19 OCTOBER 2009

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THEOREMS: SHIFT THEOREM

$$f[x] \longrightarrow F[\xi] \quad \text{(SPECTRUM)}$$

$$f[x-x_0] \longrightarrow F[\xi]\, e^{-2\pi i \xi x_0}$$

FILTER THEOREM

$$f(x] \longrightarrow F[\xi]$$

$$h[x] \longrightarrow H[\xi]$$

$$f(x] * h[x] \longrightarrow$$

$$\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\alpha]\, h[x-\alpha]\, d\alpha \right] e^{-2\pi i \xi x}\, dx$$

SWAP ORDER OF INTEGRATION

$$\int_{-\infty}^{+\infty} f(\alpha) \left[\underbrace{\int_{-\infty}^{+\infty} h[x-\alpha]\, e^{-2\pi i \xi x}\, dx}_{\text{SHIFT THEOREM}} \right] d\alpha = \int_{-\infty}^{+\infty} f(\alpha] \left[H[\xi]\, e^{-2\pi i \xi \alpha} \right] d\alpha$$

$$\int_{-\infty}^{+\infty} f(\alpha)\left[H(\xi)\, e^{-2\pi i \xi \alpha}\right] d\alpha = H(\xi) \int_{-\infty}^{+\infty} f(\alpha)\, e^{-2\pi i \xi \alpha}\, d\alpha$$

$$= H(\xi) \cdot F(\xi) = F(\xi) \cdot H(\xi)$$

$$\boxed{\mathcal{F}_1\{f[x] * h[x]\} = F[\xi] \cdot H(\xi)}$$

$$\mathcal{O}\{f[x]\} = f(x) * \underline{h(x)} = \underline{g(x)}$$

$$F[\xi] \cdot H(\xi) = G(\xi) \implies F(\xi) = \frac{G(\xi)}{H(\xi)}$$

IF $H(\xi) \neq 0$ AT ANY ξ, THEN $f(x) = \mathcal{F}^{-1}\left\{\frac{G(\xi)}{H(\xi)}\right\}$

$$h[x] = \mathcal{F}^{-1}\left\{\frac{G(\xi)}{F(\xi)}\right\}$$

Space Domain

Freq. Domain

$$
\begin{tikzcd}
f(x) \arrow[r] \arrow[d, shift right, "\mathcal{F}_1"] & * \, h(x) \arrow[r] \arrow[d, shift right, "\mathcal{F}_1"] & g(x) \arrow[d, shift right, "\mathcal{F}_1"] \arrow[d, bend left=60, leftrightarrow] \\
F(\xi) \arrow[r] \arrow[u, shift right, "\mathcal{F}_1^{-1}"] & \cdot \, H(\xi) \arrow[r] \arrow[u, shift right, "\mathcal{F}_1^{-1}"] & G(\xi) \arrow[u, shift right, "\mathcal{F}_1^{-1}"]
\end{tikzcd}
$$

Same "Function," Different Representations

Solve "Direct Task" around the block

Solve Inverse Problem or Analysis Problem

$h[x]$ is the Impulse Response
Point Spread Function (psf)

$H[\xi]$ is the Transfer Function
Optical Transfer Function (OTF)

$$H[\xi] = |H[\xi]|\, e^{+i\{\Phi_H[\xi]\}}$$

$$\left.\begin{array}{l}\Phi_H[\xi] \\ \Phi\{H(\xi)\}\end{array}\right\}$$

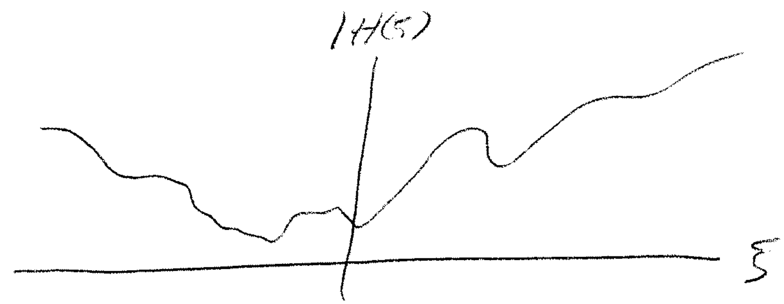
PHASE INCREMENT AT ξ DUE TO SYSTEM (CONVOLUTION)

