED CONSTANT}}$$

$$\mathcal{H}_0\left\{\frac{1}{r}\right\} = \alpha \cdot \frac{1}{\rho} \propto \frac{1}{\rho}$$

$\alpha = 1$

$$\boxed{\mathcal{H}_0\left\{\frac{1}{r}\right\} = \frac{1}{\rho}}$$

$$\mathcal{F}_2\left\{\frac{1}{\sqrt{x^2+y^2}}\right\} = \frac{1}{\sqrt{\xi^2+\eta^2}}$$

$$\cancel{\Rightarrow} \quad \cancel{\mathcal{F}_1\left\{\frac{1}{|x|}\right\} = \frac{1}{|\xi|}}$$

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SCALING THEOREM

$$\mathrm{CYL}\left(\frac{r}{d_0}\right) \xrightarrow{\mathcal{H}_0} \frac{\pi d_0^2}{4}\,\mathrm{somb}(d_0\rho)$$

$$f_r\left(\frac{r}{d_0}\right) \xrightarrow{\mathcal{H}_0} d_0^2\, F_r(d_0\rho) \longrightarrow d_0^2\, F_r(d_0\rho)$$

$$d_0^2\, F_r\left(\frac{\rho}{1/d_0}\right)$$

$$\frac{1}{r} \longrightarrow \frac{1}{(r/d_0)} \xrightarrow{\mathcal{H}_0} d_0^2\left(\frac{1}{d_0\rho}\right) = \frac{d_0}{\rho}$$

5 $\text{GAus}(r) = e^{-\pi(x^2+y^2)} = e^{-\pi x^2} e^{-\pi y^2}$

$\text{GAus}(\rho) = e^{-\pi \xi^2} e^{-\pi \eta^2} = e^{-\pi \rho^2}$

$$\mathcal{H}_0\left\{ e^{-\pi r^2} \right\} = e^{-\pi \rho^2} = \int_0^\infty e^{-\pi r^2} \cdot 2\pi r \, J_0(2\pi r \rho) \, dr$$

$$2\pi \int_0^\infty r \, e^{-\pi r^2} J_0(2\pi r \rho) \, dr = e^{-\pi \rho^2}$$

$$e^{\pm i\pi r^2} = e^{\pm i\pi(x^2+y^2)} = e^{\pm i\pi x^2} e^{\pm i\pi y^2}$$

$$= \left(e^{\pm i\frac{\pi}{4}} e^{\mp i\pi\xi^2}\right)\left(e^{\pm i\frac{\pi}{4}} e^{\mp i\pi\eta^2}\right)$$

$$= e^{\pm i\frac{\pi}{2}} e^{\mp i\pi(\xi^2+\eta^2)}$$

$$= \pm i \left(e^{\mp i\pi\rho^2}\right)$$

$$= \pm i\left(\cos(\pi\rho^2) \mp i \sin(\pi\rho^2)\right)$$

$$= \sin(\pi\rho^2) \pm i\cos(\pi\rho^2) = \cancel{\mathcal{F}_2}\ \mathcal{H}_0\left\{\cos(\pi r^2) \pm i \sin(\pi r^2)\right\}$$

$$\mathcal{H}_0\left\{\pm\cos(\pi r^2)\right\} = \pm \sin(\pi\rho^2)$$

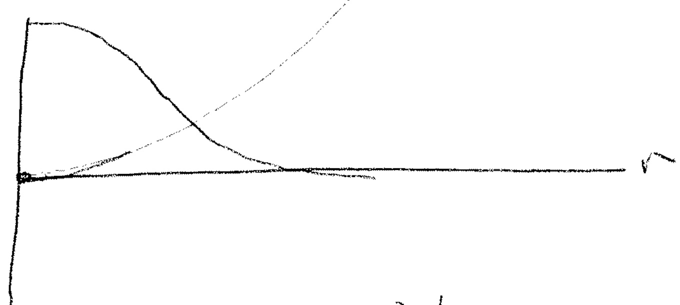
$$\mathcal{H}_0\left\{\pm\sin(\pi r^2)\right\} = \pm \cos(\pi\rho^2)$$

PROFILES OF 2-D CIRCULARLY SYMMETRIC CHIRPS

$$\int_{r=0}^{\infty} e^{\pm i\pi r^2}\, 2\pi r\, J_0(2\pi r\rho)\, dr = \pm i\, e^{\mp i\pi\rho^2}$$

$$r^2\,\mathrm{GAus}(r) = (x^2+y^2)\, e^{-\pi x^2} e^{-\pi y^2}$$

$$= \left(x^2 e^{-\pi x^2}\right) e^{-\pi y^2} + \left(y^2 e^{-\pi y^2}\right) e^{-\pi x^2}$$



$$\mathcal{F}_2\left\{x^2 e^{-\pi x^2} e^{-\pi y^2}\right\} = \mathcal{F}_2\left\{x^2 e^{-\pi(x^2+y^2)}\right\}$$

