21 October

$$f[x] \underset{\mathcal{F}_1^{-1}}{\overset{\mathcal{F}_1}{\rightleftarrows}} F[\xi]$$

$$\mathcal{O}\{f(x)\ ]\} = g(x) = f(x) * h(x) \qquad LSI$$

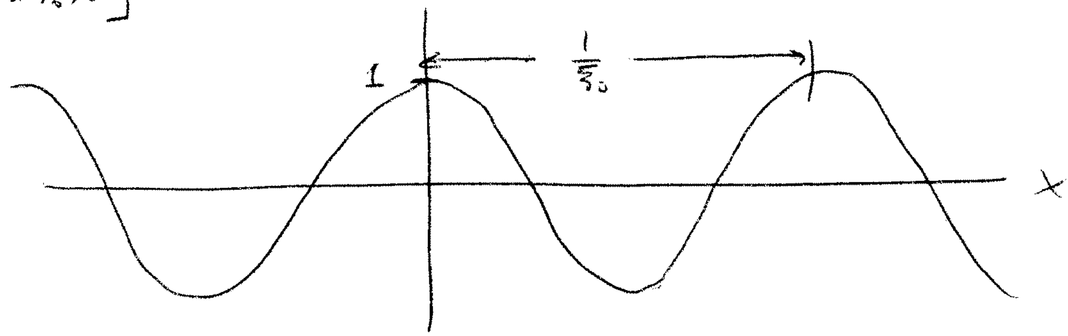
$$G(\xi) = F(\xi) \cdot H(\xi)$$

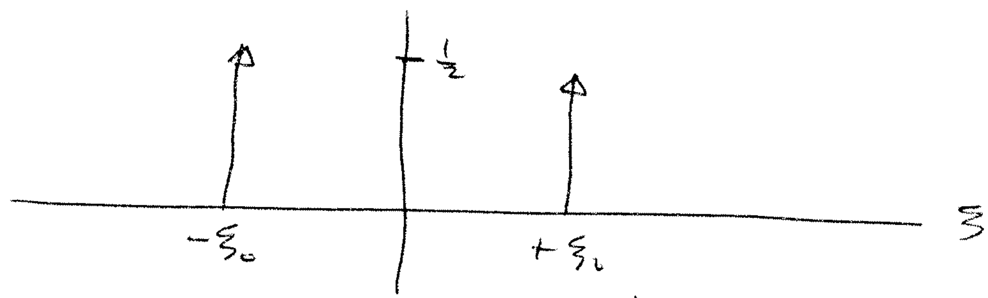
NL SV $\longrightarrow$ NL SI

$$\mathcal{O}\{f(x)\ ]\} = \left(f(x) * \delta(x-x_0)\right)^{\alpha}$$

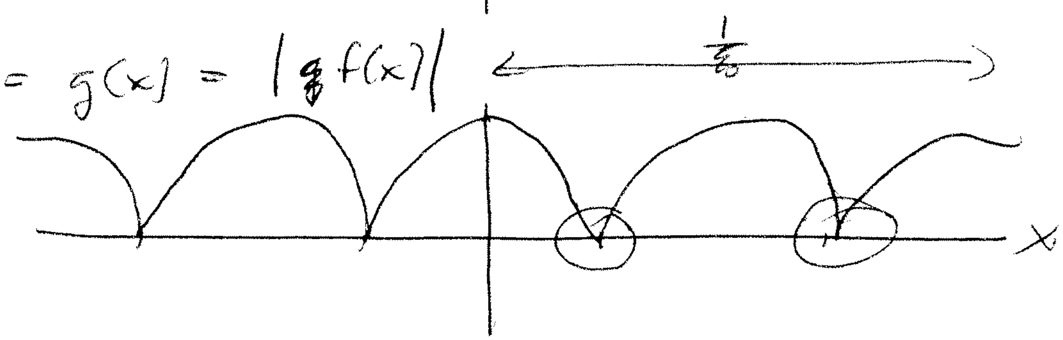
$$\left| f(x) * \delta(x-x_0) \right|$$

$$f(x) = \cos[2\pi\xi_0 x] = \frac{1}{2} e^{+2\pi i \xi_0 x} + \frac{1}{2} e^{-2\pi i \xi_0 x}$$