PHASE TRANSFER FUNCTION (ΦTF)

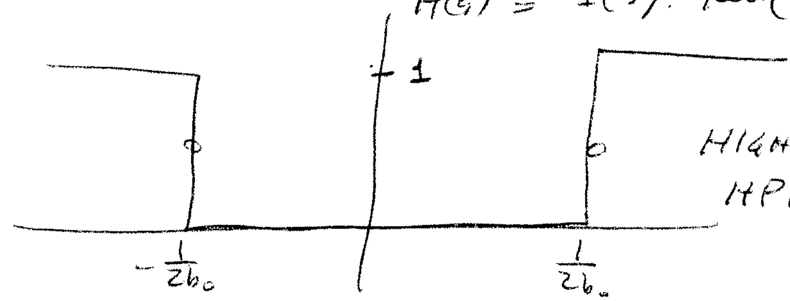
$$F(\xi) \cdot H(\xi) = |F(\xi)|\, e^{i\Phi_F(\xi)} \cdot |H(\xi)|\, e^{i\Phi_H(\xi)}$$

$$= \underbrace{|F(\xi)| \cdot |H(\xi)|}_{\text{MAGNITUDES MULTIPLIED}} \cdot e^{i\left(\underbrace{\Phi_F(\xi) + \Phi_H(\xi)}_{\text{PHASES ADDED}}\right)}$$

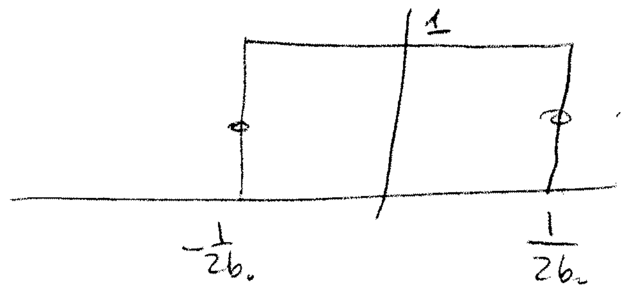
$|H(\xi)|$



$H(\xi) = 1(\xi) - \mathrm{Rect}(b_0 \xi)$



HIGHPASS FILTER
HPF



$H(\xi) = \mathrm{Rect}(b_0 \xi)$

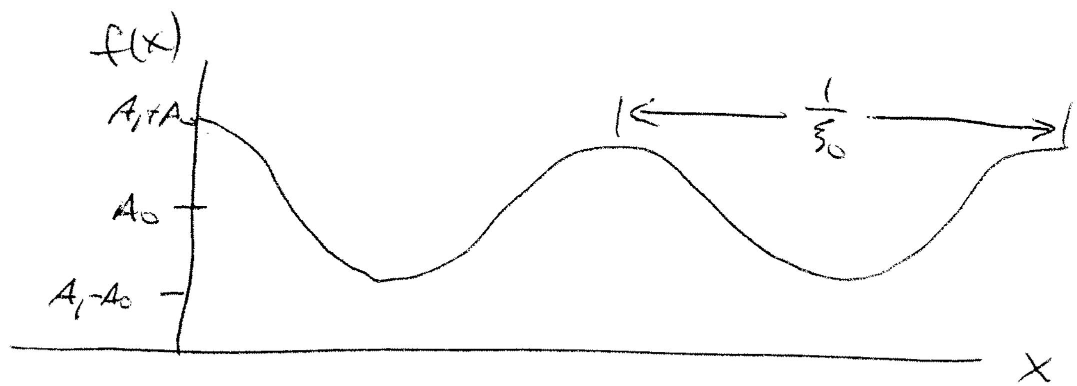
: LPF (LOWPASS FILTER)

IN OPTICS $|H(\xi = 0)| \neq 0$; $|H(0)| =$ MAXIMUM OF $|H(\xi)|$

$\dfrac{|H(\xi)|}{|H(0)|} =$ MODULATION TRANSFER FUNCTION (MTF)