$$\mathcal{F}_2\Big\{\underbrace{\big(x^2\cdot 1(y)\big)}_{2\text{-}D}\cdot \underbrace{e^{-\pi(x^2+y^2)}}_{2\text{-}D}\Big\} = \underbrace{\mathcal{F}_2\{x^2\cdot 1(y)\}}_{\cdot\,\delta(\eta)} * \underbrace{\mathcal{F}_2\{e^{-\pi(x^2+y^2)}\}}_{e^{-\pi(\xi^2+\eta^2)}}$$

$$\mathcal{F}_1\{x^2\} = \left(\frac{1}{2\pi i}\right)^2\delta''(\xi) = -\frac{1}{4\pi^2}\delta''(\xi)$$

$$\mathcal{F}_2\{x^2 e^{-\pi(x^2+y^2)}\} = -\frac{1}{4\pi^2}\,\delta''(\xi)\delta(\eta) * e^{-\pi(\xi^2+\eta^2)}$$

$$= -\frac{1}{4\pi^2}\left(\delta''(\xi) * e^{-\pi\xi^2}\right)\underbrace{\left(\delta(\eta) * e^{-\pi\eta^2}\right)}_{e^{-\pi\eta^2}}$$

$$= -\frac{1}{4\pi^2}\left(\left(-\cancel{\pi}\,2\pi\xi\right)^2 e^{-\pi\xi^2}\right)$$

$$\mathcal{F}_2\{x^2 e^{-\pi(x^2+y^2)} + y^2 e^{-\pi(x^2+y^2)}\} = -\frac{1}{4\pi^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)e^{-\pi(\xi^2+\eta^2)}$$

$$= \left(\frac{1}{\pi} - (\xi^2+\eta^2)\right)e^{-\pi(\xi^2+\eta^2)} = \left(\frac{1}{\pi}-\rho^2\right)e^{-\pi\rho^2}$$

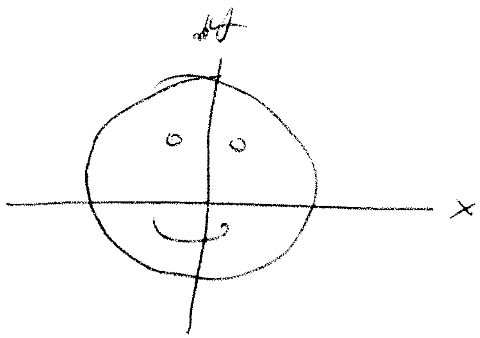
$$= \frac{1}{\pi} - \rho$$

$$\mathcal{F}_2\left\{ (x^2+y^2)\, e^{-\pi(x^2+y^2)} \right\} = \left(\frac{1}{\pi} - (\xi^2+\eta^2) \right) e^{-\pi(\xi^2+\eta^2)}$$

$$\mathcal{H}_0\left\{ r^2 e^{-\pi r^2} \right\} = \left(\frac{1}{\pi} - \rho^2 \right) e^{-\pi \rho^2}$$

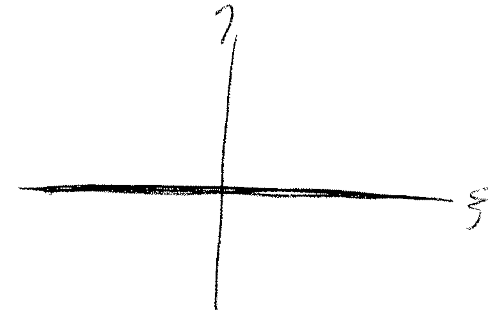
$$\mathcal{H}_0\left\{ r^2\, \mathrm{GAUS}(r) \right\} = \left(\frac{1}{\pi} - \rho^2 \right) \mathrm{GAUS}(\rho)$$

RADON TRANSFORM



$f(x,y)$

$$\xrightarrow{\mathcal{F}_2}$$



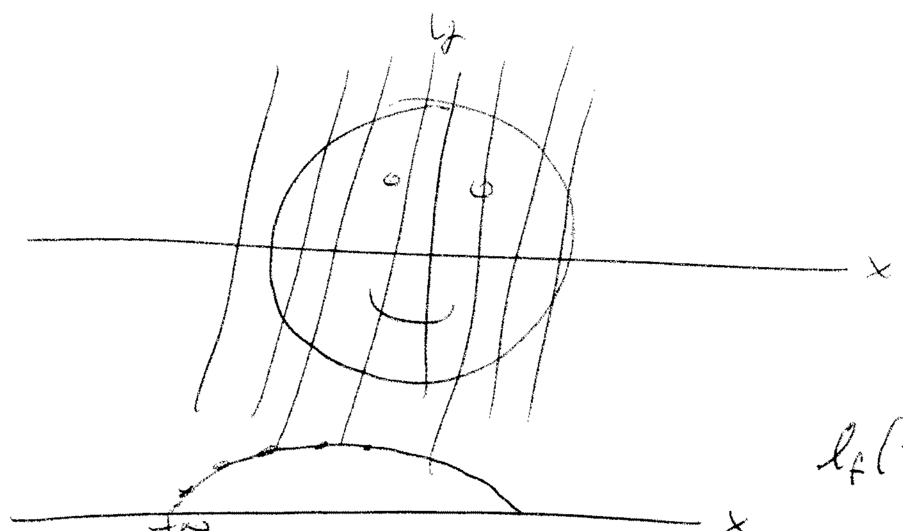
$F(\xi,\eta)$

$F[\xi,\eta=0]$

$$F[\xi,0] = \int_{-\infty}^{+\infty}\!\!\int f(x,y)\, e^{-2\pi i(\xi x + 0y)}\, dx\, dy$$