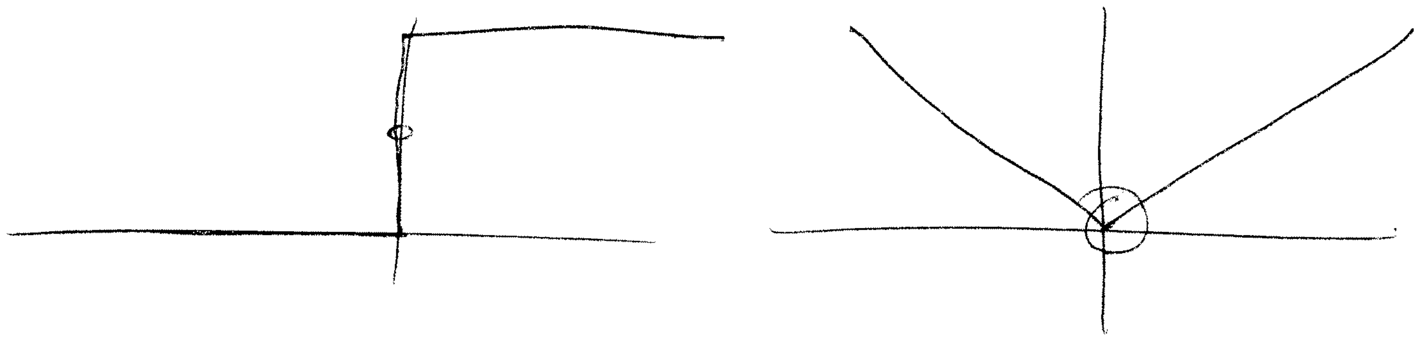




$$\mathcal{O}\{f(x)\} = g(x) = |f(x)|$$



DISCONTINUITIES IN SLOPE ⇒ HIGH-FREQUENCY SINUSOIDS



ABSOLUTE VALUE OPERATION "GENERATES" HIGH-FREQ. TERMS IN SPECTRUM $G(\xi)$

$$f[x] * h[x] = g[x]$$

$$F[\xi] \cdot H[\xi] = G[\xi]$$

NO NEW SPATIAL FREQUENCIES IN $G(\xi)$

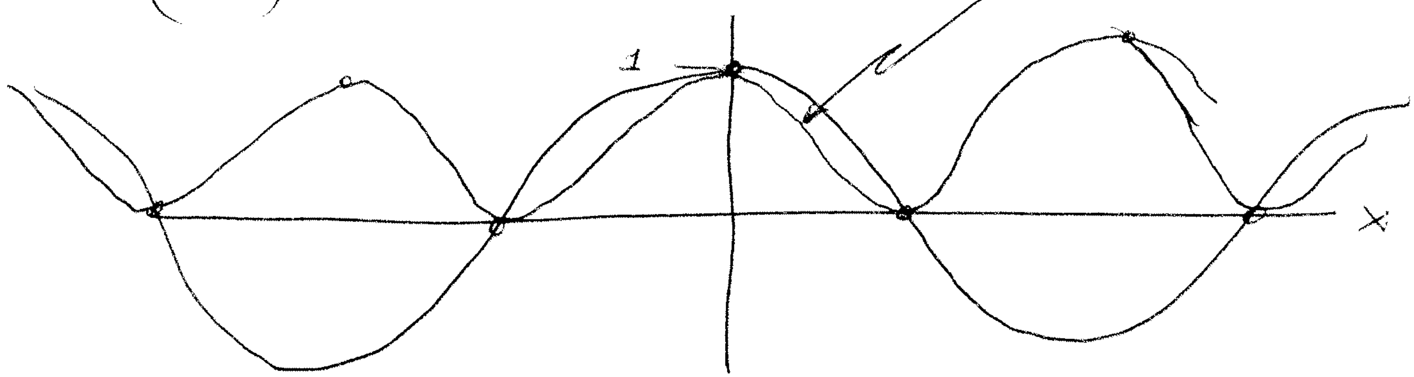
POWER-LAW NONLINEARITY

$$\mathcal{O}\{f(x)\} = \left(f(x) * \delta(x-x_0)\right)^n \longrightarrow \left(f[x]\right)^n$$