HOW WELL THE "MODULATION" OF THE INPUT IS TRANSFERRED TO THE OUTPUT

MODULATION OF SINUSOID WITH $f \geq 0$



$$f(x) = A_0 + A_1 \cos(2\pi \xi_0 x)$$

$$m_f \equiv \frac{f_{max} - f_{min}}{f_{max} + f_{min}} = \frac{(A_0 + A_1) - (A_0 - A_1)}{(A_0 + A_1) + (A_0 - A_1)} = \frac{2A_1}{2A_0} = \frac{A_1}{A_0} \begin{array}{l} \text{— AMPLITUDE} \\ \text{— BIAS} \end{array}$$

LOWPASS FILTER $\Rightarrow$ REDUCED MODULATION

$$f(x) = 1 \cdot e^{+2\pi i \xi_0 x} = \cos(2\pi \xi_0 x) + i \sin(2\pi \xi_0 x)$$

$$\downarrow \mathcal{F}_1$$

$$F(\xi) = \delta(\xi - \xi_0)$$

SINGLE-FREQUENCY SINUSOID

$$f(x)\,\delta(x - x_0) = f(x_0)\,\underline{\underline{\delta(x - x_0)}}$$

$$f(x) * h(x) = g(x)$$

$$\downarrow$$

$$\delta(\xi - \xi_0) \cdot H(\xi) = \delta(\xi - \xi_0) \cdot \underbrace{H(\xi_0)}_{\text{CONSTANT}} = G(\xi)$$

$$g(x) = \mathcal{F}_1^{-1}\{G(\xi)\} = \int_{-\infty}^{+\infty} \delta(\xi - \xi_0)\, H(\xi_0)\, e^{+2\pi i \xi x}\, d\xi$$

$$g(x) = H(\xi_0) \int_{-\infty}^{+\infty} \delta(\xi - \xi_0)\, e^{+2\pi i \xi x}\, d\xi = \underline{\underline{H(\xi_0)}} \cdot e^{+2\pi i \xi_0 x}$$

$$\mathcal{O}\left\{ \underline{\underline{e^{+2\pi i \xi_0 x}}} \right\} = e^{+2\pi i \xi_0 x} * h(x) = g(x)$$

$F(\xi)$

$$= \underline{\underline{H(\xi_0)}} \cdot e^{+2\pi i \xi_0 x} = g(x)$$

$g(x) \propto \underline{f(x)}$ OUTPUT IS PROPORTIONAL TO INPUT

$$g(x) = \lambda \cdot f(x) \longrightarrow \lambda(\xi_0) \cdot f(x)$$

EIGENFUNCTION OF CONVOLUTION

ANY FUNCTION OF FORM $e^{+2\pi i \xi_0 x}$ IS ONLY SCALED BY A MULTIPLICATIVE FACTOR $\lambda(\xi_0) = H(\xi_0)$

$H(\xi_0)$ IS THE EIGENVALUE

MODULATION THEOROM

$$\mathcal{F}_1\{\underbrace{f(x)}_{\mathcal{F}_1^{-1}\{F(\xi)\}} \cdot \underbrace{m(x)}_{\mathcal{F}_1^{-1}\{M(\xi)\}}\} = F(\xi) * M(\xi)$$

$$\int_{-\infty}^{+\infty} \left(f(x) \cdot m(x)\right) e^{-2\pi i \xi x}\, dx$$

$$f(x) = \mathcal{F}_1^{-1}\{F(\xi)\} = \int_{-\infty}^{+\infty} F(u)\, e^{+2\pi i x u}\, du$$

$$m(x) = \mathcal{F}_1^{-1}\{M(\xi)\} = \int_{-\infty}^{+\infty} M(v)\, e^{+2\pi i x v}\, dv$$

$$\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} F(u)\, e^{+2\pi i u x}\, du\right]\left[\int_{-\infty}^{+\infty} M(v)\, e^{+2\pi i v x}\, dv\right] e^{-2\pi i \xi x}\, dx$$