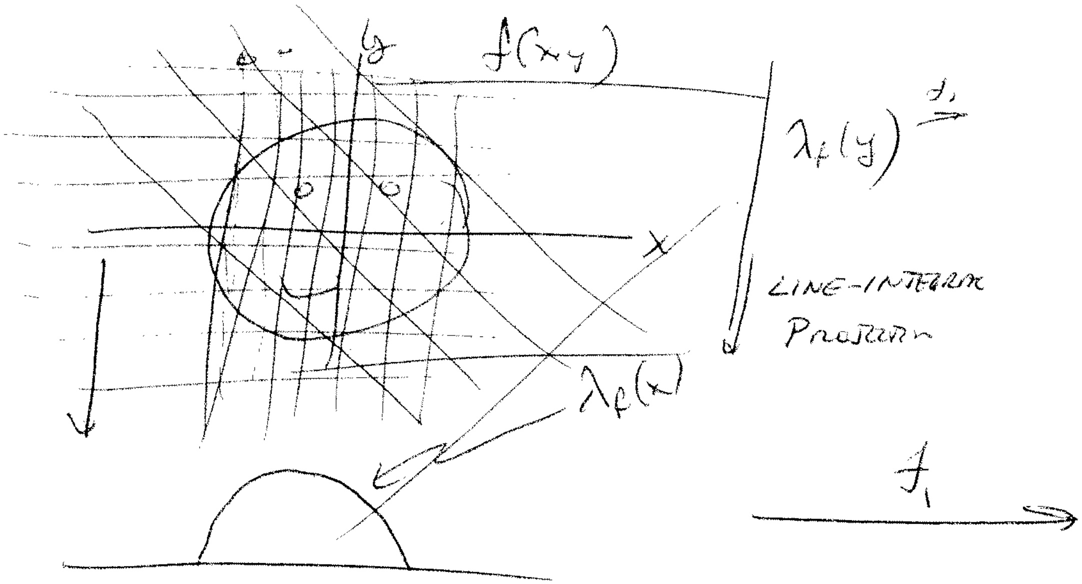
$$= \int_{-\infty}^{+\infty}\!\!\int f(x,y)\, e^{-2\pi i \xi x} \cdot 1(y)\, dx\, dy$$

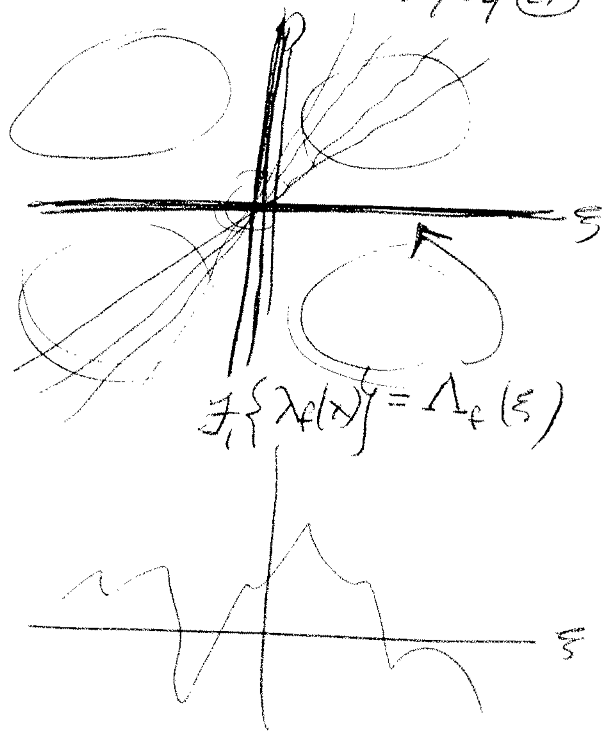
$$\int_{x=-\infty}^{+\infty} e^{-2\pi i \xi x} \left[\int_{y=-\infty}^{+\infty} f(x,y)\, dy \right] dx$$



$\ell_f(x)$; $\lambda_f(x)$

$$\lambda_f(x) \equiv \int_{-\infty}^{+\infty} f(x,y)\, dy \longrightarrow \int_{-\infty}^{+\infty} \lambda_f(x)\, e^{-2\pi i \xi x}\, dx = F(\xi, \eta = 0)$$





CAT — XRAYS

MRI — MAGNETIC FIELDS