$G[\xi] \neq$ WILL BE "RELATED TO" $F[\xi]$

$$f(x) = \cos[2\pi\xi_0 x] = \frac{1}{2}e^{+2\pi i\xi_0 x} + \frac{1}{2}e^{-2\pi i\xi_0 x} \longrightarrow F[\xi] = \frac{1}{2}\delta(\xi+\xi_0) + \frac{1}{2}\delta(\xi-\xi_0)$$

$$g[x] = \left(f(x)\right)^2 = \cos^2(2\pi\xi_0 x)$$



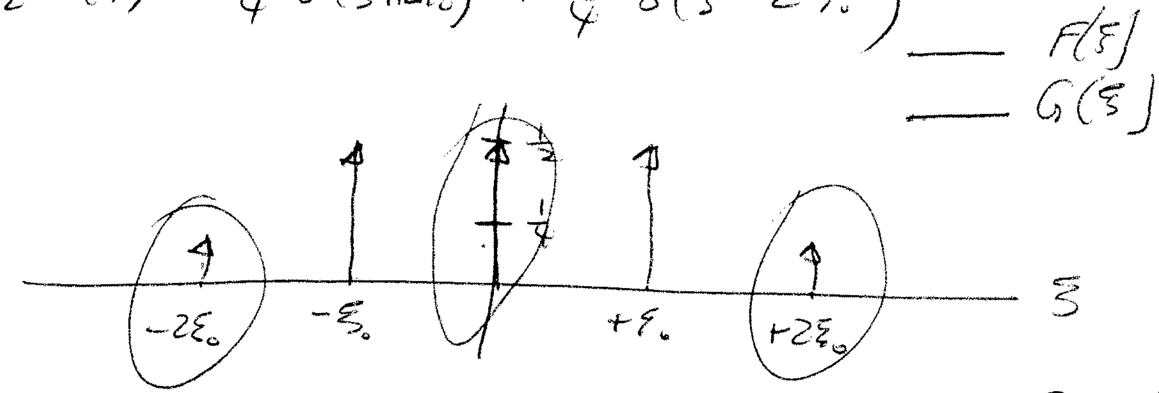
$$g(x) = \cos^2(2\pi\xi_0 x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi(2\xi_0)x)$$