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$$\int_{-\infty}^{-\infty} F[u] \int_{-\infty}^{+\infty} M(v) \left[\int_{+\infty}^{+\infty} e^{+2\pi i u x} e^{+2\pi i v x} e^{-2\pi i \xi x} \, dx \right] dv \, du$$

$$\int_{-\infty}^{+\infty} F(u) \int_{-\infty}^{+\infty} M(v) \underbrace{\int_{-\infty}^{+\infty} e^{+2\pi i x(\underline{u+v-\xi})} \, dx}_{\delta(u+v-\xi)} \, dv \, du$$

$$\int_{-\infty}^{+\infty} F[u] \left(\int_{-\infty}^{+\infty} M(v) \, \delta(v-(\xi-u)) \, dv \right) du$$

$$\int_{-\infty}^{+\infty} F[u] \, M(\xi-u) \, du = F(\xi] * M(\xi) = \mathcal{F}_1\{ f(x) \cdot m(x) \}$$

DERIVATIVE THEOREM

$$f(x) \longrightarrow F(\xi)$$

$$\frac{df}{dx} \longrightarrow ? \qquad \frac{d^n f}{dx^n} \longrightarrow ?$$

$\mathcal{F}^{-1}\{F(\xi)\}$

$$\frac{d^n}{dx^n}\left(\int_{-\infty}^{+\infty} F(\xi)\, e^{+2\pi i \xi x}\, d\xi\right) = \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}\left(F(\xi)\, e^{+2\pi i \xi x}\right) d\xi$$

$$= \int_{-\infty}^{+\infty} F(\xi) \cdot \frac{d^n}{dx^n}\left(e^{+(2\pi i \xi) x}\right) d\xi = \int_{-\infty}^{+\infty} \left[F(\xi)\,(+2\pi i \xi)^n\right] e^{+2\pi i \xi x}\, d\xi$$

$$\frac{d}{dx} e^{Ax} = A e^{Ax}$$

$$\frac{d^n}{dx^n} e^{Ax} = A^n e^{Ax}$$

$$\int_{-\infty}^{+\infty} \left(F(\xi) \cdot (+2\pi i \xi)^n \right) e^{+2\pi i \xi x} \, d\xi = \frac{d^n}{dx^n} f(x)$$

$$\mathcal{F}_1 \mathcal{F}_1^{-1} \left\{ F(\xi) \cancel{\cdot} (+2\pi i \xi)^n \right\} = \mathcal{F}_1 \left\{ \frac{d^n}{dx^n} f(x) \right\}$$

$$\mathcal{F}_1 \left\{ \frac{d^n}{dx^n} f(x) \right\} = F(\xi) \cdot \underbrace{(+2\pi i \xi)^n}_{H(\xi)}$$

TRANSFER FUNCTION OF DIFFERENTIATION

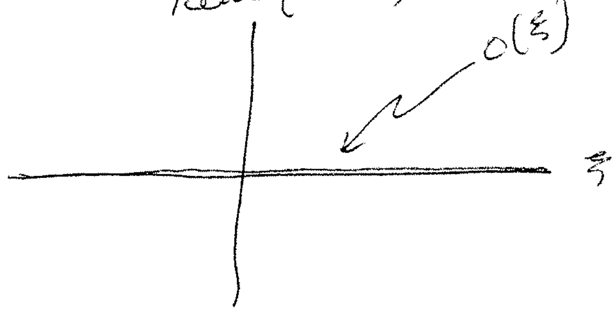
DIFFERENTIATION IS LSI $\Rightarrow$ CONVOLUTION

$$f(x) * \frac{d^n}{dx^n} \delta(x) = \frac{d^n f}{dx^n}$$

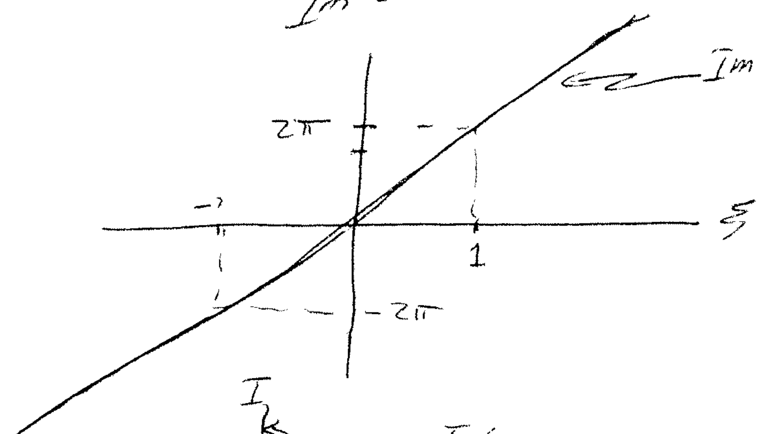
$$\mathcal{F}_1 \left\{ \frac{d^n \delta(x)}{dx^n} \right\} = (+2\pi i \xi)^n$$