$$= \frac{1}{2}\left(1 + \cos(2\pi(2\xi_0)x)\right)$$

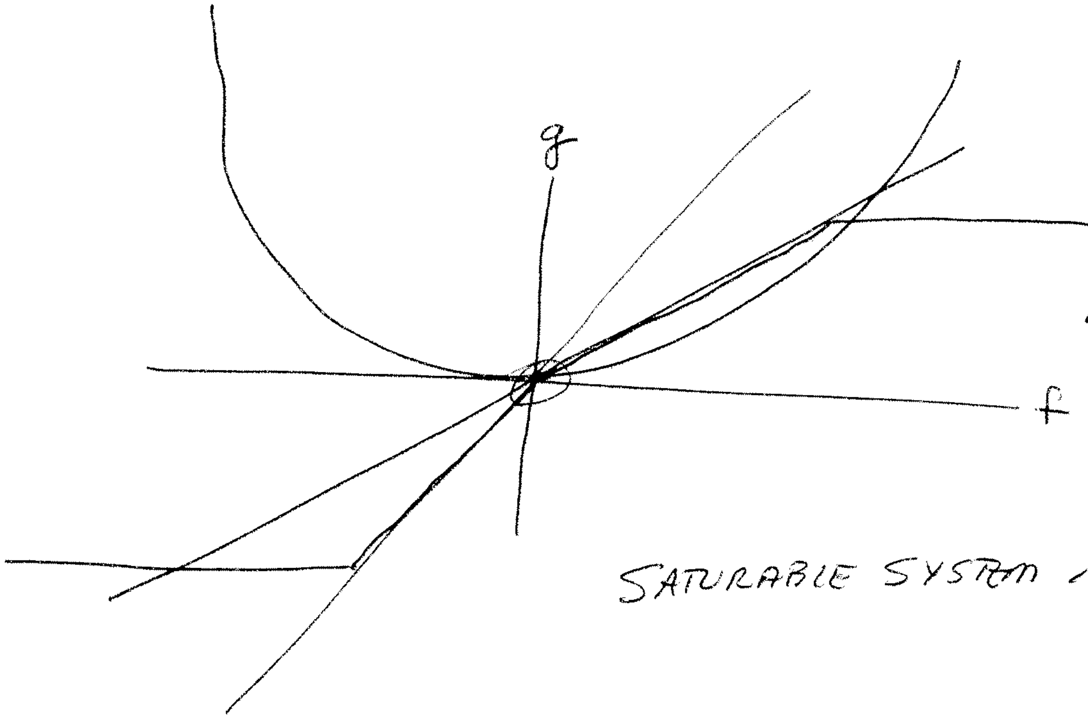
$1(x)$

$$G(\xi) = \frac{1}{2}\cdot\delta(\xi) + \frac{1}{2}\left(\frac{1}{2}\,\delta(\xi + 2\xi_0) + \frac{1}{2}\,\delta(\xi - 2\xi_0)\right)$$

$$= \frac{1}{2}\,\delta(\xi) + \frac{1}{4}\,\delta(\xi + 2\xi_0) + \frac{1}{4}\,\delta(\xi - 2\xi_0)$$



NEW SINUSOIDAL COMPONENTS GENERATED BY NONLINEARITY (OUT OF THIN AIR!)



LOOKUP TABLE FOR AMPLITUDE

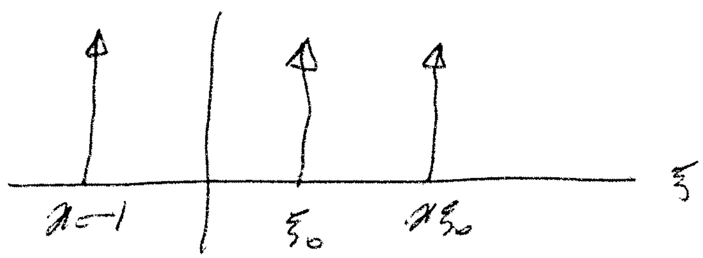
SATURABLE SYSTEM IS NONLINEAR

MORE GENERAL CASE $\quad g(x) = \left(f(x)\right)^{\varkappa}$ WHERE $\varkappa$ IS REAL

SINGLE-FREQ INPUT

$$f(x) = e^{+2\pi i \xi_0 x} \longrightarrow g(x) = \left(e^{+2\pi i \xi_0 x}\right)^{\varkappa}$$

$$= e^{+2\pi i (\varkappa \xi_0) x}$$



$$G(\xi) = \delta(\xi - \varkappa \xi_0)$$

ADD SECOND FREQ.

$$f[x] = a_0 e^{+2\pi i \xi_0 x} + a_1 e^{+2\pi i \xi_1 x}$$

IF $a_0 > a_1 \implies a_0 e^{+2\pi i \xi_0 x} \cdot \left(1 + \frac{a_1}{a_0} e^{+2\pi i (\xi_1 - \xi_0) x}\right)$

$$g(x) = \left(f(x)\right)^{\nu} = \underbrace{\left(a_0^{\nu} e^{+2\pi i (\nu \xi_0) x}\right)}_{a_0^{\nu}\,\delta(\xi - \nu\xi_0)} \cdot \underbrace{\left(1 + \frac{a_1}{a_0} e^{+2\pi i (\xi_1 - \xi_0) x}\right)^{\nu}}_{(\;?\;)}$$