$$\mathcal{F}_1\left\{\frac{df}{dx}\right\} = (+2\pi i\xi)\cdot F(\xi)$$

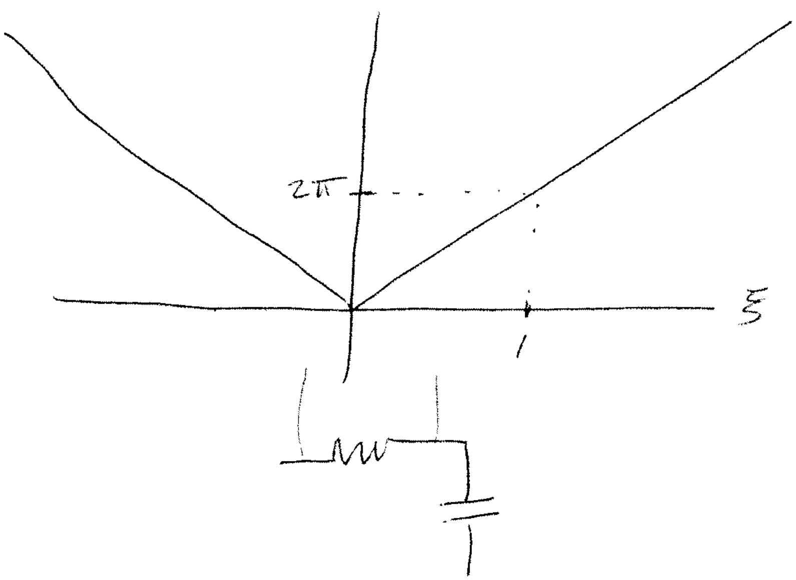
$\text{Real}\{+2\pi i\xi\} = 0(\xi)$



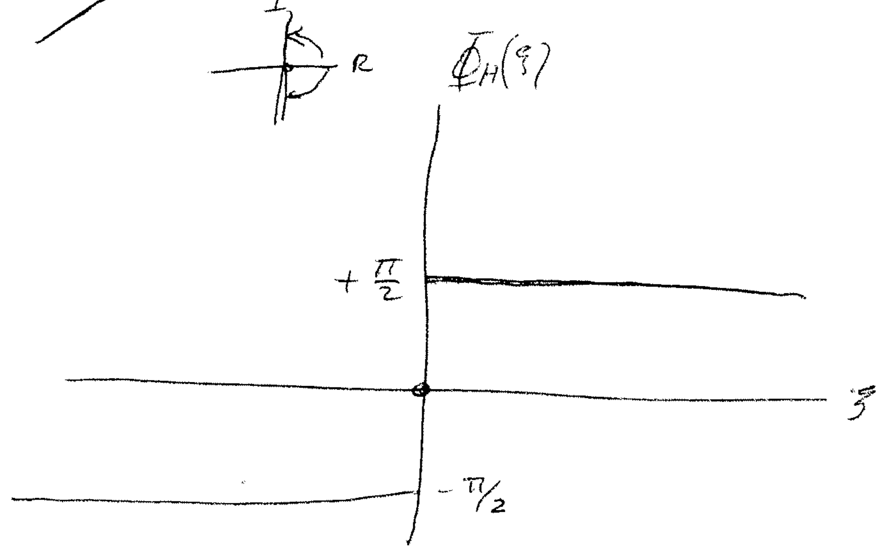
$Im = +2\pi\xi$



$|H(\xi)| = |2\pi\xi|$

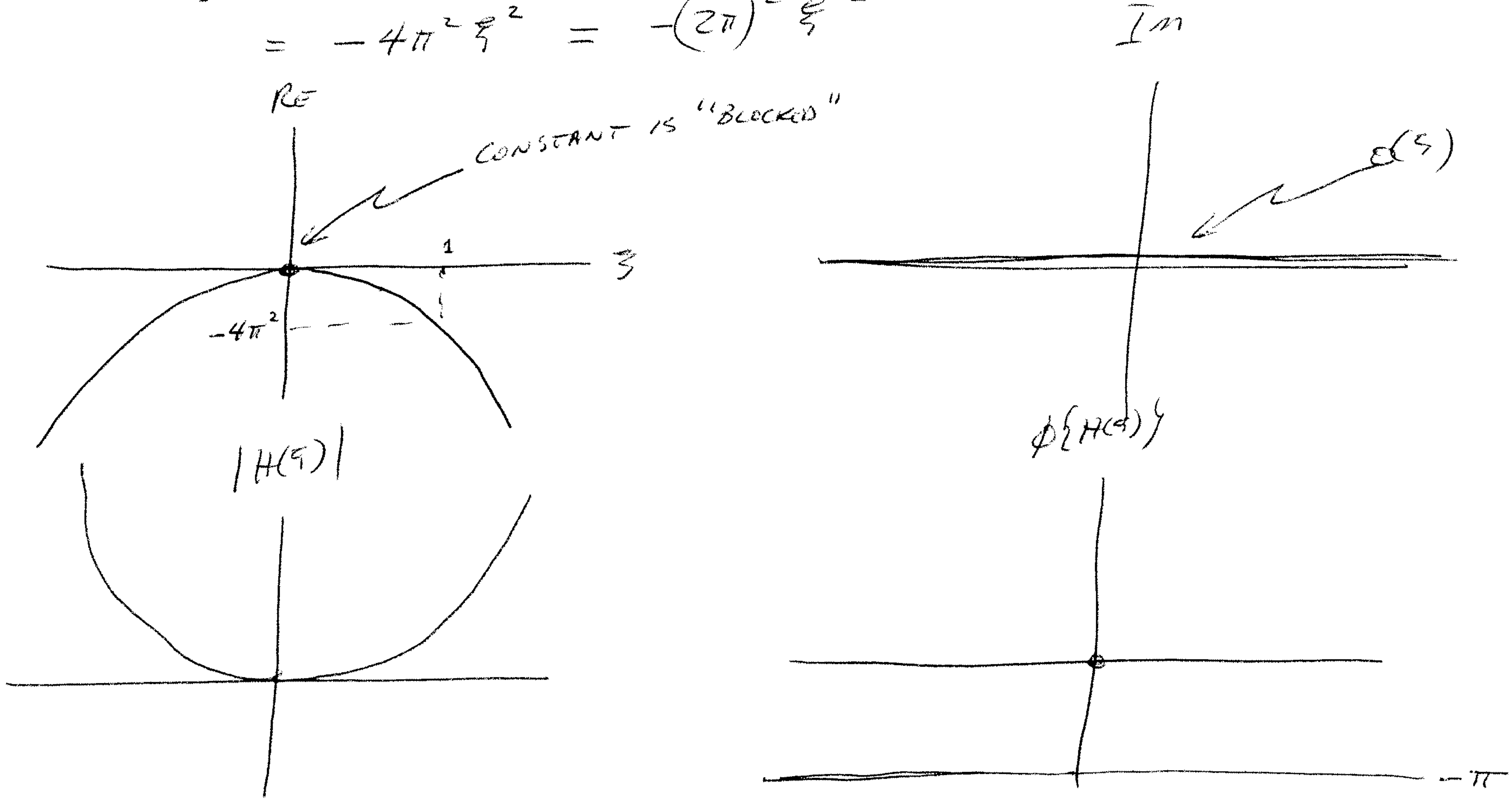


$\Phi_H(\xi)$



$$\mathcal{F}\left\{\frac{d^2f}{dx^2}\right\} = \left(+2\pi i \xi\right)^2 F(\xi)$$

$$= -4\pi^2\xi^2 = -(2\pi)^2\xi^2$$



$$\nabla^2 \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \longrightarrow -4\pi^2(\xi^2+\eta^2) = -4\pi^2(\rho^2)$$

$$\frac{d}{dx}\cos(2\pi\xi_0 x) = -2\pi\xi_0 \sin(2\pi\xi_0 x) \qquad \cos\left(\theta-\frac{\pi}{2}\right)=+\sin\theta$$

$$= -2\pi\xi_0 \cos\left(2\pi\xi_0 x - \frac{\pi}{2}\right)$$

$$= 2\pi\xi_0 \cos\left(2\pi\xi_0 x - \frac{\pi}{2} + \pi\right)$$

$$= \underbrace{2\pi\xi_0}_{MTF} \cos\Big(2\pi\xi_0 x + \underbrace{\frac{\pi}{2}}_{\phi TF}\Big)$$