$a_0^{\nu}\,\delta(\xi - \nu\xi_0) * (\;?\;)$

$$\mathcal{F}_1 \left\{ \left(1 + \frac{a_1}{a_0} e^{+2\pi i (\xi_1 - \xi_0) x}\right)^{\nu} \right\}$$

$$\frac{a_1}{a_0} < 1$$

$$\left| \frac{a_1}{a_0} e^{+2\pi i (\xi_1 - \xi_0) x} \right| < 1$$

$$(1 + \beta)^{\nu} = 1 + \nu\beta + \frac{\nu(\nu-1)}{2!}\beta^2 + \frac{\nu(\nu-1)(\nu-2)}{3!}\beta^2 + \cdots$$

10/21/2 — ⑧

$$\mathcal{F}_1\left\{ 1[x] + \varkappa \frac{a_1}{a_0} e^{+2\pi i(\xi_1-\xi_0)x} + \frac{\varkappa(\varkappa-1)}{2}\left(e^{+2\pi i(\xi_1-\xi_0)x}\right)^2 + \cdots \right\}$$

$2(\xi_1-\xi_0)$

$$= \delta[\xi] + \varkappa \frac{a_1}{a_0}\,\delta(\xi-(\xi_1-\xi_0)) + \frac{\varkappa(\varkappa-1)}{2}\left(\frac{a_1}{a_0}\right)^2 \delta(\xi-2(\xi_1-\xi_0)) + \cdots$$

$$G(\xi) = \left(a_0^{\varkappa}\,\delta(\xi-\varkappa\xi_0)\right) * \left(\delta(\xi) + \varkappa\frac{a_1}{a_0}\,\delta(\xi-(\xi_1-\xi_0)) + \frac{\varkappa(\varkappa-1)}{2}\left(\frac{a_1}{a_0}\right)^2 \delta(\xi-(2\xi_1-2\xi_0)) + \cdots\right)$$

$$= a_0^{\varkappa}\,\delta(\xi-\varkappa\xi_0) + \varkappa\frac{a_1}{a_0}a_0^{\varkappa}\,\delta(\xi-(\varkappa\xi_0+\xi_1-\xi_0)) + \frac{\varkappa(\varkappa-1)}{2}a_0^{\varkappa}\frac{a_1^2}{a_0^2}\,\delta(\xi-(\varkappa\xi_0+2\xi_1-2\xi_0)) + \cdots$$

MULTIDIMENSIONAL FOURIER TRANSFORMS

(2-D)

$f[x,y]$ →

(1) SEPARABLE (CARTESIAN) § 10

$$f[x,y] = f_1[x]\, f_2[y]$$

(2) CIRCULARLY SYMMETRIC § 11

$$f\left(\sqrt{x^2+y^2}\right) 1(\theta) = f(r)\, 1(\theta)$$

(3) ARBITRARY § 12

RADON TRANSFORM $\underset{\sim}{\mathcal{F}} \circ \hat{\underset{\sim}{P}}\ ,\ \underset{\sim}{\mathcal{F}} \circ \hat{\underset{\sim}{P}}^{-1}$

Separable 2-D Functions

$$f[x,y] = f_1[x] \cdot f_2[y] \xrightarrow{\mathcal{F}} \text{~~}F_1[\xi] * F_2[\eta]\text{~~} \; ?$$

$$\rightarrow \left(f_1[x] \cdot 1(y)\right) \cdot \left(1(x) \cdot f_2(y)\right)$$

$$\left[F_1[\xi] \cdot \delta(\eta)\right] * \left(\delta(\xi)\, F_2(\eta)\right)$$

$$= \left(F_1[\xi] * \delta(\xi)\right) \cdot \left(\delta(\eta) * F_2(\eta)\right) = F_1(\xi) \cdot F_2(\eta)$$

$$= F(\xi, \eta)$$

$$\boxed{f_1(x) \cdot f_2(y) \longrightarrow F_1(\xi) \cdot F_2(\eta)}$$

e.g.,

$$\text{RECT}[x,y] = \text{RECT}(x)\cdot\text{RECT}(y) \xrightarrow{f_2} \text{SINC}(\xi)\,\text{SINC}(\eta) = \text{SINC}[\xi,\eta]$$

$$\text{RECT}[x]\cdot\delta(y) \longrightarrow \text{SINC}(\xi)\cdot 1(\eta)$$

$$1[x]\,\delta(y) \longrightarrow \delta(x)\,1(y)$$

$$\text{CROSS}[x,y] = 1(x)\,\delta(y) + \delta(x)\,1(y) \longrightarrow \delta(\xi)\,1(\eta) + 1(\xi)\,\delta(\eta) = \text{CROSS}(\xi,\eta)$$

ROTATE $f(x,y)$ BY θ $\longrightarrow$ TRANSFORM ROTATES BY θ

THEOREMS

(1) ROTATE $f(x,y) \longrightarrow F(\xi,\eta)$ ROTATES

(2) SCALING $f_1\left[\frac{x}{b_0}\right] f_2\left(\frac{y}{d_0}\right) \longrightarrow |b_0 d_0|\, F_1[b_0\xi]\, F_2(d_0\eta) = |b_0 d_0|\, F_1[b_0\xi, d_0\eta]$

SHIFTING THEOREM

$$f_1(x-x_0)\cdot f_2(y-y_0) \xrightarrow{\mathcal{F}_2} F_1(\xi)\,F_2(\eta)\,e^{-2\pi i(\xi x_0+\eta y_0)}$$

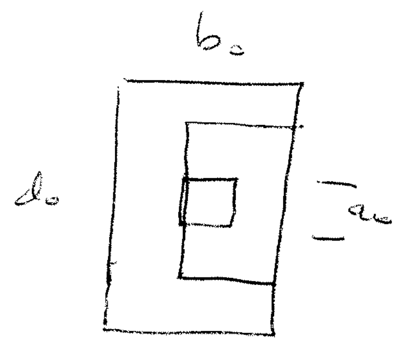
$$= F_{\mathcal{F}}(\xi,\eta)\,e^{-2\pi i(\xi x_0+\eta y_0)}$$

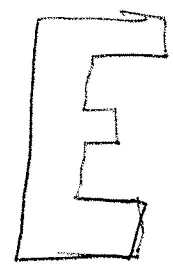
SUPERPOSITIONS

$$\mathrm{Rect}\left(\frac{x}{b_0},\frac{y}{d_0}\right)$$

$$+\ \mathrm{Rect}\left(\frac{x}{a_0},\frac{y}{a_0}\right)$$

$$-\ \mathrm{Rect}\left(\frac{x-x_0}{b_0-2}\right)\mathrm{Rect}\left(\frac{y}{d_0-2}\right)$$





$$\longrightarrow |b_0 d_0|\,\mathrm{Sinc}(b_0\xi, d_0\eta) + |a_0|^2\,\mathrm{Sinc}(a_0\xi, a_0\eta)$$

$$|(b_0-2)(d_0-2)|\,\mathrm{Sinc}\left((b_0-2)\xi, (d_0-2)\eta\right)e^{-2\pi i\xi x_0}$$

TRANSFORM-OF-TRANSFORM

$$f(x,y) = f_1(x)\, f_2(y) \longrightarrow F(\xi,\eta) = F_1(\xi) F_2(\eta)$$

$$F(x,y) = F_1(x) F_2(y) \longrightarrow f_1(-\xi)\, f_2(-\eta)$$

$$= f(-\xi, -\eta)$$

DERIVATIVE THEOREM

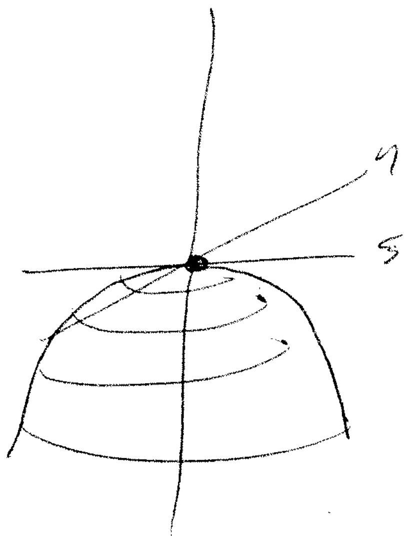
$$\left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m f(x,y) = \left(\frac{\partial}{\partial x}\right)^n f_1(x) \cdot \left(\frac{\partial}{\partial y}\right)^m f_2(y)$$