$f[x] \longrightarrow F(\xi)$

$f^*[x] \longrightarrow ?$

$\left(f[-x] \longrightarrow F(-\xi) \right)$

$f^*(-x)$

FOURIER TRANSFORM OF COMPLEX CONJUGATE

$$\int_{-\infty}^{+\infty} f^*(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int \left(f(x)\, e^{+2\pi i \xi x} \right)^* dx$$

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$$\mathcal{F}\{f^*(x)\} = \int_{-\infty}^{+\infty} \left(f(x)\, e^{+2\pi i \xi x} \right)^* dx$$

$$= \left(\int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i (-\xi) x}\, dx \right)^*$$

$$= \left(F(-\xi) \right)^* = F^*(-\xi)$$

$$\mathcal{F}_1 \{ f^*(x) \} = F^*(-\xi)$$

$$\mathcal{F}_1 \{ f(-x) \} = F(-\xi) \qquad \text{SCALING THEOREM}$$

$$\mathcal{F}_1 \{ f^*(-x) \} = F^*(\xi)$$

CROSS CORRELATION "REFERENCE FUNCTION"

$$f(x) \star m(x) = f(x) * m^*(-x)$$

$$\mathcal{F}_1\{f(x) * m^*(-x)\} = \mathcal{F}_1\{f(x)\} \cdot \mathcal{F}_1\{m^*(-x)\}$$

$$\boxed{\mathcal{F}_1\{f(x) \star m(x)\} = F(\xi) \cdot M^*(\xi)}$$

AUTOCORRELATION $\rightarrow$ WIENER - KHINTCHIN THEOREM

$$\mathcal{F}_1\{f(x) \star f(x)\} = F(\xi) \cdot F^*(\xi) = |F(\xi)|^2$$ POWER SPECTRUM

$$|F(\xi)| = \sqrt{\mathcal{F}_1\{f(x) \star f(x)\}}$$ MAGNITUDE SPECTRUM

IF $f(x) = n(x)$ STOCHASTIC SIGNAL ("NOISE")

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RAYLEIGH'S THEOREM

$$\int_{-\infty}^{+\infty} f(x)\,h^*(x)\,dx = \int_{-\infty}^{+\infty} \underbrace{\left[\int_{-\infty}^{+\infty} F(\xi)\,e^{+2\pi i \xi x}\,d\xi\right]}_{f(x)} \underbrace{\left[\int_{-\infty}^{+\infty} H(\nu)\,e^{+2\pi i \nu x}\,d\nu\right]^*}_{h^*(x)} dx$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} F(\xi)\,e^{+2\pi i \xi x}\,d\xi\right]\left[\int_{-\infty}^{+\infty} H^*(\nu)\,e^{-2\pi i \nu x}\,d\nu\right] dx$$

$$= \int_{-\infty}^{+\infty} F(\xi) \int_{-\infty}^{+\infty} H^*(\nu) \underbrace{\left(\int_{-\infty}^{+\infty} e^{+2\pi i (\xi-\nu) x}\,dx\right)}_{\delta(\xi-\nu)} d\nu\, d\xi$$

$$= \int_{-\infty}^{+\infty} F(\xi) \int_{-\infty}^{+\infty} H^*(\nu)\,\delta(\xi-\nu)\,d\nu\, d\xi = \int_{-\infty}^{+\infty} F(\xi)\,H^*(\xi)\,d\xi$$

$$\boxed{\int_{-\infty}^{+\infty} \left(f(x)\, \underline{\underline{h^*(x)}} \right) dx = \int_{-\infty}^{+\infty} F(\xi)\, H^*(\xi)\, d\xi}$$

RAYLEIGH'S THM.

IF $h(x) = f(x)$

$$\int_{-\infty}^{+\infty} f(x) \cdot f^*(x)\, dx = \boxed{\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(\xi)|^2 d\xi}$$

PARSEVAL'S THM.

CONSERVATION OF ENERGY