$$= (2\pi i\xi)^n F_1(\xi) \cdot (2\pi i\eta)^m F_2(\eta)$$

$$= \left((2\pi i\xi)^n (2\pi i\eta)^m\right) F(\xi,\eta)$$

LAPLACIAN

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



$$\nabla^2 f[x,y] = \frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y] = g(x]$$

$$\cancel{G[\xi]} = \left(\frac{\partial^2}{\partial x^2} f_1[x]\right) f_2(y) + f_1(x)\left(\frac{\partial^2}{\partial y^2} f_2(y)\right)$$

$$= (+2\pi i\xi)^2 F_1(\xi) F_2(\eta) + (2\pi i\eta)^2 F_1(\xi) F_2(\eta)$$

$$= \left(-(2\pi)^2 \xi^2 - (2\pi)^2 \eta^2\right) F[\xi,\eta]$$

$$= -(2\pi)^2 \rho^2 F(\xi,\eta)$$

WHERE $\rho^2 \equiv \xi^2 + \eta^2 \Leftarrow (r^2 = x^2 + y^2)$

CIRCULARLY SYMMETRIC CASE

$$f[x,y] \longrightarrow f\left(\sqrt{x^2+y^2}\right) \quad f(\theta)$$

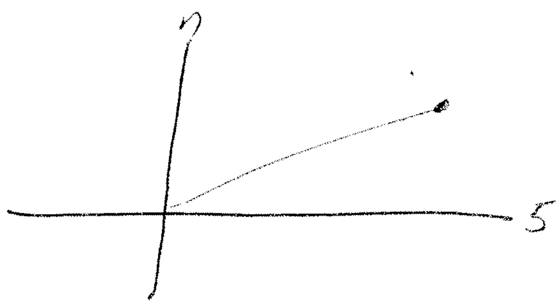
$$f(r)$$

DERIVE CIRCULARLY SYMMETRIC VERSION OF $\mathcal{F}_2 \rightarrow \mathcal{H}_0$

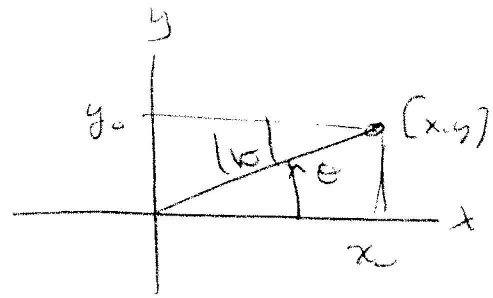
$$\iint_{-\infty}^{+\infty} f[x,y]\, e^{-2\pi i(\xi x + \eta y)}\, dx\, dy \longrightarrow \mathcal{H}_0\{f(r)\}$$

2-D SINOSOID WITH FREQ $[\xi,\eta]$

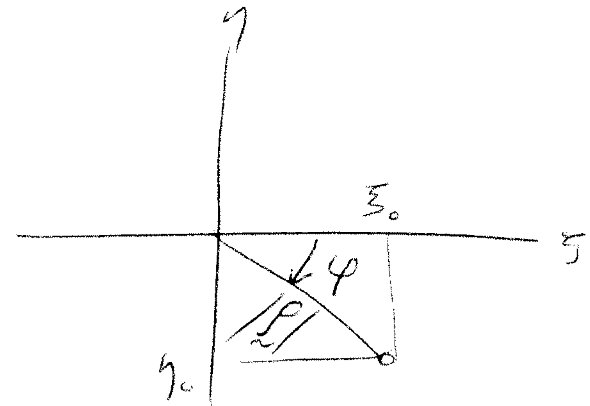
1-D TRANSFORM



$\xi x + \eta y$



$$\underset{\sim}{r} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underset{\sim}{\rho} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$



$$\underset{\sim}{r} \cdot \underset{\sim}{\rho} = x\xi + y\eta = |\underset{\sim}{r}||\underset{\sim}{\rho}| \cos \alpha$$

$$x = |\underset{\sim}{r}| \cos\theta \qquad \xi = |\underset{\sim}{\rho}| \cos\varphi$$

$$y = |\underset{\sim}{r}| \sin\theta \qquad \eta = |\underset{\sim}{\rho}| \sin\varphi$$

$$x\xi + y\eta = |\underset{\sim}{r}| \cos\theta \, |\underset{\sim}{\rho}| \cos\varphi + |\underset{\sim}{r}| \sin\theta \, |\underset{\sim}{\rho}| \sin\varphi$$

$$= |\underset{\sim}{r}||\underset{\sim}{\rho}| \left(\cos\theta \cos\varphi + \sin\theta \sin\varphi \right)$$

$$x\xi + y\eta = |\underset{\sim}{r}||\underset{\sim}{\rho}|(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$

$$= |\underset{\sim}{r}||\underset{\sim}{\rho}|\cos(\underline{\underline{\theta - \varphi}})$$

$dx\,dy = r\,dr\,d\theta$

$$\iint_{-\infty}^{+\infty} f\left(\sqrt{x^2+y^2}\right) 1(\theta)\, e^{-2\pi i(\xi x + \eta y)}\, dx\, dy$$

$$= \int_{\theta=-\pi}^{+\pi}\int_{r=0}^{r=\infty} f(r)\, 1(\theta)\, e^{-2\pi i(|\underset{\sim}{r}||\underset{\sim}{\rho}|\cos(\theta-\varphi))}\, r\,dr\,d\theta$$

$$F(\rho)1(\varphi) = \int_{r=0}^{\infty} f(r)\left(\underline{\int_{\theta=-\pi}^{+\pi} 1(\theta)\, e^{-2\pi i r\rho\cos(\theta-\varphi)}\, d\theta}\right) r\,dr$$

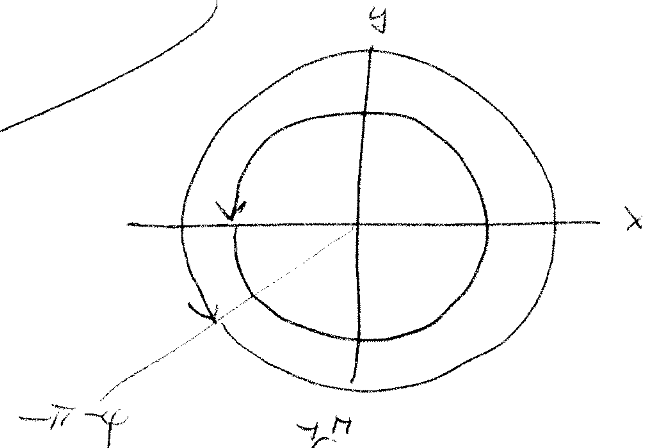
$$\int_{-\pi}^{+\pi} e^{-2\pi i r \rho \cos(\theta - \varphi)} \, d\theta = \int_{\xi = -\pi - \varphi}^{+\pi - \varphi} e^{-2\pi i r \rho \cos \xi} \, d\xi$$

$\xi \equiv \theta - \varphi$

$d\xi = d\theta$

$\theta = -\pi \Rightarrow \xi = -\pi - \varphi$

$\theta = +\pi \Rightarrow \xi = +\pi - \varphi$



$$\int_{\xi = -\pi}^{\xi = +\pi} e^{-2\pi i r \rho \cos \xi} \, d\xi \longrightarrow \int_{-\pi}^{+\pi} e^{-2\pi i r \rho \cos \theta} \, d\theta$$

$$\int_{-\pi}^{+\pi} \cos(2\pi r\rho \cos\theta)\, d\theta \;-\; i \int_{-\pi}^{+\pi} \sin(2\pi r\rho \cos\theta)\, d\theta \rightarrow 0$$

$$\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$$

$$\int_{-\pi}^{+\pi} 1\, d\theta - \int_{-\pi}^{+\pi} \frac{(2\pi r\rho\cos\theta)^2}{2}\, d\theta + \cdots$$